The group of isometries of a Banach space and duality

Miguel Martín

http://www.ugr.es/local/mmartins

Universidad de Granada

Seoul (Republic of Korea), November 2015

서울대학교

SEOUL NATIONAL UNIVERSITY

Introduction

Sección 1

Basic notation and main objective

Notation

X Banach space.

- $S_X = \{x \in X : \|x\| = 1\}$ unit sphere, $B_X = \{x \in X : \|x\| \leqslant 1\}$ closed unit ball.
- X^* dual space.
- $L(X)$ bounded linear operators.
- $W(X)$ weakly compact linear operators.
- $\mathrm{Iso}(X)$ surjective (linear) isometries group.

Note

- $T : X \longrightarrow X$ isometry $\overset{\text{def}}{\Longleftrightarrow}$ $\|Tx - Ty\| = \|x - y\|$ for all $x, y \in X$.
- (Mazur-Ulam) T surjective isometry, $T(0) = 0 \implies T$ linear.

Objective

Observation

In **concrete** non-reflexive Banach spaces X, it is usually easy to construct elements in $\mathrm{Iso}(X^*)$ which are not adjoins to members of $\mathrm{Iso}(X)$.

But, can the "size" of $\mathrm{Iso}(X)$ and $\mathrm{Iso}(X^*)$ be very different **?**

Yes, our objective is...

★ Construct a Banach space X with "small" $\mathrm{Iso}(X)$ and "big" $\mathrm{Iso}(X^*)$.

A previous attempt

M., 2008

There is X such that

- $\mathrm{Iso}(X)$ does not contains uniformly continuous semigroups of isometries;
- $\mathrm{Iso}(X^*) \supset \mathrm{Iso}(\ell_2)$ and, therefore, $\mathrm{Iso}(X^*)$ contains infinitely many uniformly continuous semigroups of isometries.
- But $\mathrm{Iso}(X)$ contains infinitely many strongly continuous semigroups of isometries.

Question we are going to solve

Is is possible to produce a space X such that $\mathrm{Iso}(X^*) \supset \mathrm{Iso}(\ell_2)$ but $\mathrm{Iso}(X)$ is "small" (for instance, it does not contain strongly continuous semigroups) **?**

Motivation

X Banach space.

Autonomous linear dynamical system

$(\diamondsuit)$ $\begin{cases} x'(t) = A\,x(t) \\ x(0) = x_0 \end{cases}$ $x_0 \in X$, $A : D(A) \longrightarrow X$ linear, closed, densely defined.

One-parameter semigroup of operators

$\Phi : \mathbb{R}_0^+ \longrightarrow L(X)$ such that $\Phi(t+s) = \Phi(t)\Phi(s)\ \forall t, s \in \mathbb{R}_0^+$, $\Phi(0) = \mathrm{Id}$.

- *Uniformly continuous*: $\Phi : \mathbb{R}_0^+ \longrightarrow (L(X), \|\cdot\|)$ continuous.
- *Strongly continuous*: $\Phi : \mathbb{R}_0^+ \longrightarrow (L(X), \mathrm{SOT})$ continuous.

Relationship (Hille-Yoshida, 1950's)

- *Bounded case:*
  - If $A \in L(X) \implies \Phi(t) = \exp(t\,A)$ solution of $(\diamondsuit)$ uniforly continuous.
  - Φ uniformly continuous $\implies A = \Phi'(0) \in L(X)$ and Φ solution of $(\diamondsuit)$.
- *Unbounded case:*
  - Φ strongly continuous $\implies A = \Phi'(0)$ closed and Φ solution of $(\diamondsuit)$.
  - If $(\diamondsuit)$ has solution Φ strongly continuous $\implies A = \Phi'(0)$ and $\Phi(t) =$ "$\exp(t\,A)$".

What we are going to show

The example

we will construct X such that

$$\mathrm{Iso}(X) = \{\pm \mathrm{Id}\} \qquad \text{but} \qquad \mathrm{Iso}(X^*) \supset \mathrm{Iso}(\ell_2).$$

The main tools

- Extremely non-complex Banach spaces:
  spaces X such that $\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$ for every $T \in L(X)$.
- Koszmider type compact spaces:
  topological compact spaces K such that $C(K)$ has few operators.

The talk is based on the papers

P. Koszmider, M. Martín, and J. Merí.
Extremely non-complex $C(K)$ spaces.
J. Math. Anal. Appl. (2009).

P. Koszmider, M. Martín, and J. Merí.
Isometries on extremely non-complex Banach spaces.
J. Inst. Math. Jussieu (2011).

M. Martín
The group of isometries of a Banach space and duality.
J. Funct. Anal. (2008).

Sketch of the talk

Extremely non-complex Banach spaces: motivation and examples

Sección 2

2. Extremely non-complex Banach spaces: motivation and examples
   - Complex structures
   - The first examples: $C(K)$ spaces with few operators
   - More $C(K)$-type examples
   - Further examples

Complex structures

Definition

X has complex structure if there is $T \in L(X)$ such that $T^2 = -\mathrm{Id}$.

Some remarks

- This gives a structure of vector space over $\mathbb{C}$:

$$(\alpha + i\,\beta)\,x = \alpha\,x + \beta\,T(x) \qquad \big(\alpha + i\,\beta \in \mathbb{C},\ x \in X\big)$$

- Defining

$$|||x||| = \max\left\{ \|e^{i\theta} x\| \ :\ \theta \in [0, 2\pi] \right\} \qquad (x \in X)$$

  one gets that $(X, |||\cdot|||)$ is a complex Banach space.
- If T is an isometry, then the given norm of X is actually complex.
- Conversely, if X is a complex Banach space, then

$$T(x) = i\,x \qquad \big(x \in X\big)$$

  satisfies $T^2 = -\mathrm{Id}$ and T is an isometry.

Complex structures II

Some examples

1. If $\dim(X) < \infty$, X has complex structure iff $\dim(X)$ is even.
2. If $X \simeq Z \oplus Z$ (in particular, $X \simeq X^2$), then X has complex structure.
3. There are infinite-dimensional Banach spaces without complex structure:
   - **Dieudonné, 1952:** the James' space $\mathcal{J}$ (since $\mathcal{J}^{**} \equiv \mathcal{J} \oplus \mathbb{R}$).
   - **Szarek, 1986:** some uniformly convex examples.
   - **Gowers-Maurey, 1993:** their H.I. space (no H.I. has complex structure) .

Definition

X is extremely non-complex if $\operatorname{dist}(T^2, -\mathrm{Id})$ is the maximum possible, i.e.

$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\| \qquad \big(T \in L(X)\big)$$

Question (Gilles Godefroy, private communication, 2005)

Is there any $X \neq \mathbb{R}$ such that $\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$ for every $T \in L(X)$?

Examples: $C(K)$ spaces with few operators

Theorem (Koszmider, 2004)

There are infinitely many different perfect compact spaces K such that all operators on $C(K)$ are weak multipliers.

- They are called weak Koszmider spaces.

Definition: weak multiplier

Let K be a compact space. $T \in L\big(C(K)\big)$ is a weak multiplier if

$$T^* = g\,\mathrm{Id} + S$$

where g is a Borel function and S is weakly compact.

Proposition

K perfect, $T \in L\big(C(K)\big)$ weak multiplier $\quad \Longrightarrow \quad \|\mathrm{Id} + T^2\| = 1 + \|T^2\|$

Corollary

There are infinitely many non-isomorphic extremely non-complex spaces.

More $C(K)$-type examples

More $C(K)$ type examples

There are perfect compact spaces $K_1,\ K_2$ such that:

- $C(K_1)$ and $C(K_2)$ are extremely non-complex,
- $C(K_1)$ contains a complemented copy of $C(\Delta)$.
- $C(K_2)$ contains a $(1$-complemented$)$ isometric copy of ℓ_∞.

Observation

- $C(K_1)$ and $C(K_2)$ have operators which are not weak multipliers.
- They are decomposable spaces.

Further examples

Spaces $C_E(K\|L)$

K compact, $L \subset K$ closed nowhere dense, $E \subset C(L)$. Define

$$C_E(K\|L) := \big\{ f \in C(K) \ : \ f|_L \in E \big\}.$$

Observation

$$C_E(K\|L)^* \equiv E^* \oplus_1 C_0(K\|L)^*$$

Theorem

K perfect weak Koszmider, L closed nowhere dense, $E \subset C(L)$
$\Longrightarrow$ $C_E(K\|L)$ is extremely non-complex.

Isometries on extremely non-complex spaces

Sección 3

Isometries on extremely non-complex spaces

Theorem

X extremely non-complex.

- $T \in \mathrm{Iso}(X) \implies T^2 = \mathrm{Id}$.
- $T_1, T_2 \in \mathrm{Iso}(X) \implies T_1 T_2 = T_2 T_1$.
- $T_1, T_2 \in \mathrm{Iso}(X) \implies \|T_1 - T_2\| \in \{0, 2\}$.
- $\Phi : \mathbb{R}_0^+ \longrightarrow \mathrm{Iso}(X)$ one-parameter semigroup $\implies \Phi(\mathbb{R}_0^+) = \{\mathrm{Id}\}$.

Consequences

- $\mathrm{Iso}(X)$ is a Boolean group with the composition.
- $\mathrm{Iso}(X)$ identifies with the set $\mathsf{Unc}(X)$ of unconditional projections on X:

$$P \in \mathsf{Unc}(X) \iff P^2 = P,\ 2P - \mathrm{Id} \in \mathrm{Iso}(X)$$
$$\iff P = \frac{1}{2}(\mathrm{Id} + T),\ T \in \mathrm{Iso}(X),\ T^2 = \mathrm{Id}.$$

Extremely non-complex $C_E(K\|L)$ spaces.

Remember

K perfect weak Koszmider, L closed nowhere dense, $E \subset C(L)$
$\implies C_E(K\|L)$ is extremely non-complex and $C_E(K\|L)^* \equiv E^* \oplus_1 C_0(K\|L)^*$.

Proposition

K perfect $\implies \exists\, L \subset K$ closed nowhere dense with $C[0,1] \subset C(L)$.

A good example

Take K perfect weak Koszmider, $L \subset K$ closed nowhere dense with
$E = \ell_2 \subset C[0,1] \subset C(L)$:

- $C_{\ell_2}(K\|L)$ has no non-trivial one-parameter semigroup of isometries.
- $C_{\ell_2}(K\|L)^* \equiv \ell_2 \oplus_1 C_0(K\|L)^* \quad \implies \quad \mathrm{Iso}\big(C_{\ell_2}(K\|L)^*\big) \supset \mathrm{Iso}(\ell_2)$.

But we are able to give a better result...

Isometries on extremely non-complex $C_E(K\|L)$ spaces

Theorem (Banach-Stone like)

$C_E(K\|L)$ extremely non-complex, $T \in \mathrm{Iso}(C_E(K\|L))$
$\Longrightarrow$ exists $\theta : K \setminus L \longrightarrow \{-1, 1\}$ continuous such that

$$[T(f)](x) = \theta(x)f(x) \qquad \big(x \in K \setminus L,\ f \in C_E(K\|L)\big)$$

Consequence: cases $E = C(L)$ and $E = 0$

- $C(K)$ extremely non-complex, $\varphi : K \longrightarrow K$ homeomorphism $\Longrightarrow$ $\varphi = \mathrm{id}$
- $C_0(K \setminus L) \equiv C_0(K\|L)$ extremely non-complex, $\varphi : K \setminus L \longrightarrow K \setminus L$ homeomorphism $\Longrightarrow$ $\varphi = \mathrm{id}$

Consequence: connected case

If $K \setminus L$ is connected, then

$$\mathrm{Iso}\big(C_E(K\|L)\big) = \{-\mathrm{Id}, +\mathrm{Id}\}$$

The main example

Koszmider, 2004

$\exists\ \mathcal{K}$ weak Koszmider space such that $\mathcal{K} \setminus F$ is connected if $|F| < \infty$.

Important observation on the construction above

There is $\mathcal{L} \subset \mathcal{K}$ closed and nowhere dense, with

- $\mathcal{K} \setminus \mathcal{L}$ connected
- $C[0,1] \subseteq C(\mathcal{L})$

Consequence: the best example

Consider $X = C_{\ell_2}(\mathcal{K} \| \mathcal{L})$. Then:

$$\operatorname{Iso}(X) = \{-\mathrm{Id}, +\mathrm{Id}\} \qquad \text{and} \qquad \operatorname{Iso}(X^*) \supset \operatorname{Iso}(\ell_2)$$