Bishop-Phelps-Bollobás moduli of a Banach space

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Manolo's 60th birthday

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Based on the papers. . .

M. Chica, V. Kadets, M. Martín, S. Moreno, and F. Rambla.
Bishop-Phelps-Bollobás moduli of Banach spaces.
J. Math. Anal. Appl. (2014)

M. Chica, V. Kadets, M. Martín, J. Merí, and M. Soloviova.
Two refinements of the Bishop-Phelps-Bollobás modulus.
Banach J. Math. Anal. (2015)

M. Chica, V. Kadets, M. Martín, and J. Merí.
Further properties of the Bishop-Phelps-Bollobás moduli.
Preprint (2015)

Summary

Introduction

Some notation

Notation

X real or complex Banach space

B_X closed unit ball

S_X unit sphere

X^* topological dual

$y^* \in X^*$ **attains its norm** if there is $y \in B_X$ such that $\|y^*\| = |y^*(y)|$

$$\Pi(X) = \{(y, y^*) \in S_X \times S_{X^*} \ : \ y^*(y) = 1\}$$

For $(x, x^*) \in X \times X^*$ its distance to $\Pi(X)$ is

$$d\big((x,x^*),\Pi(X)\big) = \inf\Big\{ \max\{\|x-y\|, \|x^*-y^*\|\} \ : \ (y,y^*) \in \Pi(X)\Big\}$$

The starting point

A PROOF THAT EVERY BANACH SPACE IS SUBREFLEXIVE

BY ERRETT BISHOP AND R. R. PHELPS

Communicated by Mahlon M. Day, August 19, 1960

A real or complex normed space is *subreflexive* if those functionals which attain their supremum on the unit sphere S of E are norm-dense in E^*, i.e., if for each f in E^* and each $\epsilon > 0$ there exist g in E^* and x in S such that $|g(x)| = \|g\|$ and $\|f - g\| < \epsilon$. There exist incomplete normed spaces which are not subreflexive [1][1] as well as incomplete spaces which *are* subreflexive (e.g., a dense subspace of a Hilbert space). It is evident that every reflexive Banach space is subreflexive. The theorem mentioned in the title will be proved for *real* Banach spaces; the result for complex spaces follows from this by considering the spaces over the real field and using the known isometry between complex functionals and the real functionals defined by their real parts.

In other words...

Norm attaining funcionals are dense in X^*.

The starting point

AN EXTENSION TO THE THEOREM OF BISHOP AND PHELPS

BÉLA BOLLOBÁS

Bishop and Phelps proved in **[1]** that every real or complex Banach space is *subreflexive*, that is the functionals (real or complex) which attain their supremum on the unit sphere of the space are dense in the dual space. We shall sharpen this result and then apply it to a problem about the numerical range of an operator.

Denote by S and S' the unit spheres in a Banach space B and its dual space B', respectively.

THEOREM 1. *Suppose $x \in S$, $f \in S'$ and $|f(x)-1| \leqslant \varepsilon^2/2$ $(0 < \varepsilon < \frac{1}{2})$. Then there exist $y \in S$ and $g \in S'$ such that $g(y) = 1$, $\|f-g\| \leqslant \varepsilon$ and $\|x-y\| < \varepsilon + \varepsilon^2$.*

This is nowadays known as the Bishop-Phelps-Bollobás theorem

The starting point

AN EXTENSION TO THE THEOREM OF BISHOP AND PHELPS

BÉLA BOLLOBÁS

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Our objective is to introduce two moduli which measure the best possible Bishop-Phelps-Bollobás theorem in a concrete Banach space

Definitions and basic properties

Definitions

X Banach space, $\delta \in (0,2)$

Bishop-Phelps-Bollobás modulus

$\Phi_X(\delta)$ is the smallest $\varepsilon > 0$ such that given $x \in B_X$, $x^* \in B_{X^*}$ with $\operatorname{Re} x^*(x) > 1 - \delta$, $\exists (y, y^*) \in \Pi(X)$ with $\|x - y\| < \varepsilon$, $\|x^* - y^*\| < \varepsilon$

- $\Phi_X(\delta) = \mathrm{d}_H\big(\Pi(X)\,,\, \{(x, x^*) \in B_X \times B_{X^*} \,:\, \operatorname{Re} x^*(x) > 1 - \delta\}\big)$

Spherical Bishop-Phelps-Bollobás modulus

$\Phi_X^S(\delta)$ is the smallest $\varepsilon > 0$ such that given $x \in S_X$, $x^* \in S_{X^*}$ with $\operatorname{Re} x^*(x) > 1 - \delta$, $\exists (y, y^*) \in \Pi(X)$ with $\|x - y\| < \varepsilon$, $\|x^* - y^*\| < \varepsilon$

- $\Phi_X^S(\delta) = \mathrm{d}_H\big(\Pi(X)\,,\, \{(x, x^*) \in S_X \times S_{X^*} \,:\, \operatorname{Re} x^*(x) > 1 - \delta\}\big)$

Observation

The smaller are Φ_X and Φ_X^S, the better is the Bishop-Phelps-Bollobás theorem in the space X

Basic properties

Properties of the moduli

- $\Phi_X^S(\delta) \leqslant \Phi_X(\delta)$ for all δ
- Φ_X and Φ_X^S are non-decreasing functions
- Φ_X and Φ_X^S are continuous on δ
- Fixed δ, $X \longmapsto \Phi_X(\delta)$ and $X \longmapsto \Phi_X^S(\delta)$ are continuous (Banach-Mazur)
- Consequence: if X_1 and X_2 are almost isometric,
  then $\Phi_{X_1}(\delta) = \Phi_{X_2}(\delta)$ and $\Phi_{X_1}^S(\delta) = \Phi_{X_2}^S(\delta)$
- $\Phi_X(\delta) \leqslant \Phi_{X^*}(\delta)$ and $\Phi_X^S(\delta) \leqslant \Phi_{X^*}^S(\delta)$ for all δ

The upper bound of the moduli

The upper bound of the moduli

X Banach space, $\delta \in (0,2)$. Then, $\Phi_X(\delta) \leqslant \sqrt{2\delta}$ (so, $\Phi^S_X(\delta) \leqslant \sqrt{2\delta}$).

- In other words, given $(x, x^*) \in B_X \times B_{X^*}$ with $\operatorname{Re} x^*(x) > 1-\delta$, there is $(y, y^*) \in \Pi(X)$ with $\|x-y\| < \sqrt{2\delta}$, $\|x^*-y^*\| < \sqrt{2\delta}$.

Idea of the Proof (for Φ^S_X)

Phelps, 1974

COROLLARY 2.2. *Suppose that C is a closed convex subset of the Banach space E, that $f \in E^*$ has norm 1, and that $\epsilon > 0$ and $z \in C$ are such that*

$$\sup f(C) \leqslant f(z) + \epsilon.$$

Then for any $0 < k < 1$ there exist $g \in E^*$ and $x_0 \in C$ such that $\sup g(C) = g(x_0)$, $\| x_0 - z \| \leqslant \epsilon/k$ and $\| f - g \| \leqslant k$.

- Use with $C = B_X$, $z = x$, $f = x^*$, $\varepsilon = \delta$, $k = \frac{\delta}{\sqrt{2\delta}}$
- Get $y = x_0 \in S_X$, $g \in Y^* \setminus \{0\}$ with $\|x-y\| < \sqrt{2\delta}$, $\|x^* - g\| < \frac{\delta}{\sqrt{2\delta}}$
- $y^* = g/\|g\|$, $(y, y^*) \in \Pi(X)$
  $$\begin{aligned} &\|x^* - y^*\| \\ &\leqslant \|x^* - g\| + \big\|g - g/\|g\|\big\| \\ &= \|x^* - g\| + \big|1 - \|g\|\big| \\ &\leqslant 2\|x^* - g\| < \sqrt{2\delta} \end{aligned}$$

The best possible general Bishop-Phelps-Bollobás theorem

Bishop-Phelps-Bollobás Theorem

X Banach space, $\varepsilon \in (0,2)$. Given $(x, x^*) \in B_X \times B_{X^*}$ with $\operatorname{Re} x^*(x) > 1 - \varepsilon^2/2$, there is $(y, y^*) \in \Pi(X)$ with $\|x - y\| < \varepsilon$, $\|x^* - y^*\| < \varepsilon$.

This is best possible

The real space $X = \ell_\infty^{(2)}$ satisfies $\Phi_X^S(\delta) = \Phi_X(\delta) = \sqrt{2\delta}$ for all $\delta \in (0,2)$.

Indeed,

$$z = (1 - \sqrt{2\delta}, 1) \in S_X \qquad z^* = \left(\frac{\sqrt{2\delta}}{2}, 1 - \frac{\sqrt{2\delta}}{2}\right) \in S_{X^*}$$

satisfy $\operatorname{Re} z^*(z) = 1 - \delta$ and $\mathrm{d}\big((z, z^*), \Pi(X)\big) = \sqrt{2\delta}$.

Examples

Hilbert spaces

Example

$$\Phi_{\mathbb{R}}(\delta) = \begin{cases} \delta & \text{if } 0 < \delta \leqslant 1 \\ \sqrt{\delta - 1} + 1 & \text{if } 1 < \delta < 2 \end{cases} \qquad \Phi^S_{\mathbb{R}}(\delta) = 0 \text{ for every } \delta \in (0,2).$$

Example

Let H be a Hilbert space of dimension over $\mathbb{R}$ greater than or equal to two. Then:

(a) $\Phi^S_H(\delta) = \sqrt{2 - \sqrt{4 - 2\delta}}$ for every $\delta \in (0,2)$.

(b) For $\delta \in (0,1]$, $\Phi_H(\delta) = \max\left\{\delta, \sqrt{2 - \sqrt{4 - 2\delta}}\right\}$.
For $\delta \in (1,2)$, $\Phi_H(\delta) = \sqrt{\delta}$.

More examples

Proposition

X Banach space satisfying one of the following conditions:

- $X = Y \oplus_1 Z$
- $X^* = V \oplus_1 W$ and V, W are not weak*-dense in X^*
- in particular, $X = Y \oplus_\infty Z$

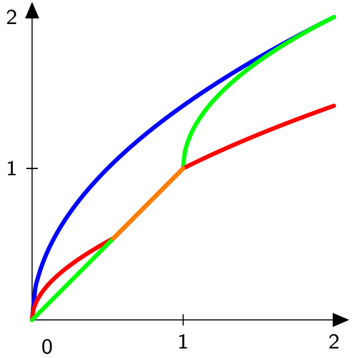
then $\Phi^S_X(\delta) = \Phi_X(\delta) = \sqrt{2\delta}$ for $\delta \in (0, 1/2)$.

Examples

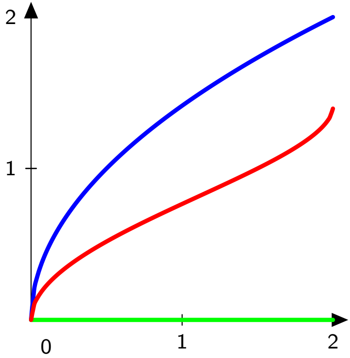
The above result applies to

1. $L_1(\mu)$, $L_\infty(\mu)$, $C(K)$.
2. C^*-algebras with non-trivial center.
3. $\mathcal{L}(H)^{**}$, but not known for $\mathcal{K}(H)$ or $\mathcal{L}(H)$.

A picture of the moduli of the known examples



The value of $\Phi_X(\delta)$ for $\mathbb{R}$, $\mathbb{C}$, $\ell_\infty^{(2)}$



The value of $\Phi_X^S(\delta)$ for $\mathbb{R}$, $\mathbb{C}$, $\ell_\infty^{(2)}$

Relation with uniformly non-squareness

Relation with uniformly non-squareness

Theorem

If X is uniformly non-square, then $\Phi_X(\delta) < \sqrt{2\delta}$.

Remarks

- X is uniformly non-square iff it does not contain almost isometric copies of the real space $\ell_\infty^{(2)}$
- X is uniformly non-square iff so is X^*
- The result can be quantified, relating $\Phi_X(\delta)$ with a modulus of uniformly non-squareness for small δ's
- The theorem reads also as: if $\Phi_X(\delta) = \sqrt{2\delta}$ for some δ, then X contains almost isometric copies of $\ell_\infty^{(2)}$.
- $\Phi_X(\delta) = \sqrt{2\delta}$ iff $\Phi_X^S(\delta) = \sqrt{2\delta}$

We are going to prove the theorem in the finite-dimensional case.

Recalling the slide "The upper bound of the moduli"

X Banach space, $\delta \in (0,2)$. Then, $\Phi_X(\delta) \leqslant \sqrt{2\delta}$ (so, $\Phi_X^S(\delta) \leqslant \sqrt{2\delta}$).

- In other words, given $(x, x^*) \in B_X \times B_{X^*}$ with $\operatorname{Re} x^*(x) > 1 - \delta$, there is $(y, y^*) \in \Pi(X)$ with $\|x - y\| < \sqrt{2\delta}$, $\|x^* - y^*\| < \sqrt{2\delta}$.

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$$\sup f(C) \leqslant f(z) + \epsilon.$$

Then for any $0 < k < 1$ there exist $g \in E^*$ and $x_0 \in C$ such that $\sup g(C) = g(x_0)$, $\| x_0 - z \| \leqslant \epsilon/k$ and $\| f - g \| \leqslant k$.

If $\Phi_X(\delta) = \sqrt{2\delta}$ and $\dim(X) < \infty$, $\exists x^*, g$ with

$$\big\|x^* - g/\|g\|\big\| = 2\|x^* - g\|$$

- Use with $C = B_X$, $z = x$, $f = x^*$, $\varepsilon = \delta$, $k = \frac{\delta}{\sqrt{2\delta}}$
- Get $y = x_0 \in S_X$, $g \in Y^* \setminus \{0\}$ with $\|x - y\| < \sqrt{2\delta}$, $\|x^* - g\| < \frac{\delta}{\sqrt{2\delta}}$
- $y^* = g/\|g\|$, $(y, y^*) \in \Pi(X)$
  $$\big\|x^* - g/\|g\|\big\| \leqslant \|x^* - g\| + \big\|g - g/\|g\|\big\| = \|x^* - g\| + \big|1 - \|g\|\big| \leqslant 2\|x^* - g\| < \frac{2\delta}{\sqrt{2\delta}}$$

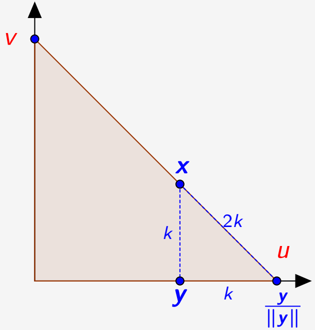
A way to find $\ell_\infty^{(2)}$

A sufficient condition

X Banach space, $k \in (0,1)$, exist $x \in S_X$, $y \in X$ with $\|x-y\| = k$ and $\left\| x - \frac{y}{\|y\|} \right\| = 2k$. Then, X contains (the real) $\ell_\infty^{(2)}$.

Proof.

1. It is enough to find $u, v \in S_X$ such that $\|u+v\| = \|u-v\| = 2$.
2. $\big|1 - \|y\|\big| = k$, so $\|y\| = 1-k$ or $\|y\| = 1+k$.
3. If $\|y\| = 1-k$, take $\boldsymbol{u} = y/(1-k)$, $\boldsymbol{v} = (x-y)/k$ in S_X
4. If $\|y\| = 1+k$, take $u = y/(1+k)$, $v = (y-x)/k$ in S_X



If $\dim(X) = \infty$, either use limits or ultrapowers

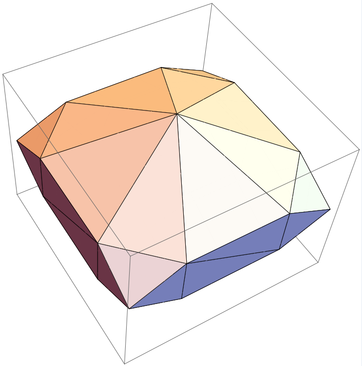
Containing $\ell_\infty^{(2)}$ is not enough

Example

Fix $\delta \in (0,2)$. Let X_δ such that B_X is the absolutely convex hull of

$$(0,0,\tfrac{3}{4}),\ (1-\varepsilon,1,\tfrac{\varepsilon}{2}),\ (1-\varepsilon,-1,\tfrac{\varepsilon}{2}),$$
$$(\varepsilon-1,1,\tfrac{\varepsilon}{2}),\ (\varepsilon-1,-1,\tfrac{\varepsilon}{2}),(1,1-\varepsilon,\tfrac{\varepsilon}{2}),$$
$$(-1,1-\varepsilon,\tfrac{\varepsilon}{2}),\ (1,\varepsilon-1,\tfrac{\varepsilon}{2}),$$
$$(-1,\varepsilon-1,\tfrac{\varepsilon}{2}),\ (1,1,0),\ (1,-1,0).$$

where $\varepsilon = \sqrt{2\delta}$.



Then, X_δ contains $\ell_\infty^{(2)}$ isometrically but $\Phi_{X_\delta}(\delta) < \sqrt{2\delta}$

Final remarks and open problems

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Remark

It is possible to get a modulus which depends on the norm of points and functionals which is more accurate than the general one.

- It is possible to get the minimum value which is valid for all Banach spaces,
- It has been calculated for Hilbert spaces, $L_1(\mu)$, $C(K)$...

Open problems

1. Calculate or estimate the moduli for other spaces like $L_p(\mu)$
2. Find a lower bound of the moduli valid for all Banach spaces with dimension greater than or equal to two
3. Can we get the same consequences if we just study the behaviour of the moduli close to 0?
   For instance, does X contain $\ell_\infty^{(2)}$ if $\limsup_{\delta\to 0} \frac{\Phi_X(\delta)}{\sqrt{2\delta}} = 1$**?**