Daugavet-like properties and numerical indices in some function spaces

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The talk is based on the papers

Vladimir Kadets, Miguel Martín, Javier Merí and Dirk Werner,
Lushness, numerical index one and the Daugavet property in rearrangement invariant spaces.
Canadian J. Math. (to appear).

Han-Ju Lee and Miguel Martín,
Polynomial numerical indices of Banach spaces with 1-unconditional bases.
Linear Algebra Appl. (2012).

Han-Ju Lee, Miguel Martín and Javier Merí,
Polynomial numerical indices of Banach spaces with absolute norm.
Linear Algebra Appl. (2011).

Sketch of the talk

1. Introduction and preliminaries
   - Notation
   - The two main properties we are dealing with
2. Sequence spaces
   - Definitions
   - Numerical index one
   - Polynomial numerical index one
3. Function spaces
   - Definitions
   - Lush spaces
   - Daugavet property
4. Open problems

Introduction and preliminaries

Basic notation

Basic notation

X real or complex Banach space.

- S_X unit sphere
- B_X closed unit ball
- $\mathbb{T}$ modulus-one scalars
- X^* dual space
- $L(X)$ bounded linear operators from X to X.
- $\mathrm{aconv}(\cdot)$ absolutely convex hull.

The two main properties we are dealing with

The Daugavet property (Kadets-Shvidkoy-Sirotkin-Werner, 1997 - 2000)

X has the Daugavet property if

$$\|\mathrm{Id} + T\| = 1 + \|T\| \qquad \text{(DE)}$$

for rank-one operators $T \in L(X)$.

- Then every $T \in L(X)$ not fixing copies of ℓ_1 also satisfies (DE).

Banach spaces with numerical index one (Lumer, 1968)

X has numerical index one if

$$\max_{\theta \in \mathbb{T}} \|\mathrm{Id} + \theta T\| = 1 + \|T\| \qquad \text{(aDE)}$$

for EVERY operator T on X.

- Equivalently,

$$\|T\| = \sup\{|x^*(Tx)| : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}$$

for EVERY $T \in L(X)$.

On the Daugavet property

Examples

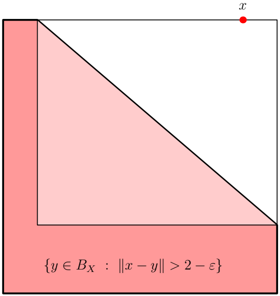
1. $C(K,E)$ when K is perfect
2. $L_1(\mu,E)$ and $L_\infty(\mu,E)$ when μ is atomless
3. the disk algebra $A(\mathbb{D})$ and H^∞
4. function algebras with perfect Choquet boundary
5. $\mathrm{Lip}(K)$ when K is a compact convex subset of ℓ_p
6. non-atomic C^*-algebras and preduals of non-atomic von Neumann algebras
7. some "big" subspaces of $C[0,1]$

Characterization

X has the Daugavet property iff

$$B_X = \overline{\mathrm{co}}\Big(\{y\in B_X \ :\ \|x-y\|\geqslant 2-\varepsilon\}\Big)$$

for every $x\in S_X$ and every $\varepsilon>0$



On the Daugavet property

Some results

X with the Daugavet property. Then:

- Every weakly-open subset of B_X has diameter 2.
- X contains a copy of ℓ_1.
- Actually, given $x_0 \in S_X$ and slices $\{S_n : n \geqslant 1\}$, one may take $x_n \in S_n$ $\forall n \geqslant 1$ such that $\{x_n : n \geqslant 0\}$ is equivalent to the ℓ_1-basis.
- X does not have unconditional basis.

This follows from the following characterization:

Characterization

X has the Daugavet property iff for every $x \in S_X$, $x^* \in S_{X^*}$ and $\varepsilon > 0$, there exists $y \in B_X$ such that

$$\|x+y\| \geqslant 2-\varepsilon \qquad \text{and} \qquad \operatorname{Re} x^*(y) > 1-\varepsilon.$$

On the numerical index one

Examples

1. $L_1(\mu)$ and their isometric preduals
2. so $C(K)$ and $L_\infty(\mu)$
3. the disk algebra $A(\mathbb{D})$ and H^∞
4. all function algebras
5. some "big" subspaces of $C[0,1]$
6. if X^* has numerical index one, so does X
7. there is X with numerical index one whose dual does not have numerical index one
8. c_0-, ℓ_1-, and ℓ_∞-sums of spaces with numerical index one

Characterization

We do not know of any operator-free characterization!!

On the numerical index one

Some results

X with numerical index one, $\dim(X) = \infty$. Then:

- X^* is not smooth and X^* is not strictly convex.
- In some particular cases, it is possible to prove that X is not smooth and that X is not strictly convex.
- Nevertheless, there is a strictly convex **non-complete** X such that $X^* \equiv L_1(\mu)$ (and so X has numerical index one).
- In the real case, $X^* \supseteq \ell_1$.
- The norm of X cannot be Fréchet smooth.
- There are no LUR points in S_X.

How to deal with numerical index one property?

One the one hand: weaker properties

- In a general Banach space, we only can construct nuclear operators.
- Actually, we only may easily calculate the norm of rank-one operators.
- All the results about Banach spaces with numerical index one are actually proved for Banach spaces with the following property:

The alternative Daugavet property (M.–Oikhberg, 2007)

A Banach space X has the alternative Daugavet property (ADP) if the norm equality

$$\max_{\theta\in\mathbb{T}} \|\mathrm{Id} + \theta T\| = 1 + \|T\| \qquad \text{(aDE)}$$

holds for every for every RANK-ONE operator $T \in L(X)$.

- Then every $T \in L(X)$ not fixing copies of ℓ_1 also satisfies (aDE).

How to deal with numerical index one property?

One the other hand: stronger properties

- When we know that a Banach space has numerical index one, we actually prove more.
- There are some sufficient geometrical conditions.
- The weakest property of this kind is the following:

Lushness (Boyko–Kadets–M.–Werner, 2007)

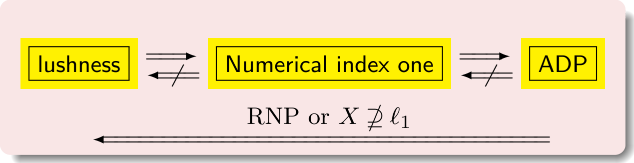
X is lush if given $x, y \in S_X$, $\varepsilon > 0$, there is $x^* \in S_{X^*}$ such that

$$x \in S := \{z \in B_X : \operatorname{Re} x^*(z) > 1 - \varepsilon\} \quad \text{and} \quad \operatorname{dist}(y, \operatorname{aconv}(S)) < \varepsilon.$$

How to deal with numerical index 1 property?

Relationship between the properties

- One of the key ideas to get interesting results for Banach spaces with numerical index one is to study when one is able to pass from the weak property to the strong one.
- This happens, for instance, when X has RNP or $X \not\supseteq \ell_1$:



Examples

- $C\big([0,1],\ell_2\big)$ has ADP but not numerical index one
- there exists $\mathcal{X}$ with numerical index one which is not lush

What we are going to present

Main objective

Determine which spaces have the Daugavet property or have numerical index one among Köthe sequence or function spaces.

We will give partial answers. . .

- For sequence spaces: we show which r.i. spaces have numerical index one and we show a results about spaces with polynomial numerical index one.
- For function spaces: we characterize separable r.i. spaces with the Daugavet property or which are lush.

Sequence spaces

1 Introduction and preliminaries

2 Sequence spaces
- Definitions
- Numerical index one
- Polynomial numerical index one

3 Function spaces

4 Open problems

Definitions and remarks

Definitions

1. A sequence space with absolute norm is a Banach subspace X of $\mathbb{K}^{\mathbb{N}}$ with
   - if $x, y \in \mathbb{K}^{\mathbb{N}}$ with $|x| \leqslant |y|$ and $y \in X$, then $x \in X$ with $\|x\| \leqslant \|y\|$,
   - for every $n \in \mathbb{N}$, $e_n := \mathbf{1}_{\{n\}} \in X$ with $\|e_n\| = 1$.

   In this case, $\ell_1 \subset X \subset \ell_\infty$ with contractive inclusions.
2. A sequence space with absolute norm X is a rearrangement invariant (r.i.) space if, in addition,
   - for every bijection $\tau : \mathbb{N} \to \mathbb{N}$ and every $x \in X$, $\|x \circ \tau\| = \|x\|$.
   - the Köthe dual X' of X is norming.

Remarks

- A separable sequence space with absolute norm is nothing than a Banach space with 1-unconditional basis.
- A separable r.i. sequence space is nothing than a Banach space with 1-symmetric basis.

Numerical index one. I

Theorem

X separable r.i. sequence space (X Banach space with 1-symmetric basis).
If X has numerical index one, then X is c_0 or ℓ_1.

The ideas behind:

1. X with 1-unconditional basis: the ADP, numerical index one and lushness are equivalent.
2. X separable lush, then there is $A \subset S_{X^*}$ norming such that $|x^{**}(x^*)| = 1$ for every $x^* \in A$ and every $x^{**} \in \operatorname{ext}(B_{X^{**}})$.
3. X separable sequence space and $x' \in S_{X'}$ with $|x^{**}(x')| = 1$ for every $x^{**} \in \operatorname{ext}(B_{X^{**}})$. Then $|x'(n)| \in \{0, 1\}$.
4. X separable r.i. with numerical index one. Two possibilities:
   - there is $a' \in A$ in 2 with $\operatorname{supp}(a')$ infinite $\Rightarrow$ $X = \ell_1$,
   - $\operatorname{supp}(a')$ finite for every $a' \in A$ in 2 $\Rightarrow$ $X = c_0$.

Numerical index one. II

Theorem

X r.i. sequence space with numerical index one.
Then $X = c_0$, $X = \ell_1$, or $X = \ell_\infty$.

The ideas behind:

1. X r.i. with ADP (in particular with numerical index one), then $Y = \mathrm{lin}\{e_n : n \in \mathbb{N}\}$ has ADP.
2. By the previous slide $\Rightarrow$ Y has numerical index one $\Rightarrow$ $Y = c_0$ or $Y = \ell_1$.
3. If $Y = \ell_1$ then $X = \ell_1$.
4. If $Y = c_0$, then $X = c_0$ or $X = \ell_\infty$.

Polynomial numerical index one

Polynomial numerical index of order 2 equal to one and the 2-ADP (Choi–Garcia–Kim–Maestre, 2006; Choi–Garcia–Maestre–M., 2007)

X has polynomial numerical index of order 2 equal to one if the norm equality

$$\max_{\theta\in\mathbb{T}} \|\mathrm{Id} + \theta\, P\| = 1 + \|P\| \qquad \text{(aDE)}$$

holds for every 2-homogeneous polynomial from X to X
(the norm in of the space of all polynomials).

- If every **rank-one** 2-homogeneous polynomial from X to X satisfies (aDE), we say that X has the 2-ADP.

Examples

- complex $C_0(L)$ has polynomial numerical index of order 2 equal to one,
- complex $C_0(L,E)$ has the 2-ADP if L is perfect,
- no real space of dimension greater than 1 is known to have the 2-ADP,
- the real or complex $L_1(\mu)$ spaces do not have the 2-ADP.

Polynomial numerical index one. II

Theorem

- c_0 and ℓ_∞^m are the only complex Banach spaces with 1-unconditional basis which have polynomial numerical index of order 2 equal to one.
- Apart of $\mathbb{R}$, there is no real Banach space with 1-unconditional basis which has polynomial numerical index of order 2 equal to one.

The ideas behind:

1. X with 1-unconditional basis and polynomial numerical index of order 2 equal to one: this implies that X has numerical index one and so, it is lush.
2. Then there is $C \subset S_{X'}$ norming such that $|x^{**}(x^*)| = 1$ for every $x^* \in C$ and every $x^{**} \in \operatorname{ext}(B_{X^{**}})$.
3. As previously, we get that for every $x' \in C$, one has $|x'(n)| \in \{0, 1\}$.
4. If $\operatorname{supp}(x')$ has more than one point for some $x' \in C$, we find a good copy of ℓ_1^2 in X.
5. Using that ℓ_1^2 does not have polynomial numerical index of order 2 equal to one, we get that every element in C has only one non-null coordinate.
6. This gives $X = c_0$ or $X = \ell_\infty^m$. In the complex case, these spaces are possible. In the real case, they are not possible.

Polynomial numerical index one. III

Corollary

X complex sequence space such that X' is norming for X, whose polynomial numerical index of order 2 is equal to one. Then $c_0 \subset X \subset \ell_\infty$ isometrically.

The ideas behind:

1. Let $E = \mathrm{lin}\{e_n : n \in \mathbb{N}\}$.
2. Using that X' is norming, we get that
$$E \subseteq X \subseteq E'' \subseteq E^{**}.$$
with equality of norms.
3. Using Aron-Berner extensions of polynomial, we get that E has the 2-ADP (i.e. rank-one 2-homogeneous polynomials satisfy (aDE)).
4. By the previous slice, we get $E = c_0$ and so $E'' = \ell_\infty$.

Conversely

If $c_0 \subseteq X \subseteq \ell_\infty$ isometrically, then X has polynomial numerical index of order 2 equal to one.

Function spaces

1 Introduction and preliminaries

2 Sequence spaces

3 Function spaces
- Definitions
- Lush spaces
- Daugavet property

4 Open problems

Definition and remark

Definition

A (separable) rearrangement invariant space on $[0,1]$ is a separable Banach space X consisting on equivalence classes of locally integrable scalar functions on $[0,1]$ satisfying

(a) if $|f| \leqslant |g|$ a.e. with f measurable and $g \in X \implies f \in X$ and $\|f\| \leqslant \|g\|$.

(b) the Köthe dual X' of X coincides with X^*

(c) as sets, $L_\infty[0,1] \subset X \subset L_1[0,1]$ with contractive inclusions.

(d) if $\tau : [0,1] \longrightarrow [0,1]$ is a measure preserving bijection and f is a measurable function, then

$$f \in X \ \Leftrightarrow \ f \circ \tau \in X \quad \text{and, in this case, } \|f\| = \|f \circ \tau\|$$

Examples

1. $L_p[0,1]$ spaces for $1 \leqslant p < \infty$
2. separable Lorentz spaces
3. separable Orlitz spaces

Lush spaces

Theorem

The only separable r.i. lush space is $L_1[0,1]$.

The ideas behind:

1. X separable lush, then there is $A \subset S_{X^*}$ norming such that $|x^{**}(g)| = 1$ for every $g \in A$ and every $x^{**} \in \operatorname{ext}(B_{X^{**}})$.
2. Key technical lemma: If $g \in A$, then $|g|$ is constant; hence $|g| = 1$.
3. Then,
$$\|x\|_1 \leqslant \|x\|_X = \sup_{g \in A} \left| \int_0^1 x(t) g(t)\, dt \right| \leqslant \|x\|_1$$
for every $x \in X$.
4. This gives that $X = L_1[0,1]$ with equality of norms.

Daugavet property

Theorem

The only separable real r.i. space with the Daugavet property is $L_1[0,1]$.

1. Remark: M. Acosta, A. Kamińska and M. Mastyło proved in 2009 under additional hypotheses that X is isomorphic to $L_1[0,1]$.
   The proof is rather technical.
2. It is only valid in the real case.
3. The same proof also gives the following result:

An stronger result

$L_1[0,1]$ is the only separable real r.i. space in which the norm equality

$$\|\mathrm{Id} - P\| \geqslant 2$$

holds for every rank-one projection P.

Let us give the proof of this result:

How to prove that $X = L_1$?

The fundamental function of X is defined by

$$\phi(t) = \|\mathbf{1}_{[0,t]}\|_X.$$

One always has

$$t \leqslant \phi(t) \leqslant 1.$$

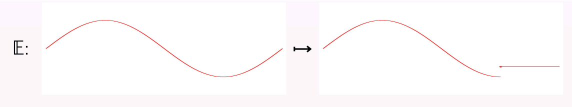
Lemma

Let X be an r.i. space on $[0,1]$. Then TFAE:

- $X = L_1$ with equality of norms.
- $\phi(t) = t$ for all t.
- $\displaystyle\lim_{t\to 0} \frac{\phi(t)}{t} = 1.$

Conditional expectations

The (simplest) conditional expectation operator $\mathbb{E}$ averages on a subset $A \subset [0,1]$:



Lemma

$$\|\mathbb{E}g\|_X \leqslant \|g\|_X \text{ for all } g.$$

Corollary

For $t \geqslant \mu(\mathrm{supp}(g))$

$$\frac{\phi(t)}{t}\|g\|_{L_1} \leqslant \|g\|_X.$$

So it remains to find $g \in X$ with small support and $\|g\|_X \approx \|g\|_{L_1} \approx 1$ in order to prove that $X = L_1$!

Sketch of proof of the Theorem

Recall geometric characterisation: If X has the Daugavet property, then for each $f_0 \in S_X$, $\ell_0 \in S_{X^*}$ and $\varepsilon > 0$ there is $f \in X$ with

- $\|f\|_X \leqslant 1$,
- $\|f_0 + f\|_X \geqslant 2 - \varepsilon$,
- $\ell_0(f) \geqslant 1 - \varepsilon$.

Here choose $f_0 = \mathbf{1}$ and $\ell_0 = -\int$; hence there exists $f \in X$ with

- $\|f\|_X \leqslant 1$,
- $\|\mathbf{1} + f\|_X \geqslant 2 - \varepsilon$,
- $\int_0^1 f(t)\,dt \leqslant -1 + \varepsilon$.

Sketch of proof of the Theorem (cont'd)

Decompose f as follows: Let $A = \{f \leqslant -2\}$, $B = \{f > -2\}$ so that $f = f\mathbf{1}_A + f\mathbf{1}_B$.

Key technical estimate

$\mu(A)$ is small and $\int_A |f(t)|\,dt \approx 1$ when ε becomes small.

Consequently, for $t = \mu(A)$ and $g = f\mathbf{1}_A$:

$$1 \approx \|g\|_{L_1} \leqslant \frac{\phi(t)}{t}\|g\|_{L_1} \leqslant \|g\|_X \leqslant \|f\|_X \leqslant 1,$$

which implies that

$$\lim_{t\to 0} \frac{\phi(t)}{t} = 1,$$

and $X = L_1$.

Open problems

Open problems

Problem 1

Is $L_\infty[0,1]$ the unique non-separable r.i. space with the Daugavet property or which is lush?

Problem 2

Are $L_1[0,1]$ and $L_\infty[0,1]$ the unique r.i. spaces with numerical index one?

Problem 3

Are $L_1[0,1]$ and $L_\infty[0,1]$ the unique r.i. spaces with the ADP?

Problem 4

- Are the ADP, numerical index one and lushness equivalent for Köthe spaces?
- Are the ADP and the Daugavet property equivalent for Köthe spaces on $[0,1]$?