The Bishop-Phelps-Bollobás modulus of a Banach space

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The coauthors

The talk is based on the following paper

M. Chica, V. Kadets, M. Martín, S. Moreno, F. Rambla
The Bishop-Phelps-Bollobás modulus of a Banach space
In preparation

Outline of the talk

1. Introduction
   - Notation
   - The starting point
2. Definition and first properties
   - Definition
   - The upper bound of the modulus
   - Some properties
3. Examples
4. Spaces with the greatest possible value of the modulus
5. Open problems

Introduction

Section 1

Basic notation

Basic notation

X real or complex Banach space.

- S_X unit sphere
- B_X closed unit ball
- $X^\star$ dual space
- An element $f \in X^\star$ attains its norm if

$$\|f\| = \max\left\{|f(x)| \,:\, x \in B_X\right\},$$

that is, there is $x_0 \in B_X$ such that $\|f\| = |f(x_0)|$.

- The above is equivalent to say that $\operatorname{Re} f$ is a supporting functional of B_X at x_0.
- $\Pi(X) := \left\{(x, x^*) \in S_X \times S_{X^*} \,:\, x^*(x) = 1\right\}$

Three theorems and one definition

James, 1957

Let X be a Banach space. Then

$$X \text{ is reflexive} \iff \text{every element of } X^\star \text{ attains its norm}$$

Bishop-Phelps, 1961

The set of norm-attaining functionals on a Banach space X is dense in $X^\star$.

Bollobás, 1970 (known as Bishop-Phelps-Bollobás theorem)

Let X be a Banach space. Suppose $x \in S_X$ and $x^* \in S_{X^\star}$ satisfy

$$|1 - x^*(x)| \leqslant \varepsilon^2/2 \qquad (0 < \varepsilon < 1/2).$$

Then there exists $(y, y^*) \in \Pi(X)$ (i.e. $y^*(y) = 1$) such that

$$\|x - y\| < \varepsilon + \varepsilon^2 \quad \text{and} \quad \|x^* - y^*\| \leqslant \varepsilon.$$

Three theorems and one definition

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Our idea

- Can the result bellow be improved for concrete Banach spaces?
- That is, for a Banach space X, we want to quantify how good or bad is the approximation in Bollobás' theorem:

Bishop-Phelps, 1961

The set of norm-attaining functionals on a Banach space X is dense in $X^\star$.

Three theorems and one definition

James, 1957

Let X be a Banach space. Then

$$X \text{ is reflexive} \iff \text{every element of } X^\star \text{ attains its norm}$$

Bishop-Phelps, 1961

The set of norm-attaining functionals on a Banach space X is dense in $X^\star$.

Bishop-Phelps-Bollobás modulus

Let X be a Banach space. For every $\delta \in (0,2)$ find the **smaller** $\varepsilon > 0$ such that whenever $x \in B_X$ and $x^* \in B_{X^\star}$ satisfy

$$\operatorname{Re} x^*(x) > 1 - \delta,$$

there exists $(y, y^*) \in \Pi(X)$ (i.e. $y^*(y) = 1$) such that

$$\|x - y\| < \varepsilon \quad \text{and} \quad \|x^* - y^*\| < \varepsilon.$$

Definition and first properties

Section 2

Definition of the Bishop-Phelps-Bollobás modulus

Bishop-Phelps-Bollobás modulus of a Banach space X

It is the function $\Phi_X : (0,2) \longrightarrow \mathbb{R}$ defined as

$$\Phi_X(\delta) := \inf\Big\{\varepsilon > 0 : \forall (x,x^*) \in B_X \times B_{X^*} \text{ with } \operatorname{Re} x^*(x) > 1-\delta, \\ \exists (y,y^*) \in \Pi(X) \text{ with } \|x-y\| < \varepsilon \text{ and } \|x^*-y^*\| < \varepsilon\Big\}$$

- In other words: if for $\delta \in (0,2)$ we write

$$A_X(\delta) := \Big\{(x,x^*) \in B_X \times B_{X^\star} : \operatorname{Re} x^*(x) > 1-\delta\Big\},$$

it is clear that

$$\Phi_X(\delta) = \sup_{(x,x^*)\in A_X(\delta)} \ \inf_{(y,y^*)\in\Pi(X)} \max\{\|x-y\|, \|x^*-y^*\|\}.$$

- Therefore,

$$\Phi_X(\delta) = d_H\big(A_X(\delta), \Pi(X)\big) \qquad (0<\delta<2)$$

where d_H is the Hausdorff distance in $X \oplus_\infty X^\star$.

A remark

$$\begin{aligned}\Phi_X(\delta) = \inf\Big\{\varepsilon > 0 : \forall (x,x^*) \in B_X \times B_{X^*} \text{ with } \operatorname{Re} x^*(x) > 1-\delta,\\ \exists (y,y^*) \in \Pi(X) \text{ with } \|x-y\| < \varepsilon \text{ and } \|x^*-y^*\| < \varepsilon\Big\}\end{aligned}$$

$$\begin{aligned}= \inf\Big\{\varepsilon > 0 : \forall (x,x^*) \in B_X \times B_{X^*} \text{ with } \operatorname{Re} x^*(x) \geqslant 1-\delta,\\ \exists (y,y^*) \in \Pi(X) \text{ with } \|x-y\| < \varepsilon \text{ and } \|x^*-y^*\| < \varepsilon\Big\}\end{aligned}$$

$$\begin{aligned}= \inf\Big\{\varepsilon > 0 : \forall (x,x^*) \in B_X \times B_{X^*} \text{ with } \operatorname{Re} x^*(x) > 1-\delta,\\ \exists (y,y^*) \in \Pi(X) \text{ with } \|x-y\| \leqslant \varepsilon \text{ and } \|x^*-y^*\| \leqslant \varepsilon\Big\}\end{aligned}$$

$$\begin{aligned}= \inf\Big\{\varepsilon > 0 : \forall (x,x^*) \in B_X \times B_{X^*} \text{ with } \operatorname{Re} x^*(x) \geqslant 1-\delta,\\ \exists (y,y^*) \in \Pi(X) \text{ with } \|x-y\| \leqslant \varepsilon \text{ and } \|x^*-y^*\| \leqslant \varepsilon\Big\}\end{aligned}$$

Three observations

Observation 1

$\Phi_X(\delta)$ is increasing in δ.

observation 2

As a consequence of the Bishop-Phelps-Bollobás theorem, we have

$$\lim_{\delta \downarrow 0} \Phi_X(\delta) = 0$$

Observation 3

The smaller is $\Phi_X(\cdot)$, the better is the approximation in the space X.

The upper bound of the modulus

Theorem

For every Banach space X and every $\delta \in (0,2)$,

$$\Phi_X(\delta) \leqslant \sqrt{2\delta}$$

Some coments:

- We prove the result using a lemma by Phelps from 1974.
- Most of the technical main difficulties come from the fact that we approximate elements from B_X and functional from $B_{X^\star}$.
- But, on the other hand, this gives a slightly improved version of Bollobás theorem:

The Bishop-Phelps-Bollobás revisited

Corollary

Let X be a Banach space.

- Let $0 < \varepsilon < 2$ and suppose that $x \in B_X$ and $x^* \in B_{X^*}$ satisfy
$$\operatorname{Re} x^*(x) > 1 - \varepsilon^2/2.$$
Then, there exists $(y, y^*) \in \Pi(X)$ such that
$$\|x - y\| < \varepsilon \quad \text{and} \quad \|x^* - y^*\| < \varepsilon.$$
- Let $0 < \delta < 2$ and suppose that $x \in B_X$ and $x^* \in B_{X^*}$ satisfy
$$\operatorname{Re} x^*(x) > 1 - \delta.$$
Then, there exists $(y, y^*) \in \Pi(X)$ such that
$$\|x - y\| < \sqrt{2\delta} \quad \text{and} \quad \|x^* - y^*\| < \sqrt{2\delta}.$$

Some properties

Proposition

The function $\delta \longmapsto \Phi_X(\delta)$ is continuous in $(0,2)$

Proposition

$$\Phi_X(\delta) \leqslant \Phi_{X^*}(\delta)$$

- We do not know whether equality holds or not

Corollary

If X is reflexive, then $\Phi_X(\delta) = \Phi_{X^*}(\delta)$.

Examples

Section 3

The one dimensional case

Example

$$\Phi_{\mathbb{R}}(\delta) = \begin{cases} \delta & \text{if } 0 < \delta \leqslant 1 \\ \sqrt{\delta - 1} + 1 & \text{if } 1 < \delta < 2 \end{cases}$$

Hilbert spaces

Example

Let H be a Hilbert space, $\dim(H) > 1$,

$$\Phi_H(\delta) \leqslant \sqrt{\delta} \quad \text{for } 0 < \delta < 2,$$
$$\Phi_H(\delta) = \sqrt{\delta} \quad \text{for } 1 \leqslant \delta < 2$$

Catching the maximum value of the modulus

Proposition

Suppose $X = Y \oplus_1 Z$. Then

$$\Phi_X(\delta) = \sqrt{2\delta} \qquad (0 < \delta < 1/2)$$

Proposition

Suppose $X = Y \oplus_\infty Z$. Then

$$\Phi_X(\delta) = \sqrt{2\delta} \qquad (0 < \delta < 1/2)$$

Examples

$$\Phi_X(\delta) = \sqrt{2\delta} \qquad (0 < \delta < 1/2)$$

for X equals c_0, ℓ_1, ℓ_∞, $L_1[0,1]$, $L_\infty[0,1]\ldots$

Catching the maximum value of the modulus II

Proposition

Suppose $X^\star = Y \oplus_1 Z$ and Y, Z are NOT w^*-dense in $X^\star$. Then

$$\Phi_X(\delta) = \sqrt{2\delta} \qquad (0 < \delta < 1/2)$$

Corollary

Suppose X contains two M-ideals J_1 and J_2 with $J_1 \cap J_2 = \{0\}$. Then

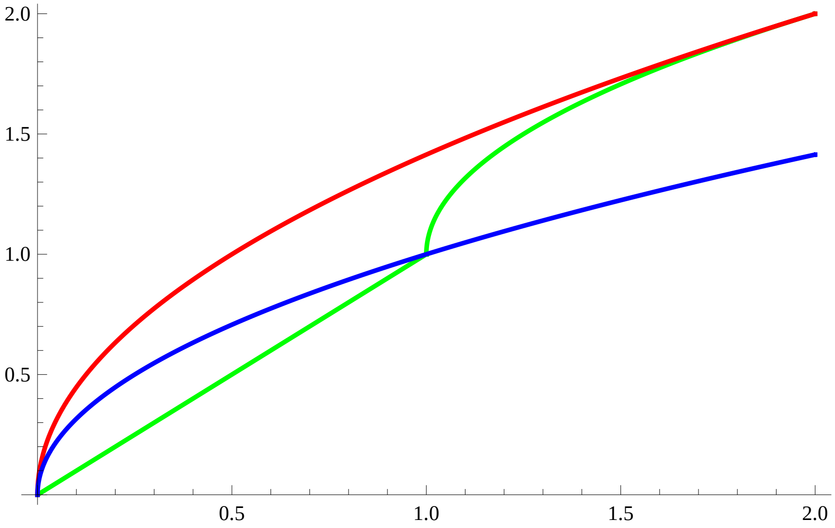
$$\Phi_X(\delta) = \sqrt{2\delta} \qquad (0 < \delta < 1/2)$$

Examples

$$\Phi_X(\delta) = \sqrt{2\delta} \qquad (0 < \delta < 1/2)$$

for X equals $C[0,1]$, $C_0(\mathbb{R})$, $C_b(\mathbb{R}^N)$...

A picture of the values of the modulus for some examples



Spaces with the greatest possible value of the modulus

Section 4

A necessary condition...

Theorem

Let X be a Banach space. Suppose there is $\delta_0 \in (0,2)$ such that $\Phi_X(\delta_0) = \sqrt{2\delta_0}$. Then $X^\star$ contains an almost isometric copy of the real two-dimensional ℓ_∞.

Some comments:

- What we show: $\forall \varepsilon > 0$, $\exists x_\varepsilon^*, y_\varepsilon^* \in S_{X^\star}$ with

$$\|x_\varepsilon^* + y_\varepsilon^*\| = 2 \quad \text{and} \quad \|x_\varepsilon^* - y_\varepsilon^*\| \geqslant 2 - \varepsilon.$$

- The proof is rather technical. It is actually an analysis of techniques used in the proof of the Bishop-Phelps theorem, but studying what happens when they give the "worst" possible value.
- In the complex case, it is not possible to get an almost isometric copy of either ℓ_1^2 or ℓ_∞^2, since they are not isometric and both have the greatest possible Bishop-Phelps-Bollobás modulus.

. . . which is not sufficient

Example

There is a real three-dimensional space X whose dual contains an isometric copy of the two-dimensional ℓ_∞ space, but for which

$$\Phi_X(\delta) < \sqrt{2\delta} \text{ for every } \delta \in (0,2).$$

Open problems

Section 5

Open problems

Problem 1

Is $\Phi_X(\delta)$ equal to $\Phi_{X^*}(\delta)$ for every Banach space **?**

Problem 2

Calculate $\Phi_H(\delta)$ for a Hilbert space H of dimension greater than one.
In particular, is $\Phi_H(\delta) = \sqrt{\delta}$ **?**

Problem 3

Is $\Phi_X(\delta) \geqslant \sqrt{\delta}$ when $\dim(X) \geqslant 2$ **?**

Problem 4

Characterize those Banach spaces for which $\Phi_X(\delta) = \sqrt{2\delta}$.