Norm attaining compact operators

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Sketch of the talk

Preliminaries

Sección 1

1. Preliminaries
   - Bishop-Phelps theorem
   - Norm attaining operators
   - Lindenstrauss properties A and B

Bishop-Phelps theorem

Norm attaining functionals

X real or complex Banach space

$$B_X = \{x \in X \,:\, \|x\| \leqslant 1\} \qquad S_X = \{x \in X \,:\, \|x\| = 1\} \qquad X^* \text{ dual space}$$

$$\|x^*\| = \sup\{|x^*(x)| \,:\, x \in B_X\} \qquad (x^* \in X^*)$$

x^* attains its norm when this supremum is a maximum:

$$\exists\, x \in S_X \,:\, |x^*(x)| = \|x^*\|$$

Theorem (E. Bishop & R. Phelps, 1961)

The set of norm attaining functionals is dense in X^*
(for the norm topology).

Norm attaining operators: posing the problem

Norm attaining operators

X, Y Banach spaces, $L(X,Y)$ (bounded linear) operators

$$\|T\| = \sup\{\|Tx\| : x \in B_X\} \qquad \big(T \in L(X,Y)\big)$$

T attains its norm when this supremum is a maximum:

$$T \in NA(X,Y) \iff \exists\, x \in S_X : \|Tx\| = \|T\|$$

Problem

$$¿\ \overline{NA(X,Y)} = L(X,Y)\ ?$$

- J. Lindenstrauss, *Israel J. Math.* (1963) started the study of this problem.
- In general, the answer is **Negative**.
- Lindenstrauss introduced properties A and B.

Lindenstrauss properties A and B

Definition

- X has property A when: $\overline{NA(X,Z)} = L(X,Z) \quad \forall Z$
- Y has property B when: $\overline{NA(Z,Y)} = L(Z,Y) \quad \forall Z$

First positive examples (Lindenstrauss)

- X reflexive $\implies$ X has property A.
- ℓ_1 has property A (property α).
- If $c_0 \subset Y \subset \ell_\infty$ or Y is finite-dimensional and polyhedral, then Y has property B (property β).

First negative examples (Lindenstrauss)

- $L_1[0,1]$ and $C_0(L)$ (L infinite) do not have property A.
- If Y is strictly convex and contains an isomorphic copy of c_0, then Y fails property B.

Relation with Radon-Nikodým property

A theorem of Bourgain (1977)

$$\text{RNP} \quad \Longrightarrow \quad \text{A} \qquad \text{(in every equivalent norm)}$$

Bourgain also gave a reciprocal of this result.

R. Huff (1980)

$$X \text{ no RNP} \quad \Longrightarrow \quad \exists\, X_1 \sim X \sim X_2 \ : \ \overline{NA(X_1, X_2)} \neq L(X_1, X_2)$$

More counterexamples

W. Gowers (1990)

No infinite-dimensional Hilbert space has property B

For $1 < p < \infty$, ℓ_p and L_p fail property B

Squeezing: if Y is strictly convex and contains an isomorphic copy of ℓ_p with $1 < p < \infty$, then Y does not have property B.

M. D. Acosta (1999)

No infinite-dimensional strictly convex Banach space has property B

ℓ_1 and $L_1[0,1]$ fail property B

W. Schachermayer (1983)

$NA(L_1[0,1], C[0,1])$ is not dense in $L(L_1[0,1], C[0,1])$

Compact operators: posing the problem

Sección 2

Posing the problem for compact operators

Question

Can every compact operator be approximated by norm-attaining operators?

Observations

- In all the negative examples of the previous section, the authors constructed NON COMPACT operators which cannot be approximated by norm attaining operators.
- Actually, in most examples it is known that compact operators attaining the norm are dense.

Where is it possed?

- Diestel-Uhl, *Rocky Mount. J. Math.*, 1976.
- Diestel-Uhl, *Vector measures* (monograph), 1977.
- Johnson-Wolfe, *Studia Math.*, 1979.
- Acosta, *RACSAM* (survey), 2006.

More observations on compact operators

Question

Can every compact operator be approximated by norm-attaining operators?

Observations

- If X is reflexive, then ALL compact operators from X into Y are norm attaining. (Indeed, compact operators carry weak convergent sequences to norm convergent sequences.)
- On the other hand, for a non reflexive space X we do not know whether there is any compact operator into a Hilbert space (with rank greater than one) attaining the norm.
- Actually, we do not know whether for every Banach space there is a norm attaining rank-two operator into a Hilbert space.

Compact operators: negative results

Sección 3

3 Compact operators: negative results

- The first negative example
- More examples: about the domain space
- More examples: about the range space
- More examples: Domain=Range
- Some by-products

Extending a result by Lindenstrauss

X, Y Banach spaces, $T \in L(X, Y)$ and $x_0 \in S_X$ with $\|T\| = \|Tx_0\| = 1$.

- If x_0 is not extreme point of B_X, there is $z \in X$ such that $\|x_0 \pm z\| \leqslant 1$, so $\|Tx_0 \pm Tz\| \leqslant 1$.
- If Tx_0 is an extreme point of B_Y, then $Tz = 0$.

First consequence (Lindenstrauss)

- $NA(c_0, Y) \subseteq F(c_0, Y)$ if Y is strictly convex.
- Therefore, c_0 fails property A.

Geometrical lemma, Lindenstrauss

X, Y Banach spaces. Suppose that

- for every $x_0 \in S_X$, $\mathrm{Lin}\{z \in X \ : \ \|x_0 \pm z\| \leqslant 1\}$ has finite codimension,
- Y is strictly convex.

Then, $NA(X, Y) \subseteq F(X, Y)$ $\quad$ ($F(X, Y)$ finite-rank operators).

Extending a result by Lindenstrauss (II)

Proposition (extension of Lindenstrauss result)

$X \leqslant c_0$. For every $x_0 \in S_X$, $\operatorname{Lin}\{z \in X \,:\, \|x_0 \pm z\| \leqslant 1\}$ has finite codimension.

Proof.

- As $x_0 \in c_0$, there exists m such that $|x_0(n)| < 1/2$ for every $n \geqslant m$.
- Let $Z = \big\{z \in X \,:\, x_0(i) = 0 \text{ for } 1 \leqslant i \leqslant m\big\}$ (finite codimension in X).
- For $z \in Z$ with $\|z\| \leqslant 1/2$, one has $\|x \pm z\| \leqslant 1$.

Main consequence

$X \leqslant c_0$, Y strictly convex. Then $NA(X,Y) \subseteq F(X,Y)$.

Question

How to use this result?

Grothendieck's approximation property

Definition (Grothendieck, 1950's)

X has the approximation property (AP) if for every $K \subset X$ compact and every $\varepsilon > 0$, there exists $F \in L(X,X)$ of finite rank such that $\|Fx - x\| < \varepsilon$ for all $x \in K$.

Notación

X and Y Banach, $F(X,Y)$ finite-rank operators $K(X,Y)$ compact operators

Basic results

- (Grothendieck) Y has AP iff $\overline{F(Z,Y)} = K(Z,Y)$ for all Z.
- (Grothendieck) X^* has AP iff $\overline{F(X,Z)} = K(X,Z)$ for all Z.
- (Grothendieck) X^* AP $\implies$ X AP.
- (Enflo, 1973) There exists $X \leqslant c_0$ without AP.

The first example

Theorem

There exists a **compact** operator which cannot be approximated by norm attaining operators.

Proof:

- Consider $X \leqslant c_0$ without AP (Enflo).
- X^* does not has AP,
  $\Longrightarrow$ there exists Y and $T \in K(X,Y)$ such that $T \notin \overline{F(X,Y)}$.
- We may suppose $Y = \overline{T(X)}$, which is separable, so admits an equivalent strictly convex renorming (Klee).
- We apply the extension of Lindenstrauss result: $NA(X,Y) \subseteq F(X,Y)$.
- Therefore, $T \notin \overline{NA(X,Y)}$.

More examples I. Domain space

Proposition

X subspace of c_0 such that X^* has no AP. Then, there is Y and $T \in K(X,Y)$ which cannot be approximated by norm attaining operators.

Example by Johnson-Schechtman, 2001

Exists X subspace of c_0 **with Schauder basis** such that X^* fails the AP.

Corolary

There exists a Banach space X with Schauder basis, a Banach space Y and a compact operator from X to Y which cannot be approximated by norm attaining operators.

Proposition

Every polyhedral space X such that X^* fails AP admits a renorming such that there is a compact operator from X which cannot be approximated by norm attaining operators (for the new norm).

More examples II. Range space

Strictly convex spaces

Y strictly convex without AP. Then there exists a compact operator into Y which cannot be approximated by norm attaining operators.

Lemma (Grothendieck)

Y has AP iff for every closed subspace X of c_0,
$F(X,Y)$ is dense in $K(X,Y)$.

Subespacios de $L_1(\mu)$

Y subspace of the complex space $L_1(\mu)$ without AP.
Then there exist X and a compact operator from X to Y which cannot be approximated by norm attaining operators.

More examples III. Domain=Range

Theorem

There exists a Banach space Z and a compact operator from Z to Z which cannot be approximated by norm attaining operators.

Actually:

X and Y Banach spaces.
If all (compact) operators from $Z = X \oplus_\infty Y$ to itself can be approximated by norm attaining (compact) operators, then the same is true for all (compact) operators from X to Y.

(this result was unknown to Lindenstrauss)

Some by-products

Subspaces of c_0

No infinite-dimensional subspace of c_0 has Lindenstrauss property A.

Proof: The inclusion from X into a strictly convex renorming of c_0 cannot be approximated by norm attaining operators.

An alternative proof of a result by Gowers

No infinite-dimensional L_p space $(1 < p < \infty)$ has Lindenstrauss property B.

Proof:

- Gasparis, 2009: there is E_p polyhedral with a quotient isomorphic to ℓ_p.
- This provides a non-compact operator from E_p into L_p.
- Let X be a renorming of E_p whose unit sphere "looks like the one of c_0" (the norm depends upon finitely many coordinates).
- Then, $NA(X, L_p) \subset F(X, L_p)$.

Properties A^k and B^k

Sección 4

4 Properties A^k and B^k
- Definición y primeros ejemplos
- Main open problems
- Positive results on property A^k
- Positive results on property B^k

Properties A^k and B^k

Definition

- X has property A^k when: $\overline{NA(X,Z)\cap K(X,Z)} = K(X,Z) \quad \forall\, Z$
- Y has property B^k when: $\overline{NA(Z,Y)\cap K(Z,Y)} = K(Z,Y) \quad \forall\, Z$

First positive examples

- All mentioned examples with property A have property A^k (ex. RNP)
- All mentioned examples with property B have property B^k (ex. property β)
- (Diestel-Uhl) $L_1(\mu)$ has A^k.
- (Johnson-Wolfe) $C_0(L)$ has A^k.
- (Johnson-Wolfe) Every isometric predual of $L_1(\mu)$ has B^k.
  In the real case, $L_1(\mu)$ has B^k.
- (Cascales-Guirao-Kadets) $A(\mathbb{D})$ has B^k (actually, every uniform algebra).

Negative examples

- For A^k: every subspace of c_0 whose dual fails AP.
- For B^k: every strictly convex space without AP.

Main open problems

Main problem

Does every finite-dimensional space have Lindenstrauss property B?

This is equivalent to:

Problem

$$AP \implies \mathrm{B}^k?$$

On the domain space, we have the following problem:

Problem

$$X^* \ AP \implies X \ \mathrm{A}^k?$$

Observation

Known positive results on properties A^k and B^k are partial answers to the above two questions, as strong forms of the AP are involved.

Positive results on property A^k.

Problem

$$X^* \ AP \implies X \ \mathrm{A}^k?$$

Partial answer:

(Johnson-Wolfe) With a stronger approximation property...

Suppose there exists a net of contractive projections $(P_\alpha)_\alpha$ in X with finite rank such that $\lim_\alpha P_\alpha^* = \mathrm{Id}_{X^*}$ in SOT. Then, X has A^k.

Consequences

- (Diestel-Uhl) $L_1(\mu)$ has A^k.
- (Johnson-Wolfe) $C_0(L)$ has A^k.
- X with monotone and "shrinking" basis $\implies$ X has A^k.
- $X^* \equiv \ell_1 \implies X$ has A^k (using a result by Gasparis).
- $X \leqslant c_0$ with monotone basis $\implies$ X has A^k (using a result by Godefroy–Saphar).

Positive results on property B^k.

Problem

$$AP \implies \mathrm{B}^k?$$

A partial answer

- If Y is polyhedral (real) and has AP $\implies$ Y has B^k.
- There is a complex analogous...

Example

$$Y \leqslant c_0 \text{ (real or complex) with AP} \implies Y \text{ has } \mathrm{B}^k.$$

A somehow reciprocal to the problem...

$$Y \text{ separable with } \mathrm{B}^k \text{ for every equivalent norm} \implies Y \text{ has AP}.$$

Open Problems and bibliography

Sección 5

Some open problems

Main problem

Can every finite-rank operator be approximated by norm-attaining operators ?

Problem

X Banach space, does there exist a norm-attaining rank-two operator from X to a Hilbert space ?

Another main problem

$$X^* \ AP \implies X \ \mathsf{A}^k?$$

Problem

$X \leqslant c_0$ with the metric AP, does it have A^k?

Problem

X such that $X^* \equiv L_1(\mu)$, does X have A^k?

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