Slicely Countably Determined Banach spaces

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Most of the results of the talk appeared in...

A. Avilés, V. Kadets, M. Martín, J. Merí, and V. Shepelska
Slicely Countably Determined Banach spaces
C. R. Acad. Sci. Paris (2009)

A. Avilés, V. Kadets, M. Martín, J. Merí, and V. Shepelska
Slicely Countably Determined Banach spaces
Trans. Amer. Math. Soc. (2010)

Introduction

Section 1

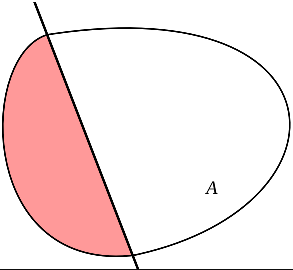
Basic notation

Basic notation

X real or complex Banach space.

- S_X unit sphere, B_X closed unit ball, $\mathbb{T}$ modulus-one scalars.
- X^* dual space, $L(X)$ bounded linear operators from X to X.
- $\operatorname{conv}(\cdot)$ convex hull, $\overline{\operatorname{conv}}(\cdot)$ closed convex hull,
- A slice of $A \subset X$ is a (nonempty) subset of the form

$$S(A,x^*,\alpha) = \{x \in A : \operatorname{Re} x^*(x) > \sup \operatorname{Re} x^*(A) - \alpha\} \quad (x^* \in X^*,\ \alpha > 0)$$



Two classical concepts: Radon-Nikodým property and Asplund spaces

The Radon-Nikodým property or RNP (1930's)

- X has the RNP iff the Radon-Nikodým theorem is valid for X-valued meassures;
- Equivalently [1960's], every bcc subset contains a denting point (i.e. a point belonging to slices of arbitrarily small diameter).

X Asplund $\iff$ X^* RNP

Reflexive (say) $\Longrightarrow$ (RNP and Asplund)

(RNP or Asplund) $\Longrightarrow$ **???**

Asplund spaces (1960's)

- X is an Asplund space if every continuous convex real-valued function defined on an open subset of X is F-differentiable on a dense subset;
- Equivalently [1970's], every separable subspace has separable dual.

The road map of the talk

The property

We introduce an isomorphic property for (separable) Banach spaces, the so-called

slicely countably determination (SCD)

such that

- it is satisfied by RNP spaces
  (actually, by strongly regular spaces – PCP in particular–);
- it is satisfied by Asplund spaces
  (actually, by spaces not containing ℓ_1).

We also present examples and stability properties.

The applications

- We apply SCD to get results for the Daugavet property, the alternative Daugavet property and spaces with numerical index 1.
- We present SCD operators and applications.

Outline

Slicely Countably Determined sets and spaces

Section 2

SCD sets: Definitions and preliminary remarks

X Banach space, $A \subset X$ bounded and convex.

SCD sets

A is Slicely Countably Determined (SCD) if there is a sequence $\{S_n : n \in \mathbb{N}\}$ of slices of A satisfying one of the following equivalent conditions:

- if $B \subseteq A$ satisfies $B \cap S_n \neq \emptyset \ \forall n$, then $A \subseteq \overline{\text{conv}}(B)$,
- given $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in S_n \ \forall n \in \mathbb{N}$, $A \subseteq \overline{\text{conv}}\big(\{x_n : n \in \mathbb{N}\}\big)$,
- every slice of A contains one of the S_n's,

Remarks

- A is SCD iff $\overline{A}$ is SCD.
- If A is SCD, then it is separable.

SCD sets: Elementary examples I

Example

A separable and $A = \overline{\mathrm{conv}}(\mathrm{dent}(A)) \Longrightarrow A$ is SCD.

Proof.

- Take $\{a_n : n \in \mathbb{N}\}$ denting points with $A = \overline{\mathrm{conv}}\big(\{a_n : n \in \mathbb{N}\}\big)$.
- For every $n, m \in \mathbb{N}$, take a slice $S_{n,m}$ containing a_n and of diameter $1/m$.
- If $B \cap S_{n,m} \neq \emptyset \Longrightarrow a_n \in \overline{B}$.
- Therefore, $A = \overline{\mathrm{conv}}\big(\{a_n : n \in \mathbb{N}\}\big) \subseteq \overline{\mathrm{conv}}(\overline{B}) = \overline{\mathrm{conv}}(B)$.

Example

In particular, A RNP separable $\Longrightarrow A$ SCD.

Corollary

- If X is separable LUR $\Longrightarrow B_X$ is SCD.
- So, every separable space can be renormed such that $B_{(X,|\cdot|)}$ is SCD.

SCD sets: Elementary examples II

Example

If X^* is separable $\Longrightarrow A$ is SCD.

Proof.

- Take $\{x_n^* : n \in \mathbb{N}\}$ dense in S_{X^*}.
- For every $n, m \in \mathbb{N}$, consider $S_{n,m} = S(A, x_n^*, 1/m)$.
- It is easy to show that any slice of A contains one of the $S_{n,m}$

Example

$B_{C[0,1]}$ and $B_{L_1[0,1]}$ are not SCD.

SCD sets: Further examples I

Convex combination of slices

$W = \sum_{k=1}^{m} \lambda_k S_k \subset A$ where $\lambda_k \geqslant 0$, $\sum \lambda_k = 1$, S_k slices.

Proposition

In the definition of SCD we can use a sequence $\{S_n : n \in \mathbb{N}\}$ of convex combination of slices.

Small combinations of slices

A has small combinations of slices iff every slice of A contains convex combinations of slices of A with arbitrary small diameter.

Example

If A has small combinations of slices + separable $\Longrightarrow$ A is SCD.

Particular case

A strongly regular (in particular, PCP) + separable $\Longrightarrow$ A is SCD.

SCD sets: Further examples II

Bourgain's lemma

Every relative weak open subset of A contains a convex combination of slices.

Corollary

In the definition of SCD we can use a sequence of relative weak open subsets: the set A is SCD iff there is a sequence $\{V_n : n \in \mathbb{N}\}$ of relative weak open subsets of A such that every slice of A contains one of the V_n's.

π-bases

A π-base of the weak topology of A is a family $\{V_i : i \in I\}$ of weak open sets of A such that every weak open subset of A contains one of the V_i's.

Proposition

If $(A, \sigma(X, X^*))$ has a countable π-base $\Longrightarrow$ A is SCD.

SCD sets: Further examples III

Theorem

A separable without ℓ_1-sequences $\Longrightarrow (A, \sigma(X, X^*))$ has a countable π-base.

Proof.

- We see $(A, \sigma(X, X^*)) \subset C(T)$ where $T = (B_{X^*}, \sigma(X^*, X))$.
- By Rosenthal ℓ_1 theorem, $(A, \sigma(X, X^*))$ is a relatively compact subset of the space of first Baire class functions on T.
- By a result of Todorčević, $(A, \sigma(X, X^*))$ has a σ-disjoint π-base.
- $\{V_i : i \in I\}$ is *σ-disjoint* if $I = \bigcup_{n\in\mathbb{N}} I_n$ and each $\{V_i : i \in I_n\}$ is pairwise disjoint.
- A σ-disjoint family of open subsets in a separable space is countable. ✓

Main example

A separable without ℓ_1-sequences $\Longrightarrow A$ is SCD.

SCD spaces: definition and examples

SCD space

X is Slicely Countably Determined (SCD) if so are its convex bounded subsets.

Examples of SCD spaces

1. X separable strongly regular. In particular, RNP, PCP spaces.
2. X separable $X \not\supseteq \ell_1$. In particular, if X^* is separable.

Examples of NOT SCD spaces

1. $C[0,1]$, $L_1[0,1]$
2. Actually, every X containing (an isomorphic copy of) $C[0,1]$ or $L_1[0,1]$.

Example (and question), Kadets-M.-Merí-Werner, 2013

- X Banach space with 1-unconditional basis $\implies$ B_X is SCD.
- We do not know whether X is SCD.

SCD spaces: stability properties

Remark

- Every subspace of a SCD space is SCD.
- This is false for quotients.

Theorem

$Z \subset X$. If Z and X/Z are SCD $\Longrightarrow$ X is SCD.

Corollary

X separable NOT SCD $\Longrightarrow$ $X \supset \ell_1$ and

- If $\ell_1 \simeq Y \subset X \Longrightarrow X/Y$ contains a copy of ℓ_1.
- If $\ell_1 \simeq Y_1 \subset X \Longrightarrow$ there is $\ell_1 \simeq Y_2 \subset X$ with $Y_1 \cap Y_2 = 0$.

Corollary

$X_1, \ldots, X_m$ SCD $\Longrightarrow X_1 \oplus \cdots \oplus X_m$ SCD.

SCD spaces: stability properties II

Theorem

$X_1, X_2, \ldots$ SCD, E with 1-unconditional basis.

- $E \not\supseteq c_0 \Longrightarrow \left[\bigoplus_{n\in\mathbb{N}} X_n\right]_E$ SCD.
- $E \not\supseteq \ell_1 \Longrightarrow \left[\bigoplus_{n\in\mathbb{N}} X_n\right]_E$ SCD.

Examples

1. $c_0(\ell_1)$ and $\ell_1(c_0)$ are SCD.
2. $c_0 \otimes_\varepsilon c_0$, $c_0 \otimes_\pi c_0$, $c_0 \otimes_\varepsilon \ell_1$, $c_0 \otimes_\pi \ell_1$, $\ell_1 \otimes_\varepsilon \ell_1$, and $\ell_1 \otimes_\pi \ell_1$ are SCD.
3. $K(c_0)$ and $K(c_0, \ell_1)$ are SCD.
4. $\ell_2 \otimes_\varepsilon \ell_2 \equiv K(\ell_2)$ and $\ell_2 \oplus_\pi \ell_2 \equiv \mathcal{L}_1(\ell_2)$ are SCD

Applications

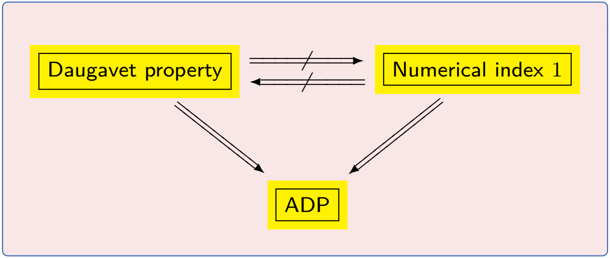
Section 3

The DPr, the ADP and numerical index 1

Definition of the properties

1. **Kadets-Shvidkoy-Sirotkin-Werner, 1997:**
   X has the Daugavet property (DPr) if
   $$\|\mathrm{Id}+T\| = 1+\|T\| \qquad \text{(DE)}$$
   for every rank-one $T \in L(X)$.
   - Then every T not fixing copies of ℓ_1 also satisfies (DE).
2. **Lumer, 1968:** X has numerical index 1 ($n(X)=1$) if
   $$\max_{\theta\in\mathbb{T}} \|\mathrm{Id}+\theta T\| = 1+\|T\| \qquad \text{(aDE)}$$
   for EVERY operator on X.
   - Equivalently,
     $$\|T\| = \sup\{|x^*(Tx)| : x\in S_X,\, x^*\in S_{X^*},\, x^*(x)=1\}$$
     for EVERY $T \in L(X)$.
3. **M.-Oikhberg, 2004:** X has the alternative Daugavet property (ADP) if every rank-one $T \in L(X)$ satisfies (aDE).
   - Then every weakly compact T also satisfies (aDE).

Relations between these properties



Examples

- $C\big([0,1],K(\ell_2)\big)$ has DPr, but has not numerical index 1
- c_0 has numerical index 1, but has not DPr
- $c_0 \oplus_\infty C\big([0,1],K(\ell_2)\big)$ has ADP, neither DPr nor numerical index 1

Remark

For RNP or Asplund spaces, $\boxed{\text{ADP}} \Longrightarrow \boxed{\text{numerical index } 1}$.

A sufficient condition for numerical index 1: lushness

Lushness (Boyko-Kadets-M.-Werner, 2007)

X is lush if given $x, y \in S_X$, $\varepsilon > 0$, there is $y^* \in S_{X^*}$ such that

$$x \in S = S(B_X, y^*, \varepsilon) \qquad \operatorname{dist}(y, \operatorname{conv}(\mathbb{T}S)) < \varepsilon.$$

Theorem (Boyko-Kadets-M.-Werner, 2007)

If X is lush, then X has numerical index 1

Example (Kadets-M.-Merí-Shepelska, 2009)

There is X with numerical index 1 which is not lush.

ADP + SCD $\Longrightarrow$ lushness

Characterization of ADP

X Banach space. TFAE:

- X has ADP (i.e. $\max_{\theta\in\mathbb{T}}\|\mathrm{Id}+\theta T\|=1+\|T\|$ for all T rank-one).
- Given $x\in S_X$, a slice S of B_X and $\varepsilon>0$, there is $y\in S$ with

$$\max_{\theta\in\mathbb{T}}\|x+\theta y\|>2-\varepsilon.$$

- Given $x\in S_X$, a sequence $\{S_n\}$ of slices of B_X, and $\varepsilon>0$, there is $y^*\in S_{X^*}$ such that $x\in S(B_X,y^*,\varepsilon)$ and

$$\overline{\mathrm{conv}}\big(\mathbb{T}\,S(B_X,y^*,\varepsilon)\big)\bigcap S_n\neq\emptyset \qquad (n\in\mathbb{N}).$$

Theorem

X ADP + B_X SCD $\Longrightarrow$ given $x\in S_X$ and $\varepsilon>0$, there is $y^*\in S_{X^*}$ such that

$$x\in S(B_X,y^*,\varepsilon) \quad \text{and} \quad B_X=\overline{\mathrm{conv}}\big(\mathbb{T}\,S(B_X,y^*,\varepsilon)\big).$$

- This clearly implies lushness, and so numerical index 1 (i.e. $\max_{\theta\in\mathbb{T}}\|\mathrm{Id}+\theta T\|=1+\|T\|$ for all T).

Some consequences

Corollary

$$\text{ADP} + \Big[\, X \not\supseteq \ell_1 \;\text{ OR }\; \text{strongly regular} \;\text{ OR }\; 1\text{-unconditional basis} \,\Big] \Longrightarrow \text{lushness (so numerical index } 1).$$

Corollary

X real $+ \dim(X) = \infty +$ ADP $\Longrightarrow X^* \supseteq \ell_1$.

In particular,

Corollary

X real $+ \dim(X) = \infty +$ numerical index $1 \Longrightarrow X^* \supseteq \ell_1$.

SCD operators

Section 4

SCD operators

SCD operator

$T \in L(X)$ is an SCD-operator if $T(B_X)$ is an SCD-set.

Examples

T is an SCD-operator when $T(B_X)$ is separable and

1. $T(B_X)$ is RNP,
2. $T(B_X)$ has no ℓ_1 sequences,
3. T does not fix copies of ℓ_1

Theorem

- X ADP + T SCD-operator $\implies$ $\max_{\theta\in\mathbb{T}} \|\mathrm{Id} + \theta T\| = 1 + \|T\|$.
- X DPr + T SCD-operator $\implies$ $\|\mathrm{Id} + T\| = 1 + \|T\|$.

Main corollary

X ADP + T does not fix copies of ℓ_1 $\implies$ $\max_{\theta\in\mathbb{T}} \|\mathrm{Id} + \theta T\| = 1 + \|T\|$.

HSCD-majorized operators (Kadets-Shepelska, 2010)

HSCD and HSDC-majorized operator

- $T \in L(X,Y)$ is an Hereditary-SCD-operator if every convex subset of $T(B_X)$ is an SCD-set.
- $T \in L(X,Y)$ is an HSCD-majorized operator if there is $S \in L(X,Z)$ HSCD-operator such that $\|Tx\| \leqslant \|Sx\|$ for every $x \in X$.

Theorem

X DPr + $T \in L(X)$ HSCD-majorized operator $\implies$ $\|\mathrm{Id} + T\| = 1 + \|T\|$.

Proposition

The class of HSCD-majorized operators is a two-sided operator ideal.

Remark

The class of operators satisfying (DE) is not even a subspace.

Open problems

Section 5

Open questions

1. Find more sufficient conditions for a set to be SCD.
2. Is SCD equivalent to the existence of a countable π-base for the weak topology **?**
3. E with (1)-unconditional basis. Is E SCD **?**
4. E with 1-unconditional basis, $\{X_n\}$ a family of SCD spaces.
   Is $[\oplus X_n]_E$ SCD **?**
5. X, Y SCD. Are $X \otimes_\varepsilon Y$ and $X \otimes_\pi Y$ SCD **?**
6. Find a good extension of the SCD property to the nonseparable case.
7. Clarify the relationship between SCD and the Daugavet property.
8. X ADP, $T \in L(X)$ HSCD-majorized, does T satisfies (aDE) **?**