The Bishop-Phelps-Bollobás point property

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Roadmap of the talk

Introducing the topic

Section 1

Introducing the topic

Section 1

Notation

X, Y real or complex Banach spaces

- $\mathbb{K}$ base field $\mathbb{R}$ or $\mathbb{C}$,
- $B_X = \{x \in X \colon \|x\| \leqslant 1\}$ closed unit ball of X,
- $S_X = \{x \in X \colon \|x\| = 1\}$ unit sphere of X,
- $\mathcal{L}(X,Y)$ bounded linear operators from X to Y,
  - $\|T\| = \sup\{\|T(x)\| \colon x \in S_X\}$ for $T \in \mathcal{L}(X,Y)$,
- $\mathcal{K}(X,Y)$ compact linear operators from X to Y,
- $X^* = \mathcal{L}(X,\mathbb{K})$ topological dual of X.

Introducing the topic

Section 1

Norm attaining functionals

Norm attaining functionals

$x^* \in X^*$ attains its norm when

$$\exists\, x \in S_X \ : \ |x^*(x)| = \|x^*\|$$

★ $\mathrm{NA}(X,\mathbb{K}) = \{x^* \in X^* : x^* \text{ attains its norm}\}$

First examples

- $\dim(X) < \infty \implies \mathrm{NA}(X,\mathbb{K}) = \mathcal{L}(X,\mathbb{K})$ (Heine-Borel).
- X reflexive $\implies \mathrm{NA}(X,\mathbb{K}) = \mathcal{L}(X,\mathbb{K})$ (Hahn-Banach).
- X non-reflexive $\implies \mathrm{NA}(X,\mathbb{K}) \neq \mathcal{L}(X,\mathbb{K})$ (James),
- but $\mathrm{NA}(X,\mathbb{K})$ always separates the points of X (Hahn-Banach).

- $\mathrm{NA}(c_0,\mathbb{K}) = c_{00} \leqslant \ell_1$,
- $\mathrm{NA}(\ell_1,\mathbb{K}) = \big\{x \in \ell_\infty : \|x\|_\infty = \max_n\{|x(n)|\}\big\} \leqslant \ell_\infty$, not subspace, contains c_0,
- $\mathrm{NA}(X,\mathbb{K})$ may contain no two-dimensional subspaces (Rmoutil, 2017).

Norm attaining operators

Norm attaining operators

$T \in \mathcal{L}(X,Y)$ attains its norm when

$$\exists\, x \in S_X \;:\; \|T(x)\| = \|T\|$$

★ $\mathrm{NA}(X,Y) = \{T \in \mathcal{L}(X,Y) \colon T$ attains its norm$\}$

First examples

- $\dim(X) < \infty \implies \mathrm{NA}(X,Y) = \mathcal{L}(X,Y)$ for every Y (Heine-Borel).
- $\mathrm{NA}(X,Y) \neq \emptyset$ (Hahn-Banach),
- X reflexive $\implies \mathcal{K}(X,Y) \subseteq \mathrm{NA}(X,Y)$ for every Y (by complete continuity),
- X non-reflexive $\implies \mathcal{K}(X,Y) \nsubseteq \mathrm{NA}(X,Y)$ for any Y (James),
- $\dim(X) = \infty \implies \mathrm{NA}(X,c_0) \neq \mathcal{L}(X,c_0)$ (see M.-Merí-Payá, 2006).

Preliminaries

Section 2

Preliminaries

Section 2

Bishop-Phelps theorem and the question of density of norm attaining operators

Theorem (Bishop–Phelps, 1961)

Norm attaining functionals are dense in X^*
(in the norm topology)

Problem

$$¿\ \overline{NA(X,Y)} = L(X,Y)\ ?$$

- Lindenstrauss, *Israel J. Math.* (1963) started the study of this problem.
- The answer is **Negative** in general.
- For the study of this problem, Lindenstrauss introduced properties A and B.

Lindenstrauss properties A and B

Definition

- X has **property A** if $\overline{NA(X,Y)} = L(X,Y) \quad \forall Y$.
- Y has **property B** if $\overline{NA(X,Y)} = L(X,Y) \quad \forall X$.

First positive examples (Lindenstrauss)

- Reflexive spaces have property A.
- ℓ_1 has property A (property α).
- If $c_0 \subset Y \subset \ell_\infty$ or Y finite dimensional and polyhedral,
  $\Longrightarrow \quad Y$ has property B (property β).

First negative examples (Lindenstrauss)

- X separable with property A, then B_X is the closed convex hull of its strongly exposed points.
- $L_1[0,1]$ and $C_0(L)$ (L infinite) fails property A.
- Y strictly convex containing c_0 fails property B.

Lindenstrauss properties A and B: further examples

Relationship with RNP (Bourgain, 1977 & Huff, 1980)

- RNP $\Longrightarrow$ A
- X no RNP $\Longrightarrow$ $\exists\ X_1 \sim X \sim X_2\ :\ \overline{NA(X_1, X_2)} \neq L(X_1, X_2)$

Examples (Gowers, 1990 & Acosta, 1999)

- (Gowers) Infinite-dimensional $L_p(\mu)$ $(1 < p < \infty)$ spaces fail property B.
  Squeezing, strictly convex spaces containing ℓ_p $(1 < p < \infty)$ fail property B.
- (Acosta) Infinite-dimensional strictly convex spaces fail property B.
- (Acosta) Infinite-dimensional $L_1(\mu)$ spaces fail property B.

Compact operators (Martín, 2014)

Exist compact operators which cannot be approximated by norm attaining operators.

Main open question

Can finite-rank operators be approximated by norm attaining operators**?**

Preliminaries

Section 2

Bishop-Phelps-Bollobás theorem

Theorem (Bishop–Phelps, 1961)

Norm attaining functionals are dense in X^*

Bollobás contribution, 1970

Fix $0 < \varepsilon < 2$.

If $x_0 \in B_X$ and $x_0^* \in B_{X^*}$ satisfy $\operatorname{Re} x_0^*(x_0) > 1 - \varepsilon^2/2$,

there exist $x \in S_X$, $x^* \in S_{X^*}$ with

$$x^*(x) = 1, \qquad \|x_0 - x\| < \varepsilon, \qquad \|x_0^* - x^*\| < \varepsilon.$$

(see Chica-Kadets-Martín-Moreno-Rambla 2014 for this version)

Bishop-Phelps-Bollobás property

Bishop-Phelps-Bollobás property (Acosta-Aron-García-Maestre, 2008)

A pair of Banach spaces (X, Y) has the **Bishop-Phelps-Bollobás property** (**BPBp**) if given $\varepsilon \in (0, 1)$ there is $\eta(\varepsilon) > 0$ such that whenever

$$T_0 \in S_{L(X,Y)}, \quad x_0 \in S_X, \quad \|T_0 x_0\| > 1 - \eta(\varepsilon),$$

there exist $S \in L(X, Y)$ and $x \in S_X$ such that

$$1 = \|S\| = \|Sx\|, \qquad \|x_0 - x\| < \varepsilon, \qquad \|T_0 - S\| < \varepsilon.$$

Observation

If (X, Y) has the BPBp $\Longrightarrow$ $\overline{NA(X, Y)} = L(X, Y)$.
★ Does this implication reverse? **No**

First examples

- There is Y_0 such that (ℓ_1, Y_0) fails BPBp.
- X, Y finite-dimensional, then (X, Y) has BPBp.
- Y with property β (example $c_0 \leqslant Y \leqslant \ell_\infty$), then (X, Y) has BPBp $\forall X$.

More examples

Pairs of classical spaces

- (Aron-Choi-García-Maestre, 2011) $(L_1[0,1], L_\infty[0,1])$ has BPBp.
- (Acosta + 7) $(C(K_1), C(K_2))$ has BPBp (in the real case).
- (Choi-Kim-Lee-Martín, 2014) $(L_1(\mu), L_1(\nu))$ has BPBp.

Other examples

- (Acosta-Becerra-García-Maestre, 2013; Kim-Lee, 2014) X uniformly convex $\implies$ (X,Y) has BPBp for all Y.
- (Cascales-Guirao-Kadets, 2013) X Asplund $\implies$ (X,A) has BPBp for every uniform algebra A (in particular, $A = C_0(L)$ or $A = A(\mathbb{D})$).
- (Choi-Kim, 2011) $(L_1(\mu), Y)$ has the BPBp when Y has the RNP and the AHSP.
- (Kim-Lee, 2015) $(C(K), Y)$ has the BPBp when Y is uniformly convex.
- (Acosta, 2016) $(C(K), Y)$ has the BPBp when Y is uniformly complex convex (e.g. complex $L_1(\mu)$).

The BPB version of Lindenstrauss properties A and B

Universal BPB domain and range spaces

- X is a **universal BPB domain space** if (X,Y) has BPBp $\forall\, Y$.
- Y is a **universal BPB range space** if (X,Y) has BPBp $\forall\, X$.

Observations

- X universal BPB domain space $\Longrightarrow$ X has property A.
- Y universal BPB range space $\Longrightarrow$ Y has property B.
- Do these implications reverse**? No:**
  ★ There is $\mathcal{Y}$ with property B such that $(\ell_1^2, \mathcal{Y})$ fails BPBp
  (Aron-Choi-Kim-Lee-Martín, 2015).

Positive examples

- Uniformly convex spaces are universal BPB domain spaces.
- Property β implies being universal BPB range space.

These are, up to now, the only known positive examples.

Universal BPB domain spaces: some necessary conditions

Theorem (Aron-Choi-Lim-Lee-Martín, 2015)

X universal BPB domain space. Then,

1. (real case) no face of S_X contains a non-empty relatively open subset of S_X;
2. if X is separable, then extreme points of B_X are dense in S_X;
3. if X is superreflexive, then strongly exposed points of B_X are dense in S_X.
4. In particular, if X is a real 2-dimensional Banach space which is a universal BPB domain space, then X is uniformly convex.

Question

Is uniformly convex every universal BPBp domain space **?**

Theorem (Aron-Choi-Kim-Lee-Martín, 2015)

X universal BPB domain space in every equivalent renorming $\Longrightarrow$ $\dim X = 1$.

The Bishop-Phelps-Bollobás point property

Section 3

The Bishop-Phelps-Bollobás point property

Section 3

Definition

Recalling the Bishop-Phelps-Bollobás property

(X,Y) has the **BPBp** if given $\varepsilon \in (0,1)$ there is $\eta(\varepsilon) > 0$ such that whenever

$$T_0 \in S_{L(X,Y)}, \quad x_0 \in S_X, \quad \|T_0 x_0\| > 1 - \eta(\varepsilon),$$

there exist $S \in L(X,Y)$ and $x \in S_X$ such that

$$1 = \|S\| = \|Sx\|, \qquad \|x_0 - x\| < \varepsilon, \qquad \|T_0 - S\| < \varepsilon.$$

Bishop-Phelps-Bollobás point property (Dantas-Kim-Lee, 2016)

(X,Y) has the **Bishop-Phelps-Bollobás point property** (**BPBpp**) if given $\varepsilon \in (0,1)$ there is $\widetilde{\eta}(\varepsilon) > 0$ such that whenever

$$T_0 \in S_{L(X,Y)}, \quad x_0 \in S_X, \quad \|T_0 x_0\| > 1 - \widetilde{\eta}(\varepsilon),$$

there exists $S \in L(X,Y)$ such that

$$1 = \|S\| = \|Sx_0\|, \qquad \|T_0 - S\| < \varepsilon.$$

Obviously, BPBpp $\Longrightarrow$ BPBp.

First results

Bishop-Phelps-Bollobás point property (Dantas-Kim-Lee, 2016)

(X, Y) has the **BPBpp** if given $\varepsilon \in (0, 1)$ there is $\widetilde{\eta}(\varepsilon) > 0$ such that whenever

$$T_0 \in S_{L(X,Y)}, \quad x_0 \in S_X, \quad \|T_0 x_0\| > 1 - \widetilde{\eta}(\varepsilon),$$

there exists $S \in L(X, Y)$ such that

$$1 = \|S\| = \|Sx_0\|, \qquad \|T_0 - S\| < \varepsilon.$$

Main known results (Dantas-Kim-Lee, 2016)

- $(X, \mathbb{K})$ has BPBpp $\iff$ X is uniformly smooth,
- (X, Y) has BPBpp for some Y $\implies$ X is uniformly smooth,
- Taking X with $\dim(X) = 2$, uniformly smooth, not strictly convex, then there is Y such that (X, Y) fails BPBp and so BPBpp,
- H Hilbert space $\implies$ (H, Y) has BPBpp for every Y,
- X uniformly smooth, Y property β or uniform algebra $\implies$ (X, Y) has BPBpp,
- $X = X_1 \oplus_1 X_2$ or $X = X_1 \oplus_\infty X_2$, $\quad Y = Y_1 \oplus_1 Y_2$ or $Y = Y_1 \oplus_\infty Y_2$, (X, Y) BPBpp $\implies$ (X_i, Y_j) BPBpp.

Some stability results

Domain spaces

X_1 contractively complemented in X, (X,Y) BPBpp $\implies$ (X_1,Y) BPBpp.

Question

Is the analogous result true for the BPBp or for the density of norm-attaining operators**?**

Range spaces (adaptation of a result of Dantas-García-Maestre-Martín)

$Y = Y_1 \oplus_a Y_2$ (absolute sum), (X,Y) BPBpp $\implies$ (X,Y_j) BPBpp.

The Bishop-Phelps-Bollobás point property

Section 3

The BPBpp version of Lindenstrauss properties A and B

Universal BPBpp domain and range spaces

- X **universal BPBpp domain space** if (X, Y) has BPBpp $\forall Y$.
- Y **universal BPBpp range space** if (X, Y) has BPBpp $\forall X$ uniformly smooth.

Observations

- X universal BPBpp domain space $\implies$ X universal BPBp domain,
- Y universal BPBpp range space $\implies$ Y universal BPBp range for uniformly smooth X's.
- Do these implications reverse**?**
  - We will show that the first question has a negative answer.

Positive examples

- Hilbert spaces are universal BPBpp domain spaces,
- property β and being uniform algebra implies being universal BPBpp range space.

Question

Are Hilbert spaces the only universal BPBpp domain spaces**?**

Universal BPBpp domain spaces

An important property

If X is universal BPBpp domain space, then there is a universal function $\widetilde{\eta}_X : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that (X, Y) has BPBpp with $\widetilde{\eta}_X$ for every Y.

Theorem

X universal BPBpp domain space $\implies$ X uniformly convex.

Theorem

X universal BPBpp domain space,
X isomorphic to a Hilbert space $\implies$ $\delta_X(\varepsilon) \geqslant C\,\varepsilon^2$.

Consequence

$L_p(\mu)$ is NOT a universal BPBpp domain space for $p > 2$.

Question

Is $L_p(\mu)$ a universal BPBpp domain space for $1 < p < 2$**?**

Universal BPBpp range spaces I

Some sufficient conditions (adaptation of Cascales-Guirao-Kadets-Soloviova)

Y having ACK_ρ-structure $\implies$ Y universal BPBpp range space.

Examples:

- Spaces with property β,
- uniform algebras (in particular, $C(K)$ spaces),
- stability by finite injective tensor product,
- stability by $C(K,Y)$, $C_w(K,Y)$, some vector-valued function algebras...

Universal BPBpp range spaces II

Example

For $p \geqslant 2$ $\exists\, X_p$ uniformly convex and uniformly smooth such that (X_p, ℓ_p^2) fails BPBpp

Remarks on the example

- The pair (X_p, ℓ_p^2) has the BPBp (as X_p is uniformly convex).
- It is also true that (X_p, ℓ_p) fails BPBpp (but has BPBp), but, in this case, it is known that ℓ_p fails Lindenstrauss property B.
- Recall that it is not known whether ℓ_p^2 has Lindenstrauss property B.
- It also shows that the BPBpp is not stable by finite ℓ_p-sum of the range; it is not known if the analogous result for the BPBp is true.

Universal BPBpp range spaces III

Example

$\exists\ X$ uniformly smooth with $\dim(X) = 2$ and a sequence $\{Y_n : n \in \mathbb{N}\}$ of polyhedral spaces such that

$$\widetilde{\eta}(X, Y_n)(\varepsilon) \longrightarrow 0 \quad \text{when } n \to \infty.$$

Consequences

Write

$$\mathcal{Y} = \left[\bigoplus_{n\in\mathbb{N}} Y_n\right]_{c_0} \qquad \text{and} \qquad \mathcal{Z} = \left[\bigoplus_{n\in\mathbb{N}} Y_n\right]_{\ell_\infty},$$

then $(X, \mathcal{Y})$ and $(X, \mathcal{Z})$ fail the BPBpp.

Therefore:

- Being universal BPBpp range space is NOT stable by c_0 or ℓ_∞ sums.
- Property B (actually quasi-β) is NOT enough to be universal BPBpp range space.

The Bishop-Phelps-Bollobás point property

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BPBpp for compact operators

Bishop-Phelps-Bollobás point property (Dantas-Kim-Lee, 2016)

(X,Y) has the **Bishop-Phelps-Bollobás point property for compact operators** (**BPBpp-K**) if given $\varepsilon \in (0,1)$ there is $\widetilde{\eta}(\varepsilon) > 0$ such that whenever

$$T_0 \in S_{K(X,Y)}, \quad x_0 \in S_X, \quad \|T_0 x_0\| > 1 - \widetilde{\eta}(\varepsilon),$$

there exist $S \in K(X,Y)$ such that

$$1 = \|S\| = \|Sx_0\|, \qquad \|T_0 - S\| < \varepsilon.$$

First examples (adapting proofs for all operators)

- H Hilbert $\implies$ (H,Y) has BPBpp-K for every Y,
- Y having ACK_ρ-structure $\implies$ (X,Y) BPBpp-K for every X uniformly convex,
- this include Y's with property β, Y uniform algebra...

Results which are only for compact operators

For the BPBp-K –only one "p"– (Dantas-García-Maestre-Martín, 201?)

- X with a net of contractive projections $\{P_\alpha\}$ with SOT $-\lim P_\alpha = \mathrm{Id}_X$ and SOT $-\lim P_\alpha^* = \mathrm{Id}_{X^*}$.
  If all the pairs $(P_\alpha(X), Y)$ have the BPBp-K with a common η
  $\Longrightarrow$ (X, Y) has BPBp-K.
- Y with a net of contractive projections $\{Q_\alpha\}$ with SOT $-\lim P_\alpha = \mathrm{Id}_Y$.
  If all the pairs $(X, Q_\alpha(Y))$ have the BPBp-K with a common η
  $\Longrightarrow$ (X, Y) has BPBp-K.

BPBpp-K for domain spaces

We do not know if the first part translated to the BPBpp-K.

BPBpp-K for range spaces

Y with a net of contractive projections $\{Q_\alpha\}$ with SOT $-\lim P_\alpha = \mathrm{Id}_Y$.
If all the pairs $(X, Q_\alpha(Y))$ have the BPBpp-K with a common η
$\Longrightarrow$ (X, Y) has BPBpp-K.

Results which are only for compact operators II

BPBpp-K for range spaces

Y with a net of contractive projections $\{Q_\alpha\}$ with $\text{SOT} - \lim P_\alpha = \mathrm{Id}_Y$.
If all the pairs $(X, Q_\alpha(Y))$ have the BPBpp-K with a common η
$\implies$ (X, Y) has BPBpp-K.

Consequences

- $1 \leqslant p < \infty$, $(X, \ell_p(Y))$ BPBpp-K $\implies$ $(X, L_p(\mu, Y))$ BPBpp-K.
- (X, Y) BPBpp-K $\implies$ $(X, L_\infty(\mu, Y))$ BPBpp-K.
- (X, Y) BPBpp-K $\implies$ $(X, C(K, Y))$ BPBpp-K.
- X uniformly convex, Y predual of L_1 $\implies$ (X, Y) BPBpp-K.

The dual property is not possible

Section 4

A dual property

Recalling the BPBpp

(X,Y) has the **BPBpp** if given $\varepsilon \in (0,1)$ there is $\widetilde{\eta}(\varepsilon) > 0$ such that whenever

$$T_0 \in S_{L(X,Y)}, \quad x_0 \in S_X, \quad \|T_0 x_0\| > 1 - \widetilde{\eta}(\varepsilon),$$

there exists $S \in L(X,Y)$ such that $\quad 1 = \|S\| = \|Sx_0\|, \quad \|T_0 - S\| < \varepsilon.$

A possible dual property...

(X,Y) has property **(P)** iff given $\varepsilon \in (0,1)$ there is $\widehat{\eta}(\varepsilon) > 0$ such that whenever

$$T_0 \in S_{L(X,Y)}, \quad x_0 \in S_X, \quad \|T_0 x_0\| > 1 - \widehat{\eta}(\varepsilon),$$

there exists $x \in S_X$ such that $\quad \|T_0 x\| = 1, \quad \|x_0 - x\| < \varepsilon.$

- Introduced by Dantas in 2016, as an auxiliary definition to study an analogous property where $\widehat{\eta}$ depends on ε and on T_0,
- in the same paper, it is shown that many pairs fail it,
- for $Y = \mathbb{K}$, it characterizes uniform convexity:

Characterizing uniform convexity

A possible dual property...

(X, Y) has property **(P)** iff given $\varepsilon \in (0, 1)$ there is $\widehat{\eta}(\varepsilon) > 0$ such that whenever

$$T_0 \in S_{L(X,Y)}, \quad x_0 \in S_X, \quad \|T_0 x_0\| > 1 - \widehat{\eta}(\varepsilon),$$

there exists $x \in S_X$ such that $\quad \|T_0 x\| = 1, \quad \|x_0 - x\| < \varepsilon.$

Theorem (Kim-Lee, 2014)

X is uniformly convex iff $(X, \mathbb{K})$ has property (P).

Observation

For $X = \mathbb{K}$, the property is trivial:

- For every Y, the pair $(\mathbb{K}, Y)$ has property (P).

Question

Are there more cases**?**

There are no more cases...

Theorem

$$\dim(X) > 1 \text{ and } \dim(Y) > 1 \implies (X, Y) \text{ **fails** property (P).}$$

- The proof reduces to the case of $\dim(X) = \dim(Y) = 2$:

Proposition

(X, Y) with property (P), $X_0 \leqslant X$ and $Y_0 \leqslant Y$ such that $\dim(X/X_0) = \dim(Y_0) = 2$.
$\implies$ (X_0, Y_0) has property (P).

- even in this case, the proof is rather involved,
- it needs to construct an operator $T \in L(X, Y)$ such that $T(B_X)$ has an special *"position"* in Y:

There are no more cases. . .

Proposition

If $\dim(X) = \dim(Y) = 2$, then there exists $T \in L(X,Y)$ with $\|T\| = 1$ such that there are $y_1, y_2 \in T(B_X) \cap S_Y$ and $y_1^* \in S_{Y^*}$ such that

$$y_1^*(y_1) = 1 \qquad \text{and} \qquad y_2 \notin \big[y_1^*\big]^{-1}\big(\{1,-1\}\big).$$

- Two cases: X is Hilbertian (use John's maximal ellipsoid); X is not Hilbertian (use Day's and Nordlander's theorems on the modulus of convexity of X).

Proposition

X, Y 2-dimensional $\implies$ $\exists \delta > 0$ and $(T_n) \subset L(X,Y)$ with $\|T_n\| = 1$, $x_0 \in S_X$ with

$$\|T_n(x_0)\| \longrightarrow 1 \qquad \operatorname{dist}\big(x_0, \{x \in S_X : \|T_n x\| = 1\}\big) \geqslant \delta.$$

Proof. Take T from the previous proposition, write

$$x_0 = T^{-1}(y_2) \in S_X, \quad \delta = \operatorname{dist}(y_2, \big[y_1^*\big]^{-1}\big(\{1,-1\}\big), \quad P = y_1^* \otimes y_1 \in L(Y,Y).$$

We just have to define

$$T_n = \frac{n}{n+1}T + \frac{1}{n+1}PT \in L(X,Y).$$