Bishop-Phelps-Bollobás version of Lindenstrauss properties A and B

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THE BISHOP-PHELPS-BOLLOBÁS VERSION OF LINDENSTRAUSS PROPERTIES A AND B

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Dedicated to the memory of Joram Lindenstrauss and Robert Phelps

Abstract. We study a Bishop-Phelps-Bollobás version of Lindenstrauss properties A and B. For domain spaces, we study Banach spaces X such that (X, Y) has the Bishop-Phelps-Bollobás property (BPBp) for every Banach space Y. We show that in this case, there exists a universal function $\eta_X(\varepsilon)$ such that for every Y, the pair (X, Y) has the BPBp with this function. This allows us to prove some necessary isometric conditions for X to have the property. We also prove that if X has this property in every equivalent norm, then X is one-dimensional. For range spaces, we study Banach spaces Y such that (X, Y) has the Bishop-Phelps-Bollobás property for every Banach space X. In this case, we show that there is a universal function $\eta_Y(\varepsilon)$ such that for every X, the pair (X, Y) has the BPBp with this function. This implies that this property of Y is strictly stronger than Lindenstrauss property B. The main tool to get these results is the study of the Bishop-Phelps-Bollobás property for c_0-, ℓ_1- and ℓ_∞-sums of Banach spaces.

Sketch of the talk

Preliminaries: Lindenstrauss world

Sección 1

Bishop-Phelps theorem

Norm attaining functionals

X real or complex Banach space

$$B_X = \{x \in X \,:\, \|x\| \leqslant 1\} \qquad S_X = \{x \in X \,:\, \|x\| = 1\} \qquad X^* \text{ dual of } X$$

$$\|x^*\| = \sup\{|x^*(x)| \,:\, x \in B_X\} \qquad (x^* \in X^*)$$

x^* attains its norm when this supremum is a maximum: $\exists\, x \in S_X \,:\, |x^*(x)| = \|x^*\|$

Theorem (E. Bishop & R. Phelps, 1961)

Norm attaining functionals are dense in X^*
(in the norm topology)

Lindenstrauss: Norm attaining operators

Norm attaining operator

X , Y Banach spaces, $L(X,Y)$ (bounded linear) operators

$$\|T\| = \sup\{\|Tx\| : x \in B_X\} \qquad \big(T \in L(X,Y)\big)$$

T attains its norm when this supremum is a maximum:

$$T \in NA(X,Y) \iff \exists\, x \in S_X : \|Tx\| = \|T\|$$

Problem

$$¿\ \overline{NA(X,Y)} = L(X,Y)\ ?$$

- J. Lindenstrauss, *Israel J. Math.* (1963) started the study of this problem.
- The answer is **Negative** in general.
- For the study of this problem, Lindenstrauss introduced properties A and B.

Lindenstrauss properties A and B

Definition

- X has **property A** if $\overline{NA(X,Y)} = L(X,Y) \quad \forall\, Y$.
- Y has **property B** if $\overline{NA(X,Y)} = L(X,Y) \quad \forall\, X$.

First positive examples (Lindenstrauss)

- Reflexive spaces have property A.
- ℓ_1 has property A (property α).
- If $c_0 \subset Y \subset \ell_\infty$ or Y finite dimensional and polyhedral,
  $\Longrightarrow \quad Y$ has property B (property β).

First negative examples (Lindenstrauss)

- $L_1[0,1]$ and $C_0(L)$ (L infinite) fails property A.
- Y strictly convex containing c_0 fails property B.

Lindenstrauss properties A and B: further examples

Relationship with RNP (J. Bourgain, 1977 & R. Huff, 1980)

- RNP $\implies$ A
- X no RNP $\implies$ $\exists\, X_1 \sim X \sim X_2\ :\ \overline{NA(X_1, X_2)} \neq L(X_1, X_2)$

W. Gowers, 1990 & M. D. Acosta, 1999

- (Gowers) Infinite-dimensional $L_p(\mu)$ $(1 < p < \infty)$ spaces fail property B.
  Squeezing, strictly convex spaces containing ℓ_p $(1 < p < \infty)$ fail property B.
- (Acosta) Infinite-dimensional strictly convex spaces fail property B.
- (Acosta) Infinite-dimensional $L_1(\mu)$ spaces fail property B.

A pair of classical spaces (W. Schachermayer, 1983)

$NA(L_1[0,1], C[0,1])$ is not dense in $L(L_1[0,1], C[0,1])$

Preliminaries: Bishop-Phelps-Bollobás world

Sección 2

Bishop-Phelps-Bollobás theorem

Theorem (E. Bishop & R. Phelps, 1961)

Norm attaining functionals are dense in X^*

B. Bollobás contribution, 1970

Fix $0 < \varepsilon < 2$.

If $x_0 \in B_X$ and $x_0^* \in B_{X^*}$ satisfy $\operatorname{Re} x_0^*(x_0) > 1 - \varepsilon^2/2$,

there exist $x \in S_X$, $x^* \in S_{X^*}$ with

$$x^*(x) = 1, \qquad \|x_0 - x\| < \varepsilon, \qquad \|x_0^* - x^*\| < \varepsilon.$$

(see Chica-Kadets-Martín-Moreno-Rambla 2014 for this version)

Bishop-Phelps-Bollobás property

Bishop-Phelps-Bollobás property (M. Acosta, R. Aron, D. García & M. Maestre, 2008)

A pair of Banach spaces (X, Y) has the **Bishop-Phelps-Bollobás property** if given $\varepsilon \in (0, 1)$ there is $\eta(\varepsilon) > 0$ such that whenever

$$T_0 \in S_{L(X,Y)}, \quad x_0 \in S_X, \quad \|T_0 x_0\| > 1 - \eta(\varepsilon),$$

there exist $S \in L(X, Y)$ and $x \in S_X$ such that

$$1 = \|S\| = \|Sx\|, \qquad \|x_0 - x\| < \varepsilon, \qquad \|T_0 - S\| < \varepsilon.$$

Observation

If (X, Y) has the BPBp $\implies$ $\overline{NA(X, Y)} = L(X, Y)$.

★ Does this implication reverse? **No**

First examples

- There is Y_0 such that (ℓ_1, Y_0) fails BPBp.
- X, Y finite-dimensional, then (X, Y) has BPBp.
- Y with property β (example $c_0 \leqslant Y \leqslant \ell_\infty$), then (X, Y) has BPBp $\forall X$.

More examples

Pairs of classical spaces

- (Aron-Choi-García-Maestre, 2011) $(L_1[0,1], L_\infty[0,1])$ has BPBp.
- (Acosta + 7) $(C(K_1), C(K_2))$ has BPBp (in the real case).
- (Choi-Kim-Lee-Martín, 2014) $(L_1(\mu), L_1(\nu))$ has BPBp.

Other examples

- (Acosta + 3, 2013; Kim-Lee, 2014) X uniformly convex, then (X, Y) has BPBp for all Y.
- (Cascales-Guirao-Kadets, 2013) X Asplund, then (X, A) has BPBp for every uniform algebra A (in particular, $A = C_0(L)$ or $A = A(\mathbb{D})$).
- (Choi-Kim, 2011) $(L_1(\mu), Y)$ has the BPBp when Y has the RNP and the AHSP.
- (Kim-Lee, 2015) $(C(K), Y)$ has the BPBp when Y is uniformly convex.
- (Acosta, 201?) $(C(K), Y)$ has the BPBp when Y is uniformly complex convex (e.g. complex $L_1(\mu)$).

The BPB version of Lindenstrauss properties A and B

Universal BPB domain and range spaces

- X is a **universal BPB domain space** if (X,Y) has BPBp $\forall Y$.
- Y is a **universal BPB range space** if (X,Y) has BPBp $\forall X$.

Observations

- X universal BPB domain space $\implies$ X has property A.
- This implication does not reverse: ℓ_1 is not a universal BPB domain space. Evenmore, ℓ_1^2 fails to be a universal BPB domain space (we'll see later).
- Y universal BPB range space $\implies$ Y has property B.
- Does this implication reverse? **No**, as we will see later.

Examples

- Uniformly convex spaces are universal BPB domain spaces.
- Property β implies being universal BPB range space.

These are, up to now, **the only known examples**.

The results

Sección 3

The best function for the BPBp

Bishop-Phelps-Bollobás property (Acosta, Aron, García, Maestre, 2008)

A pair of Banach spaces (X,Y) has the **Bishop-Phelps-Bollobás property** if given $\varepsilon\in(0,1)$ there is $\eta(\varepsilon)>0$ such that whenever

$$T_0\in S_{L(X,Y)},\quad x_0\in S_X,\quad \|T_0x_0\|>1-\eta(\varepsilon),$$

there exist $S\in L(X,Y)$ and $x\in S_X$ such that

$$1=\|S\|=\|Sx\|,\qquad \|x_0-x\|<\varepsilon,\qquad \|T_0-S\|<\varepsilon.$$

In this case, (X,Y) has the **BPBp** with the function $\varepsilon\longmapsto\eta(\varepsilon)$.

The best BPBp function

We write $\eta(X,Y)(\varepsilon)$ for the best (the greatest) function η that we may use in the definition of BPBp. Equivalently,

$$\eta(X,Y)(\varepsilon)=\inf\Big\{1-\|T\|\ :\ x\in S_X,\ T\in S_{L(X,Y)},\ \operatorname{dist}\Big((x,T),\Pi(X,Y)\Big)\geqslant\varepsilon\Big\},$$

- $\Pi(X,Y)=\Big\{(x,S)\ :\ \|Sx\|=\|S\|=\|x\|=1\Big\},$
- $\operatorname{dist}\Big((x,T),\Pi(X,Y)\Big)=\inf\Big\{\max\{\|x-y\|,\|T-S\|\}\ :\ (y,S)\in\Pi(X,Y)\Big\}.$

(X,Y) has BPBp iff $\eta(X,Y)(\varepsilon)>0$ for every $\varepsilon\in(0,1)$

BPB property and direct sums

Theorem

$\{X_i\ :\ i \in I\}$, $\{Y_j\ :\ j \in J\}$ families of Banach spaces;
let X be the c_0-, ℓ_1-, or ℓ_∞-sum of $\{X_i\}$ and let Y be the c_0-, ℓ_1-, or ℓ_∞-sum of $\{Y_j\}$.

$$\Longrightarrow \qquad \eta(X,Y) \leqslant \eta(X_i,Y_j) \qquad \big(i \in I,\ j \in J\big).$$

Recent extension (Dantas)

The result extends to arbitrary absolute sums of range spaces and to some absolute sums of domain spaces.

Main consequence

1. X universal BPB domain space, then there exists $\eta^X : (0,1) \longrightarrow \mathbb{R}^+$ such that $\eta(X,Z) \geqslant \eta^X$ for every Banach space Z.
2. Y universal BPB range space, then there exists $\eta_Y : (0,1) \longrightarrow \mathbb{R}^+$ such that $\eta(Z,Y) \geqslant \eta_Y$ for every Banach space Z.

Universal BPB domain spaces: necessary conditions

Theorem

X universal BPB domain space. Then,

1. (real case) no face of S_X contains a non-empty relatively open subset of S_X;
2. if X is isomorphic to a strictly convex Banach space, then extreme points of B_X are dense in S_X;
3. if X is superreflexive, then strongly exposed points of B_X are dense in S_X.
4. In particular, if X is a real 2-dimensional Banach space which is a universal BPB domain space, then X is uniformly convex.
5. More in particular, ℓ_1^2 is not a universal BPB domain space.

It was proved by Kim-Lee (2015) under the assumption of the existence of a “universal” function η^X, now unnecessary.

Open question

X universal BPB domain space $\Longrightarrow$ X uniformly convex **?**

Universal BPB domain spaces: renorming

Theorem

X universal BPB domain space in every equivalent renorming $\Longrightarrow$ $\dim X = 1$.

This comes from...

Lemma

$X = X_1 \oplus_1 X_2$, Y strictly convex. If (X, Y) has the BPBp $\Longrightarrow$ Y uniformly convex.

Other consequences of the Lemma

- ℓ_1^2 is not a universal BPB domain space
- There exists $X \simeq \ell_2$ such that (X, X) fails the BPBp (just take $X = \ell_1^2 \oplus_1 Y$ where $Y \simeq \ell_2$ strictly convex not uniformly convex.)

Universal BPB range spaces: a counterexample

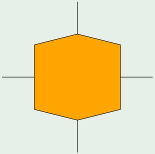
Main result on range spaces

Lindenstrauss property B **does not imply** being a universal BPB range space.

This comes from...

Example

For $k \in \mathbb{N}$, consider $Y_k = \mathbb{R}^2$ endowed with the norm



$$\|(x,y)\| = \max\{|x|, |y| + \tfrac{1}{k}|x|\} \quad (x, y \in \mathbb{R}).$$

- Y_k is polyhedral and so it is a universal BPB range space.
- $\inf_{k\in\mathbb{N}} \eta(\ell_1^2, Y_k)(\varepsilon) = 0$ for $0 < \varepsilon < 1/2$.
- Therefore, if we consider $\mathcal{Y} = \left[\bigoplus_{i=1}^{\infty} Y_k\right]_{c_0}$, then $(\ell_1^2, \mathcal{Y})$ fails the BPBp.
- On the other hand, $\mathcal{Y}$ has Lindenstrauss property B (it has property quasi-β).