

# Numerical indices and Daugavet-like properties in some function spaces

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Vladimir Kadets, Miguel Martín, Javier Merí and Dirk Werner,  
Lushness, numerical index one and the Daugavet property in  
rearrangement invariant spaces.  
*Canadian J. Math.* (to appear).



Han-Ju Lee and Miguel Martín,  
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with 1-unconditional bases.  
*Linear Algebra Appl.* (to appear).



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Polynomial numerical indices of Banach spaces with absolute norm.  
*Linear Algebra Appl.* (2011).

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  - The two main properties we are dealing with
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# *Introduction and preliminaries*

## Basic notation

$X$  real or complex Banach space.

- $S_X$  unit sphere
- $B_X$  closed unit ball
- $\mathbb{T}$  modulus-one scalars
- $X^*$  dual space
- $L(X)$  bounded linear operators from  $X$  to  $X$ .
- $\text{aconv}(\cdot)$  absolutely convex hull.

## The two main properties we are dealing with

## The Daugavet property (Kadets-Shvidkoy-Sirotkin-Werner, 1997)

$X$  has the **Daugavet property** if

$$\|\text{Id} + T\| = 1 + \|T\| \quad (\text{DE})$$

for rank-one operators  $T \in L(X)$ .

- Then every  $T \in L(X)$  not fixing copies of  $\ell_1$  also satisfies (DE).

## Banach spaces with numerical index one (Lumer, 1968)

$X$  has **numerical index one** if

$$\max_{\theta \in \mathbb{T}} \|\text{Id} + \theta T\| = 1 + \|T\| \quad (\text{aDE})$$

for EVERY operator  $T$  on  $X$ .

- Equivalently,

$$\|T\| = \sup\{|x^*(Tx)| : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}$$

for EVERY  $T \in L(X)$ .

# On the Daugavet property

## Examples

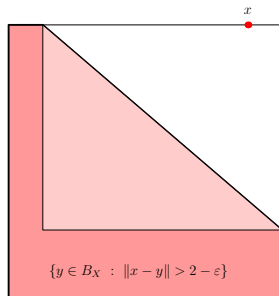
- ①  $C(K, E)$  when  $K$  is perfect
- ②  $L_1(\mu, E)$  and  $L_\infty(\mu, E)$  when  $\mu$  is atomless
- ③ the disk algebra  $A(\mathbb{D})$  and  $H^\infty$
- ④ function algebras with perfect Choquet boundary
- ⑤  $\text{Lip}(K)$  when  $K$  is a compact convex subset of  $\ell_p$
- ⑥ non-atomic  $C^*$ -algebras and preduals of non-atomic von Neumann algebras
- ⑦ some “big” subspaces of  $C[0, 1]$

## Characterization

$X$  has the Daugavet property iff

$$B_X = \overline{\text{co}} \left( \{y \in B_X : \|x - y\| \geq 2 - \varepsilon\} \right)$$

for every  $x \in S_X$  and every  $\varepsilon > 0$



## On the Daugavet property

## Some results

$X$  with the Daugavet property. Then:

- $X$  does not have the Radon-Nikodým property.
- Actually, every weakly-open subset of  $B_X$  has diameter 2.
- $X$  contains a copy of  $\ell_1$ .
- Actually, given  $x_0 \in S_X$  and slices  $\{S_n : n \geq 1\}$ , one may take  $x_n \in S_n$   $\forall n \geq 1$  such that  $\{x_n : n \geq 0\}$  is equivalent to the  $\ell_1$ -basis.
- $X^*$  contains a copy of  $L_1[0, 1]$ .
- $X$  does not have unconditional basis.
- $X$  does not embed into a unconditional sum of Banach spaces without a copy of  $\ell_1$ .



## On the numerical index one

## Examples

- ①  $L_1(\mu)$  and their isometric preduals
- ② so  $C(K)$  and  $L_\infty(\mu)$
- ③ the disk algebra  $A(\mathbb{D})$  and  $H^\infty$
- ④ all function algebras
- ⑤ some “big” subspaces of  $C[0,1]$
- ⑥ if  $X^*$  has numerical index one, so does  $X$
- ⑦ there is  $X$  with numerical index one whose dual does not have numerical index one
- ⑧  $c_0$ -,  $\ell_1$ -, and  $\ell_\infty$ -sums of spaces with numerical index one

## Characterization

We do not know of any operator-free characterization!!

## On the numerical index one

## Some results

$X$  with numerical index one,  $\dim(X) = \infty$ . Then:

- $X^*$  is not smooth and  $X^*$  is not strictly convex.
- In some particular cases, it is possible to prove that  $X$  is not smooth and that  $X$  is not strictly convex.
- Nevertheless, there is a strictly convex **non-complete**  $X$  such that  $X^* \equiv L_1(\mu)$  (and so  $X$  has numerical index one).
- In the real case,  $X^* \supseteq \ell_1$ .
- The norm of  $X$  cannot be Fréchet smooth.
- There are no LUR points in  $S_X$ .

## How to deal with numerical index one property?

## One the one hand: weaker properties

- In a general Banach space, we only can construct nuclear operators.
- Actually, we only may easily calculate the norm of **rank-one** operators.
- All the results about Banach spaces with numerical index one are actually proved for Banach spaces with the following property:

## The alternative Daugavet property (M.–Oikhberg, 2007)

A Banach space  $X$  has the **alternative Daugavet property (ADP)** if the norm equality

$$\max_{\theta \in \mathbb{T}} \|\text{Id} + \theta T\| = 1 + \|T\| \quad (\text{aDE})$$

holds for every for every RANK-ONE operator  $T \in L(X)$ .

- Then every  $T \in L(X)$  not fixing copies of  $\ell_1$  also satisfies (aDE).

## How to deal with numerical index one property?

## One the other hand: stronger properties

- When we know that a Banach space has numerical index one, we actually prove more.
- There are some sufficient geometrical conditions.
- The weakest property of this kind is the following:

## Lushness (Boyko–Kadets–M.–Werner, 2007)

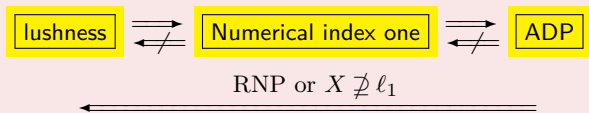
$X$  is **lush** if given  $x, y \in S_X$ ,  $\varepsilon > 0$ , there is  $x^* \in S_{X^*}$  such that

$$x \in S := \{z \in B_X : \operatorname{Re} x^*(z) > 1 - \varepsilon\} \quad \text{and} \quad \operatorname{dist}(y, \operatorname{aconv}(S)) < \varepsilon.$$

## How to deal with numerical index 1 property?

## Relationship between the properties

- One of the key ideas to get interesting results for Banach spaces with numerical index one is to study when one is able to pass from the weak property to the strong one.
- This happens, for instance, when  $X$  has RNP or  $X \not\cong \ell_1$ :



## Examples

- $C([0, 1], \ell_2)$  has ADP but not numerical index one
- there exists  $\mathcal{X}$  with numerical index one which is not lush

# What we are going to present

## Main objective

Determine which spaces have the Daugavet property or have numerical index one among Köthe sequence or function spaces.

We will give partial answers. . .

- **For sequence spaces:** we show which r.i. spaces have numerical index one and we show a results about spaces with polynomial numerical index one.
- **For function spaces:** we characterize separable r.i. spaces with the Daugavet property or which are lush.

# Sequence spaces

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## Definitions and remarks

## Definitions

- ① A **sequence space with absolute norm** is a Banach subspace  $X$  of  $\mathbb{K}^{\mathbb{N}}$  with
- if  $x, y \in \mathbb{K}^{\mathbb{N}}$  with  $|x| \leq |y|$  and  $y \in X$ , then  $x \in X$  with  $\|x\| \leq \|y\|$ ,
  - for every  $n \in \mathbb{N}$ ,  $e_n := \mathbf{1}_{\{n\}} \in X$  with  $\|e_n\| = 1$ .

In this case,  $\ell_1 \subset X \subset \ell_\infty$  with contractive inclusions.

- ② A sequence space with absolute norm  $X$  is a **rearrangement invariant (r.i.)** space if, in addition,
- for every bijection  $\tau : \mathbb{N} \rightarrow \mathbb{N}$  and every  $x \in X$ ,  $\|x \circ \tau\| = \|x\|$ .
  - the Köthe dual  $X'$  of  $X$  is norming.

## Remarks

- A **separable** sequence space with absolute norm is nothing than a Banach space with **1-unconditional basis**.
- A **separable** r.i. sequence space is nothing than a Banach space with **1-symmetric basis**.



## Numerical index one. I

## Theorem

$X$  separable r.i. sequence space ( $X$  Banach space with 1-symmetric basis).  
If  $X$  has numerical index one, then  $X$  is  $c_0$  or  $\ell_1$ .

## The ideas behind:

- ①  $X$  with 1-unconditional basis: the ADP, numerical index one and lushness are equivalent.
- ②  $X$  separable lush, then there is  $A \subset S_{X^*}$  norming such that  $|x^{**}(x^*)| = 1$  for every  $x^* \in A$  and every  $x^{**} \in \text{ext}(B_{X^{**}})$ .
- ③  $X$  separable sequence space and  $x' \in S_{X'}$  with  $|x^{**}(x')| = 1$  for every  $x^{**} \in \text{ext}(B_{X^{**}})$ . Then  $|x'(n)| \in \{0, 1\}$ .
- ④  $X$  separable r.i. with numerical index one. Two possibilities:
  - there is  $a' \in A$  in ② with  $\text{supp}(a')$  infinite  $\Rightarrow X = \ell_1$ ,
  - $\text{supp}(a')$  finite for every  $a' \in A$  in ②  $\Rightarrow X = c_0$ .

## Numerical index one. II

## Theorem

$X$  r.i. sequence space with numerical index one.

Then  $X = c_0$ ,  $X = \ell_1$ , or  $X = \ell_\infty$ .

The ideas behind:

- ①  $X$  r.i. with ADP (in particular with numerical index one), then  $Y = \text{lin}\{e_n : n \in \mathbb{N}\}$  has ADP.
- ② By the previous slide  $\Rightarrow Y$  has numerical index one  $\Rightarrow Y = c_0$  or  $Y = \ell_1$ .
- ③ If  $Y = \ell_1$  then  $X = \ell_1$ .
- ④ If  $Y = c_0$ , then  $X = c_0$  or  $X = \ell_\infty$ .

# Polynomial numerical index one

Polynomial numerical index of order 2 equal to one and the 2-ADP  
(Choi–García–Kim–Maestre, 2006; Choi–García–Maestre–M., 2007)

$X$  has **polynomial numerical index of order 2 equal to one** if the norm equality

$$\max_{\theta \in \mathbb{T}} \|\text{Id} + \theta P\| = 1 + \|P\| \quad (\text{aDE})$$

holds for every 2-homogeneous polynomial from  $X$  to  $X$   
(the norm in of the space of all polynomials).

- If every **rank-one** 2-homogeneous polynomial from  $X$  to  $X$  satisfies (aDE), we say that  $X$  has the **2-ADP**.

## Examples

- complex  $C_0(L)$  has polynomial numerical index of order 2 equal to one,
- complex  $C_0(L, E)$  has the 2-ADP if  $L$  is perfect,
- no real space of dimension greater than 1 is known to have the 2-ADP,
- the real or complex  $L_1(\mu)$  spaces do not have the 2-ADP.

## Polynomial numerical index one. II

## Theorem

- $c_0$  and  $\ell_\infty^m$  are the only complex Banach spaces with 1-unconditional basis which have polynomial numerical index of order 2 equal to one.
- Apart of  $\mathbb{R}$ , there is no real Banach space with 1-unconditional basis which has polynomial numerical index of order 2 equal to one.

## The ideas behind:

- 1  $X$  with 1-unconditional basis and polynomial numerical index of order 2 equal to one: this implies that  $X$  has numerical index one and so, [it is lush](#).
- 2 Then there is  $C \subset S_{X'}$  norming such that  $|x^{**}(x^*)| = 1$  for every  $x^* \in C$  and every  $x^{**} \in \text{ext}(B_{X^{**}})$ .
- 3 As previously, we get that for every  $x' \in C$ , one has  $|x'(n)| \in \{0, 1\}$ .
- 4 If  $\text{supp}(x')$  has more than one point for some  $x' \in C$ , we find a good copy of  $\ell_1^2$  in  $X$ .
- 5 Using that  $\ell_1^2$  does not have polynomial numerical index of order 2 equal to one, we get that every element in  $C$  has only one non-null coordinate.
- 6 This gives  $X = c_0$  or  $X = \ell_\infty^m$ . In the complex case, these spaces are possible. In the real case, they are not possible.

## Polynomial numerical index one. III

## Corollary

$X$  complex sequence space such that  $X'$  is norming for  $X$ , whose polynomial numerical index of order 2 is equal to one. Then  $c_0 \subset X \subset \ell_\infty$  isometrically.

The ideas behind:

- 1 Let  $E = \text{lin}\{e_n : n \in \mathbb{N}\}$ .
- 2 Using that  $X'$  is norming, we get that

$$E \subseteq X \subseteq E'' \subseteq E^{**}.$$

with equality of norms.

- 3 Using Aron-Berner extensions of polynomial, we get that  $E$  has the 2-ADP (i.e. rank-one 2-homogeneous polynomials satisfy (aDE)).
- 4 By the previous slice, we get  $E = c_0$  and so  $E'' = \ell_\infty$ .

## Conversely

If  $c_0 \subseteq X \subseteq \ell_\infty$  isometrically, then  $X$  has polynomial numerical index of order 2 equal to one.

# *Function spaces*

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## Definition and remark

## Definition

A (separable) **rearrangement invariant space** on  $[0, 1]$  is a separable Banach space  $X$  consisting on equivalence classes of locally integrable scalar functions on  $[0, 1]$  satisfying

- (a) if  $|f| \leq |g|$  a.e. with  $f$  measurable and  $g \in X \implies f \in X$  and  $\|f\| \leq \|g\|$ .
- (b) the Köthe dual  $X'$  of  $X$  coincides with  $X^*$
- (c) as sets,  $L_\infty[0, 1] \subset X \subset L_1[0, 1]$  with contractive inclusions.
- (d) if  $\tau : [0, 1] \rightarrow [0, 1]$  is a measure preserving bijection and  $f$  is a measurable function, then

$$f \in X \Leftrightarrow f \circ \tau \in X \quad \text{and, in this case, } \|f\| = \|f \circ \tau\|$$

## Examples

- ①  $L_p[0, 1]$  spaces for  $1 \leq p < \infty$
- ② separable Lorentz spaces
- ③ separable Orlicz spaces

## Theorem

The only separable r.i. lush space is  $L_1[0,1]$ .

The ideas behind:

- ①  $X$  separable lush, then there is  $A \subset S_{X^*}$  norming such that  $|x^{**}(x^*)| = 1$  for every  $x^* \in A$  and every  $x^{**} \in \text{ext}(B_{X^{**}})$ .
- ② Prove that every  $g \in A$  satisfies  $|g| = 1$ .

③ Then,

$$\|x\|_1 \leq \|x\|_X = \sup_{g \in A} \left| \int_0^1 x(t)g(t) dt \right| \leq \|x\|_1$$

for every  $x \in X$ .

- ④ This gives that  $X = L_1[0,1]$  with equality of norms.



## Daugavet property

## Theorem

The only separable **real** r.i. space with the Daugavet property is  $L_1[0,1]$ .

- 1 The proof is rather technical.
- 2 It is only valid in the real case.
- 3 The same proof also gives the following result:

## An stronger result

$L_1[0,1]$  is the only separable real r.i. space in which the norm equality

$$\|\text{Id} - P\| \geq 2$$

holds for every rank-one projection  $P$ .

# *Open problems*

## Open problems

## Problem 1

Is  $L_\infty[0,1]$  the unique non-separable r.i. space with the Daugavet property or which is lush?

## Problem 2

Are  $L_1[0,1]$  and  $L_\infty[0,1]$  the unique r.i. spaces with numerical index one?

## Problem 3

Are  $L_1[0,1]$  and  $L_\infty[0,1]$  the unique r.i. spaces with the ADP?

## Problem 4

- Are the ADP, numerical index one and lushness equivalent for Köthe spaces?
- Are the ADP and the Daugavet property equivalent for Köthe spaces on  $[0,1]$ ?