Norm attaining compact operators

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Roadmap of the talk

1. Introducing the topic
2. An quick overview on norm attaining operators
3. Norm attaining compact operators

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Introducing the topic

Section 1

Introducing the topic

Section 1

Notation

X, Y real or complex Banach spaces

- $\mathbb{K}$ base field $\mathbb{R}$ or $\mathbb{C}$,
- $B_X = \{x \in X : \|x\| \leqslant 1\}$ closed unit ball of X,
- $S_X = \{x \in X : \|x\| = 1\}$ unit sphere of X,
- $\mathcal{L}(X, Y)$ bounded linear operators from X to Y,
  - $\|T\| = \sup\{\|T(x)\| : x \in S_X\}$ for $T \in \mathcal{L}(X, Y)$,
- $\mathcal{K}(X, Y)$ compact linear operators from X to Y,
- $\mathcal{F}(X, Y)$ bounded linear operators from X to Y with finite rank (i.e. dimension of the range is finite),
- $X^* = \mathcal{L}(X, \mathbb{K})$ topological dual of X.

Introducing the topic

Section 1

Norm attaining functionals

Norm attaining functionals

$x^* \in X^*$ attains its norm when

$$\exists\, x \in S_X \ : \ |x^*(x)| = \|x^*\|$$

★ $\mathrm{NA}(X,\mathbb{K}) = \{x^* \in X^* : x^*$ attains its norm$\}$

First examples

- $\dim(X) < \infty \implies \mathrm{NA}(X,\mathbb{K}) = \mathcal{L}(X,\mathbb{K})$ (Heine-Borel).
- X reflexive $\implies \mathrm{NA}(X,\mathbb{K}) = \mathcal{L}(X,\mathbb{K})$ (Hahn-Banach).
- X non-reflexive $\implies \mathrm{NA}(X,\mathbb{K}) \neq \mathcal{L}(X,\mathbb{K})$ (James),
- but $\mathrm{NA}(X,\mathbb{K})$ always separates the points of X (Hahn-Banach).

Norm attaining operators

Norm attaining operators

$T \in \mathcal{L}(X,Y)$ **attains its norm** when

$$\exists\, x \in S_X \;:\; \|T(x)\| = \|T\|$$

★ $\mathrm{NA}(X,Y) = \{T \in \mathcal{L}(X,Y) \colon T \text{ attains its norm}\}$

First examples

- $\dim(X) < \infty \implies \mathrm{NA}(X,Y) = \mathcal{L}(X,Y)$ for every Y (Heine-Borel).
- $\mathrm{NA}(X,Y) \neq \emptyset$ (Hahn-Banach),
- X reflexive $\implies \mathcal{K}(X,Y) \subseteq \mathrm{NA}(X,Y)$ for every Y (we will comment),
- X non-reflexive $\implies \mathcal{K}(X,Y) \nsubseteq \mathrm{NA}(X,Y)$ for any Y (James),
- $\dim(X) = \infty \implies \mathrm{NA}(X,c_0) \neq \mathcal{L}(X,c_0)$ (see M.-Merí-Payá, 2006).

The problem of density of norm attaining functionals

Problem

Is $\mathrm{NA}(X,\mathbb{K})$ always dense in X^***?**

Theorem (E. Bishop & R. Phelps, 1961)

The set of norm attaining functionals is dense in X^* (for the norm topology).

Problem

Is $\mathrm{NA}(X,Y)$ always dense in $\mathcal{L}(X,Y)$**?**

The answer is **No**, and this is the origin of the study of norm attaining operators.

Modified problem

When is $\mathrm{NA}(X,Y)$ dense in $\mathcal{L}(X,Y)$**?**

The study of this problem was initiated by J. Lindenstrauss in 1963,
who provided the first negative and positive examples.

An quick overview on norm attaining operators

Section 2

2 An quick overview on norm attaining operators
- First results: Lindenstrauss
- The relation with the RNP: Bourgain
- Counterexamples for property B: Gowers and Acosta
- Some results on pairs of classical spaces
- Main open problems

An quick overview on norm attaining operators

Section 2

Lindenstrauss' seminal paper of 1963

Negative answer

There are bounded linear operators which cannot be approximated by norm-attaining operators:

- the domain can be c_0 (usual norm),
- the range can be any strictly convex renorming of c_0,
- the domain and the range may coincide.

The result for c_0 (we will give a detailed proof later)

$$Y \text{ strictly convex, } T \in \mathrm{NA}(c_0, Y) \quad \Longrightarrow \quad Te_n = 0 \text{ for } n \text{ big enough}$$

Observation

- The question then is for which X and Y the density holds.
- As this problem is too general, Lindenstrauss introduced two properties.

Lindenstrauss properties A and B

Definition

X, Y Banach spaces,

- X has (Lindenstrauss) property A iff $\overline{\mathrm{NA}(X,Z)} = \mathcal{L}(X,Z) \quad \forall Z$
- Y has (Lindenstrauss) property B iff $\overline{\mathrm{NA}(Z,Y)} = \mathcal{L}(Z,Y) \quad \forall Z$

Examples

- If X is finite-dimensional, then X has property A,
- Actually, reflexive spaces have property A,
- ℓ_1 has property A,
- c_0 fails property A,
- $\mathbb{K}$ has property B (Bishop-Phelps theorem),
- every Y such that $c_0 \subset Y \subset \ell_\infty$ has property B,
- finite-dimensional polyhedral spaces have property B,
- every strictly convex renorming of c_0 fails property B.

An quick overview on norm attaining operators

Section 2

The Radon-Nikodým property

Definitions

X Banach space.

- X has the Radon-Nikodým property (RNP) if the Radon-Nikodým theorem is valid for X-valued vector measures (with respect to every finite positive measure).
- $C \subset X$ is dentable if for every $\varepsilon > 0$ there is $x \in C$ which does not belong to the closed convex hull of $C \setminus (x + \varepsilon B_X)$.
- $C \subset X$ is subset-dentable if every subset of C is dentable.

Theorem (Rieffel, Maynard, Huff, David, Phelps, 1970's)

X RNP $\iff$ every bounded $C \subset X$ is dentable $\iff$ B_X subset-dentable.

Remark

In the book

J. Diestel and J. J. Uhl
Vector Measures
Math. Surveys **15**, AMS, Providence 1977.

there are more than 30 different reformulations of the RNP.

The RNP and property A: positive results

Theorem (Bourgain, 1977)

X Banach space, $C \subset X$ absolutely convex closed bounded subset-dentable, Y Banach space. Then

$$\{T \in \mathcal{L}(X,Y) \colon \text{the norm of } T \text{ attains its supremum on } C\}$$

is dense in $\mathcal{L}(X,Y)$.

★ In particular, RNP $\Longrightarrow$ property A.

Non-linear Bourgain-Stegall variational principle (Stegall, 1978)

X, Y Banach spaces, $C \subset X$ bounded subset-dentable, $\varphi : C \longrightarrow Y$ uniformly bounded such that $x \longmapsto \|\varphi(x)\|$ is upper semicontinuous.
Then for every $\delta > 0$, there exists $x_0^* \in X^*$ with $\|x_0^*\| < \delta$ and $y_0 \in S_Y$ such that the function $x \longmapsto \|\varphi(x) + x_0^*(x)y_0\|$ attains its supremum on C.

The RNP and property A: negative results

Theorem (Bourgain, 1977)

$C \subset X$ separable, bounded, closed and convex,
$\{T \in \mathcal{L}(X,Y) \colon$ the norm of T attains its supremum on $C\}$ dense in $\mathcal{L}(X,Y)$.
$\Longrightarrow$ C is dentable.

★ In particular, if X is separable and has property A $\Longrightarrow$ B_X is dentable.

A refinement (Huff, 1980)

X Banach space failing the RNP.

Then there exist X_1 and X_2 equivalent renorming of X such that

$$\mathrm{NA}(X_1, X_2) \text{ is NOT dense in } \mathcal{L}(X,Y).$$

Main consequence

Every renorming of X has property A $\Longleftrightarrow$ X has the RNP.

Another consequence

Every renorming of Y has property B $\Longrightarrow$ Y has the RNP.

An quick overview on norm attaining operators

Section 2

Gowers' result

Observation

It was an open question in the 1970's and 1980's whether

$$\text{RNP} \quad \Longrightarrow \quad \text{property B}$$

But. . .

Theorem (Gowers, 1990)

ℓ_p does not have property B for any $1 < p < \infty$.

Extending the result (Acosta-Aguirre-Payá, 1990's)

- Infinite-dimensional L_p-spaces fails property B for $1 < p < \infty$.
- Actually, if Y is strictly convex and contains an isomorphic copy of ℓ_p with $1 < p < \infty$, then Y does not have property B.

Acosta's results

Theorem (Acosta, 1999)

Every infinite-dimensional strictly convex space fails property B.

Consequence

Y separable, every renorming of Y has property B $\implies$ Y is finite-dimensional

Theorem (Acosta, 1999)

Every infinite-dimensional $L_1(\mu)$ space fails property B.

An quick overview on norm attaining operators

Section 2

Pairs of classical spaces: positive results

Example (Johnson-Wolfe, 1979)

In the real case, $\mathrm{NA}(C(K_1), C(K_2))$ is dense in $\mathcal{L}(C(K_1), C(K_2))$.

Example (Iwanik, 1979)

$\mathrm{NA}(L_1(\mu), L_1(\nu))$ is dense in $\mathcal{L}(L_1(\mu), L_1(\nu))$.

Theorem (Schachermayer, 1983)

Every weakly compact operator from $C(K)$ can be approximated by (weakly compact) norm attaining operators.

Consequence (Schachermayer, 1983)

$\mathrm{NA}(C(K), L_p(\mu))$ is dense in $\mathcal{L}(C(K), L_p(\mu))$ for $1 \leqslant p < \infty$.

Example (Finet-Payá, 1998)

$\mathrm{NA}(L_1[0,1], L_\infty[0,1])$ is dense in $\mathcal{L}(L_1[0,1], L_\infty[0,1])$.

Pairs of classical spaces: negative results

Example (Schachermayer, 1983)

$\mathrm{NA}(L_1[0,1], C[0,1])$ is NOT dense in $\mathcal{L}(L_1[0,1], C[0,1])$.

Consequence

$C[0,1]$ does not have property B and it was the first “classical” example.

Example (Aron-Choi-Kim-Lee-M., 2015; M., 2014)

$$\left.\begin{array}{c} Z = C[0,1] \oplus_1 L_1[0,1] \\ \text{or} \\ Z = C[0,1] \oplus_\infty L_1[0,1] \end{array}\right\} \implies \mathrm{NA}(Z,Z) \text{ not dense in } \mathcal{L}(Z).$$

An quick overview on norm attaining operators

Section 2

Main open problems

The main open problem

★ Do finite-dimensional spaces have Lindenstrauss property B**?**

(Stunning) open problem

Do finite-dimensional Hilbert spaces have Lindenstrauss property B**?**

Open problem

Characterize the topological compact spaces K such that $C(K)$ has property B.

Open problem

X Banach space without the RNP, does there exists a renorming of X such that $\mathrm{NA}(X,X)$ is not dense in $\mathcal{L}(X,X)$**?**

Remark

If $X \simeq Z \oplus Z$, then the answer to the question above is positive (use Bourgain-Huff).

Norm attaining compact operators

Section 3

3 Norm attaining compact operators
- Posing the problem for compact operators
- The easiest negative example
- More negative examples
- Positive results on property AK
- Positive results on property BK
- Open Problems

Norm attaining compact operators

Section 3

Posing the problem for compact operators

Question

Can every compact operator be approximated by norm-attaining operators?

Observations

- In all the negative examples of the previous section, the authors constructed NON COMPACT operators which cannot be approximated by norm attaining operators.
- Actually, the idea of the proofs is to use that the operator which is not going to be approximated is not compact or, even, it is an isomorphism.
- In most examples, it was even known that compact operators attaining the norm are dense.

Where was it explicitly possed?

- Diestel-Uhl, *Rocky Mount. J. Math.*, 1976.
- Diestel-Uhl, *Vector measures* (monograph), 1977.
- Johnson-Wolfe, *Studia Math.*, 1979.
- Acosta, *RACSAM* (survey), 2006.

More observations on compact operators

Question

Can every compact operator be approximated by norm-attaining operators?

Observations

- If X is reflexive, then ALL compact operators from X into Y are norm attaining. (Indeed, compact operators carry weak convergent sequences to norm convergent sequences.)
- It is known from the 1970's that whenever $X = C_0(L)$ or $X = L_1(\mu)$ (and Y arbitrary) or $Y = L_1(\mu)$ or $Y^* \equiv L_1(\mu)$ (and X arbitrary),
$\Longrightarrow \ \mathrm{NA}(X,Y) \cap \mathcal{K}(X,Y)$ is dense in $\mathcal{K}(X,Y)$.

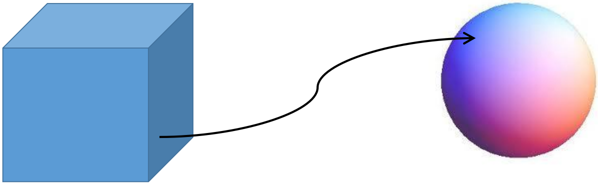
Norm attaining compact operators

Section 3

Extending a result by Lindenstrauss

X, Y Banach spaces, $T \in \mathcal{L}(X,Y)$ and $x_0 \in S_X$ with $\|T\| = \|Tx_0\| = 1$.

- If x_0 is not extreme point of B_X, there is $z \in X$ such that $\|x_0 \pm z\| \leqslant 1$, so $\|Tx_0 \pm Tz\| \leqslant 1$.
- If Tx_0 is an extreme point of B_Y, then $Tz = 0$.



Extending a result by Lindenstrauss

X, Y Banach spaces, $T \in \mathcal{L}(X,Y)$ and $x_0 \in S_X$ with $\|T\| = \|Tx_0\| = 1$.

- If x_0 is not extreme point of B_X, there is $z \in X$ such that $\|x_0 \pm z\| \leqslant 1$, so $\|Tx_0 \pm Tz\| \leqslant 1$.
- If Tx_0 is an extreme point of B_Y, then $Tz = 0$.

Geometrical lemma (abstract version of a Lindenstrauss' result)

X, Y Banach spaces. Suppose that

- for every $x_0 \in S_X$, $\operatorname{lin}\{z \in X \colon \|x_0 \pm z\| \leqslant 1\}$ has finite codimension,
- Y is strictly convex.

Then, $\operatorname{NA}(X,Y) \subseteq \mathcal{F}(X,Y)$.

First consequence (recalling, Lindenstrauss, 1963)

- $\operatorname{NA}(c_0,Y) \subseteq \mathcal{F}(c_0,Y)$ if Y is strictly convex.
- Therefore, c_0 fails property A.

Extending a result by Lindenstrauss (II)

Proposition (extension of Lindenstrauss result)

$X \leqslant c_0$. For every $x_0 \in S_X$, $\operatorname{lin}\{z \in X \colon \|x_0 \pm z\| \leqslant 1\}$ has finite codimension.

Proof.

- as $x_0 \in c_0$, there exists m such that $|x_0(n)| < 1/2$ for every $n \geqslant m$;
- let $Z = \big\{z \in X \colon x_0(i) = 0 \text{ for } 1 \leqslant i \leqslant m\big\}$ (finite codimension in X);
- for $z \in Z$ with $\|z\| \leqslant 1/2$, one has $\|x_0 \pm z\| \leqslant 1$.

Main consequence

$X \leqslant c_0$, Y strictly convex. Then $\mathrm{NA}(X,Y) \subseteq \mathcal{F}(X,Y)$.

Question

What's next? How to use this result?

Grothendieck's approximation property

Definition (Grothendieck, 1950's)

Z has the approximation property (AP) if for every $K \subset Z$ compact and every $\varepsilon > 0$, there exists $F \in \mathcal{F}(Z)$ such that $\|Fz - z\| < \varepsilon$ for all $z \in K$.

Basic results

X, Y Banach spaces.

- (Grothendieck) Y has AP $\iff$ $\overline{\mathcal{F}(Z,Y)} = \mathcal{K}(Z,Y)$ for all Z.
- (Grothendieck) X^* has AP $\iff$ $\overline{\mathcal{F}(X,Z)} = \mathcal{K}(X,Z)$ for all Z.
- (Grothendieck) X^* AP $\implies$ X AP.
- (Enflo, 1973) There exists $X \leqslant c_0$ without AP.

The first example

Theorem

There exists a **compact** operator which cannot be approximated by norm attaining operators.

Proof:

- consider $X \leqslant c_0$ without AP (Enflo);
- X^* does not has AP
  $\Longrightarrow$ there exists Y and $T \in \mathcal{K}(X, Y)$ such that $T \notin \overline{\mathcal{F}(X, Y)}$;
- we may suppose $Y = \overline{T(X)}$, which is separable;
- so Y admits an equivalent strictly convex renorming (Klee);
- we apply the extension of Lindenstrauss result: $\mathrm{NA}(X, Y) \subseteq \mathcal{F}(X, Y)$;
- therefore, $T \notin \overline{\mathrm{NA}(X, Y)}$.

Two useful definitions

Definitions

X and Y Banach spaces.

- X has property AK when $\overline{\mathrm{NA}(X,Z)\cap\mathcal{K}(X,Z)} = \mathcal{K}(X,Z) \quad \forall\, Z$;
- Y has property BK when $\overline{\mathrm{NA}(Z,Y)\cap\mathcal{K}(Z,Y)} = \mathcal{K}(Z,Y) \quad \forall\, Z$.

Some basic results

- Finite-dimensional spaces have property AK;
- $Y = \mathbb{K}$ has property BK;
- Real finite-dimensional polyhedral spaces have property BK.

Our negative example (recalling)

There exists $X \leqslant c_0$ failing AK and there exits Y failing BK.

Norm attaining compact operators

Section 3

More examples: Domain space

Proposition (what we have proved so far...)

$X \leqslant c_0$ such that X^* fails AP $\implies$ X does not have AK.

Example by Johnson-Schechtman, 2001

Exists X subspace of c_0 **with Schauder basis** such that X^* fails the AP.

Corolary

There exists a Banach space X with Schauder basis failing property AK.

More examples: Range space

Strictly convex spaces

Y strictly convex without AP $\implies$ Y fails BK.

Lemma (Grothendieck)

Y has AP iff $\mathcal{F}(X,Y)$ is dense in $\mathcal{K}(X,Y)$ for every $X \leqslant c_0$.

Subspaces of $L_1(\mu)$

$Y \leqslant L_1(\mu)$ (complex case) without AP $\implies$ Y fails BK.

Observation (Globevnik, 1975)

Complex $L_1(\mu)$ spaces are complex strictly convex:

$$f, g \in L_1(\mu),\ \|f\| = 1 \text{ and } \|f + \theta g\| \leqslant 1\, \forall \theta \in B_{\mathbb{C}} \implies g = 0.$$

More examples: Domain=Range

Theorem

There exists a Banach space Z and a compact operator from Z to Z which cannot be approximated by norm attaining operators.

Proposition

X and Y Banach spaces, $Z = X \oplus_1 Y$ or $Z = X \oplus_\infty Y$.
$\mathrm{NA}(Z,Z) \cap \mathcal{K}(Z,Z)$ dense in $\mathcal{K}(Z,Z) \implies \mathrm{NA}(X,Y) \cap \mathcal{K}(X,Y)$ dense in $\mathcal{K}(X,Y)$.

Norm attaining compact operators

Section 3

Property AK: leading open problem

Problem

$$X^* \text{ AP} \implies X \text{ AK}\textbf{?}$$

Observation

Known positive results on property AK are partial answers to the above question, as strong forms of the AP for the dual are involved.

Old known examples

- (Diestel-Uhl, 1976) $L_1(\mu)$ has AK;
- (Johnson-Wolfe, 1979) $C_0(L)$ has AK.

An interesting observation

If X^* has AP and X has property A $\implies$ X has property AK.

Positive results on property AK

Problem

$$X^* \text{ AP} \implies X \text{ AK?}$$

Partial answer:

(Johnson-Wolfe) With a strong approximation property of the dual...

Suppose there exists a net of contractive projections $(P_\alpha)_\alpha$ in X with finite rank such that $\lim_\alpha P_\alpha^* = \mathrm{Id}_{X^*}$ in SOT. Then, X has AK.

Proof. Fix $T \in \mathcal{K}(X,Y)$.

- $TP_\alpha(B_X) = T(B_{P_\alpha(X)})$ (we need $P_\alpha^2 = P_\alpha$ and $\|P_\alpha\| = 1$).
- Then, TP_α attains the norm.
- As T^* is compact, $P_\alpha^* T^* \longrightarrow T^*$ in norm, so $TP_\alpha \longrightarrow T$ in norm.

Positive results on property AK

Problem

$$X^* \text{ AP} \implies X \text{ AK?}$$

Partial answer:

(Johnson-Wolfe) With a strong approximation property of the dual...

Suppose there exists a net of contractive projections $(P_\alpha)_\alpha$ in X with finite rank such that $\lim_\alpha P_\alpha^* = \mathrm{Id}_{X^*}$ in SOT. Then, X has AK.

Consequences

- (Diestel-Uhl) $L_1(\mu)$ has AK.
- (Johnson-Wolfe) $C_0(L)$ has AK.
- X with monotone and shrinking basis $\implies$ X has AK.
- X with monotone unconditional basis, $X \not\supseteq \ell_1 \implies X$ has AK.
- $X^* \equiv \ell_1 \implies X$ has AK (using a result by Gasparis).
- $X \leqslant c_0$ with monotone basis $\implies X$ has AK (using a result by Godefroy–Saphar).

Norm attaining compact operators

Section 3

Positive results on property BK I

Main open question

$$\text{AP} \implies \text{BK}\textbf{?}$$

A partial answer (Johnson-Wolfe)

- If Y is polyhedral (real) and has AP $\implies$ Y has BK.
- X (complex) space with AP such that the norm of every finite-dimensional subspace can be calculated as the maximum of a finite set of functionals $\implies$ Y has BK.

Example (Johnson-Wolfe)

$$Y \leqslant c_0 \text{ (real or complex) with AP} \implies Y \text{ has BK.}$$

A somehow reciprocal to the problem. . .

Y separable with BK for every equivalent norm $\implies$ Y has AP.

Positive results on property BK II

Main open question

$$\text{AP} \implies \text{BK}\textbf{?}$$

Another partial answer (Johnson-Wolfe)

Y Banach space. Suppose there exists a uniformly bounded net of projections $(Q_\alpha)_\alpha$ in Y such that $\lim_\alpha Q_\alpha = \mathrm{Id}_Y$ in SOT and $Q_\alpha(Y)$ has property BK.
Then, Y has property BK.

Examples (Johnson-Wolfe)

- Y predual of $L_1(\mu)$ (real or complex) $\implies$ Y has BK;
- in particular, real or complex $C_0(L)$ spaces have property BK;
- real $L_1(\mu)$ spaces have property BK.

Norm attaining compact operators

Section 3

Some open problems

Main open problem

★ Can every finite-rank operator be approximated by norm-attaining operators **?**

Open problem

X Banach space, does there exist a norm-attaining rank-two operator from X to a Hilbert space**?**

Another main open problem

★ X^* AP $\implies$ X AK**?**

Open problem

$X \leqslant c_0$ with the metric AP, does it have AK**?**

Open problem

X such that $X^* \equiv L_1(\mu)$, does X have AK**?**

Open problem

Y subspace of the real $L_1(\mu)$ without the AP, does Y fail property BK**?**