Slicely Countably Determined Banach spaces

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Introduction

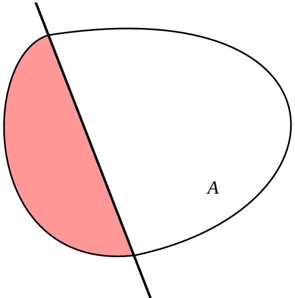
Basic notation

Basic notation

X real or complex Banach space.

- S_X unit sphere, B_X closed unit ball, $\mathbb{T}$ modulus-one scalars.
- X^* dual space, $L(X)$ bounded linear operators from X to X.
- $\operatorname{conv}(\cdot)$ convex hull, $\overline{\operatorname{conv}}(\cdot)$ closed convex hull,
- A slice of $A \subset X$ is a (nonempty) subset of the form

$$S(A, x^*, \alpha) = \{x \in A : \operatorname{Re} x^*(x) > \sup \operatorname{Re} x^*(A) - \alpha\} \quad (x^* \in X^*,\ \alpha > 0)$$



Two classical concepts: Radon-Nikodým property and Asplund spaces

The Radon-Nikodým property or RNP (1930's)

- X has the RNP iff the Radon-Nikodým theorem is valid for X-valued meassures;
- Equivalently [1960's], every bcc subset contains a denting point (i.e. a point belonging to slices of arbitrarily small diameter).

X Asplund $\Longleftrightarrow X^*$ RNP

Reflexive (say) $\Longrightarrow$ RNP and Asplund

RNP or Asplund $\Longrightarrow$ **???**

Asplund spaces (1960's)

- X is an Asplund space if every continuous convex real-valued function defined on an open subset of X is F-differentiable on a dense subset;
- Equivalently [1970's], every separable subspace has separable dual.

The road map of the talk

The property

We introduce an isomorphic property for (separable) Banach spaces, the so-called

slicely countably determination (SCD)

such that

- it is satisfied by RNP spaces
  (actually, by strongly regular spaces – PCP in particular–);
- it is satisfied by Asplund spaces
  (actually, by spaces not containing ℓ_1).

We also present examples and stability properties.

The applications

- We apply SCD to get results for the Daugavet property, the alternative Daugavet property and spaces with numerical index 1.
- We present SCD operators and applications.

Outline

Slicely Countably Determined sets and spaces

SCD sets: Definitions and preliminary remarks

X Banach space, $A \subset X$ bounded and convex.

SCD sets

A is Slicely Countably Determined (SCD) if there is a sequence $\{S_n : n \in \mathbb{N}\}$ of slices of A satisfying one of the following equivalent conditions:

- if $B \subseteq A$ satisfies $B \cap S_n \neq \emptyset\ \forall n$, then $A \subseteq \overline{\text{conv}}(B)$,
- given $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in S_n\ \forall n \in \mathbb{N}$, $A \subseteq \overline{\text{conv}}\big(\{x_n : n \in \mathbb{N}\}\big)$,
- every slice of A contains one of the S_n's,

Remarks

- A is SCD iff $\overline{A}$ is SCD.
- If A is SCD, then it is separable.

SCD sets: Elementary examples I

Example

A separable and $A = \overline{\text{conv}}(\text{dent}(A)) \implies A$ is SCD.

Proof.

- Take $\{a_n : n \in \mathbb{N}\}$ denting points with $A = \overline{\text{conv}}\big(\{a_n : n \in \mathbb{N}\}\big)$.
- For every $n, m \in \mathbb{N}$, take a slice $S_{n,m}$ containing a_n and of diameter $1/m$.
- If $B \cap S_{n,m} \neq \emptyset \implies a_n \in \overline{B}$.
- Therefore, $A = \overline{\text{conv}}\big(\{a_n : n \in \mathbb{N}\}\big) \subseteq \overline{\text{conv}}(\overline{B}) = \overline{\text{conv}}(B)$.

Example

In particular, A RNP separable $\implies A$ SCD.

Corollary

- If X is separable LUR $\implies B_X$ is SCD.
- So, every separable space can be renormed such that $B_{(X,|\cdot|)}$ is SCD.

SCD sets: Elementary examples II

Example

If X^* is separable $\Longrightarrow$ A is SCD.

Proof.

- Take $\{x_n^* : n \in \mathbb{N}\}$ dense in S_{X^*}.
- For every $n, m \in \mathbb{N}$, consider $S_{n,m} = S(A, x_n^*, 1/m)$.
- It is easy to show that any slice of A contains one of the $S_{n,m}$

Example

$B_{C[0,1]}$ and $B_{L_1[0,1]}$ are not SCD.

SCD sets: Further examples I

Convex combination of slices

$$W = \sum_{k=1}^{m} \lambda_k S_k \subset A \text{ where } \lambda_k \geqslant 0,\ \sum \lambda_k = 1,\ S_k \text{ slices.}$$

Proposition

In the definition of SCD we can use a sequence $\{S_n : n \in \mathbb{N}\}$ of convex combination of slices.

Small combinations of slices

A has small combinations of slices iff every slice of A contains convex combinations of slices of A with arbitrary small diameter.

Example

If A has small combinations of slices + separable $\Longrightarrow$ A is SCD.

Particular case

A strongly regular (in particular, PCP) + separable $\Longrightarrow$ A is SCD.

SCD sets: Further examples II

Bourgain's lemma

Every relative weak open subset of A contains a convex combination of slices.

Corollary

In the definition of SCD we can use a sequence of relative weak open subsets: the set A is SCD iff there is a sequence $\{V_n : n \in \mathbb{N}\}$ of relative weak open subsets of A such that every slice of A contains one of the V_n's.

π-bases

A π-base of the weak topology of A is a family $\{V_i : i \in I\}$ of weak open sets of A such that every weak open subset of A contains one of the V_i's.

Proposition

If $(A, \sigma(X, X^*))$ has a countable π-base $\Longrightarrow$ A is SCD.

SCD sets: Further examples III

Theorem

A separable without ℓ_1-sequences $\Longrightarrow$ $(A, \sigma(X, X^*))$ has a countable π-base.

Proof.

- We see $(A, \sigma(X, X^*)) \subset C(T)$ where $T = (B_{X^*}, \sigma(X^*, X))$.
- By Rosenthal ℓ_1 theorem, $(A, \sigma(X, X^*))$ is a relatively compact subset of the space of first Baire class functions on T.
- By a result of Todorčević, $(A, \sigma(X, X^*))$ has a σ-disjoint π-base.
- $\{V_i : i \in I\}$ is *σ-disjoint* if $I = \bigcup_{n \in \mathbb{N}} I_n$ and each $\{V_i : i \in I_n\}$ is pairwise disjoint.
- A σ-disjoint family of open subsets in a separable space is countable. ✓

Main example

A separable without ℓ_1-sequences $\Longrightarrow$ A is SCD.

SCD spaces: definition and examples

SCD space

X is Slicely Countably Determined (SCD) if so are its convex bounded subsets.

Examples of SCD spaces

1. X separable strongly regular. In particular, RNP, PCP spaces.
2. X separable $X \not\supseteq \ell_1$. In particular, if X^* is separable.

Examples of NOT SCD spaces

1. $C[0,1]$, $L_1[0,1]$
2. Actually, every X containing (an isomorphic copy of) $C[0,1]$ or $L_1[0,1]$.
3. There is X with the Schur property which is not SCD.

Remark

- Every subspace of a SCD space is SCD.
- This is false for quotients.

SCD spaces: stability properties

Theorem

$Z \subset X$. If Z and X/Z are SCD $\Longrightarrow X$ is SCD.

Corollary

X separable NOT SCD $\Longrightarrow\ X \supset \ell_1$ and

- If $\ell_1 \simeq Y \subset X \implies X/Y$ contains a copy of ℓ_1.
- If $\ell_1 \simeq Y_1 \subset X \implies$ there is $\ell_1 \simeq Y_2 \subset X$ with $Y_1 \cap Y_2 = 0$.

Corollary

$X_1, \ldots, X_m$ SCD $\Longrightarrow X_1 \oplus \cdots \oplus X_m$ SCD.

SCD spaces: stability properties II

Theorem

$X_1, X_2, \ldots$ SCD, E with unconditional basis.

- $E \not\supseteq c_0 \implies \left[\bigoplus_{n\in\mathbb{N}} X_n\right]_E$ SCD.
- $E \not\supseteq \ell_1 \implies \left[\bigoplus_{n\in\mathbb{N}} X_n\right]_E$ SCD.

Examples

1. $c_0(\ell_1)$ and $\ell_1(c_0)$ are SCD.
2. $c_0 \otimes_\varepsilon c_0$, $c_0 \otimes_\pi c_0$, $c_0 \otimes_\varepsilon \ell_1$, $c_0 \otimes_\pi \ell_1$, $\ell_1 \otimes_\varepsilon \ell_1$, and $\ell_1 \otimes_\pi \ell_1$ are SCD.
3. $K(c_0)$ and $K(c_0, \ell_1)$ are SCD.
4. $\ell_2 \otimes_\varepsilon \ell_2 \equiv K(\ell_2)$ and $\ell_2 \oplus_\pi \ell_2 \equiv \mathcal{L}_1(\ell_2)$ are SCD

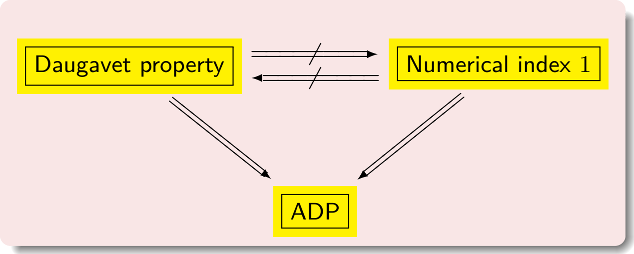
Applications

The DPr, the ADP and numerical index 1

Definition of the properties

1. **Kadets-Shvidkoy-Sirotkin-Werner, 1997:**
   X has the Daugavet property (DPr) if
   $$\|\mathrm{Id}+T\| = 1+\|T\| \qquad \text{(DE)}$$
   for every rank-one $T \in L(X)$.
   - Then every T not fixing copies of ℓ_1 also satisfies (DE).
2. **Lumer, 1968:** X has numerical index 1 ($n(X)=1$) if
   $$\max_{\theta\in\mathbb{T}} \|\mathrm{Id}+\theta T\| = 1+\|T\| \qquad \text{(aDE)}$$
   for EVERY operator on X.
   - Equivalently,
     $$\|T\| = \sup\{|x^*(Tx)| : x\in S_X,\, x^*\in S_{X^*},\, x^*(x)=1\}$$
     for EVERY $T\in L(X)$.
3. **M.-Oikhberg, 2004:** X has the alternative Daugavet property (ADP) if every rank-one $T\in L(X)$ satisfies (aDE).
   - Then every weakly compact T also satisfies (aDE).

Relations between these properties



Examples

- $C\big([0,1],K(\ell_2)\big)$ has DPr, but has not numerical index 1
- c_0 has numerical index 1, but has not DPr
- $c_0 \oplus_\infty C\big([0,1],K(\ell_2)\big)$ has ADP, neither DPr nor numerical index 1

Remarks

- For RNP or Asplund spaces, $\boxed{\text{ADP}} \Longrightarrow \boxed{\text{numerical index } 1}$.
- Every Banach space with the ADP can be renormed still having the ADP but failing the Daugavet property.

For C^*-algebras and preduals

Let V_* be the predual of the von Neumann algebra V.

The Daugavet property of V_* is equivalent to:

- V has no atomic projections, or
- the unit ball of V_* has no extreme points.

V_* has numerical index 1 iff:

- V is commutative, or
- $|v^*(v)| = 1$ for $v \in \operatorname{ext}(B_V)$ and $v^* \in \operatorname{ext}(B_{V^*})$.

The alternative Daugavet property of V_* is equivalent to:

- the atomic projections of V are central, or
- $|v(v_*)| = 1$ for $v \in \operatorname{ext}(B_V)$ and $v_* \in \operatorname{ext}(B_{V_*})$, or
- $V = C \oplus_\infty N$, where C is commutative and N has no atomic projections.

Let X be a C^*-algebra.

The Daugavet property of X is equivalent to:

- X does not have any atomic projection, or
- the unit ball of X^* does not have any w^*-strongly exposed point.

X has numerical index 1 iff:

- X is commutative, or
- $|x^{**}(x^*)| = 1$ for $x^{**} \in \operatorname{ext}(B_{X^{**}})$ and $x^* \in \operatorname{ext}(B_{X^*})$.

The alternative Daugavet property of X is equivalent to:

- the atomic projections of X are central, or
- $|x^{**}(x^*)| = 1$, for $x^{**} \in \operatorname{ext}(B_{X^{**}})$, and $x^* \in B_{X^*}$ w^*-strongly exposed, or
- $\exists$ a commutative ideal Y such that X/Y has the Daugavet property.

A sufficient condition for numerical index 1: lushness

Lushness (Boyko-Kadets-M.-Werner, 2007)

X is lush if given $x, y \in S_X$, $\varepsilon > 0$, there is $y^* \in S_{X^*}$ such that

$$x \in S = S(B_X, y^*, \varepsilon) \qquad \operatorname{dist}(y, \operatorname{conv}(\mathbb{T}S)) < \varepsilon.$$

Theorem (Boyko-Kadets-M.-Werner, 2007)

If X is lush, then X has numerical index 1

Example (Kadets-M.-Merí-Shepelska, 2009)

There is X with numerical index 1 which is not lush.

ADP + SCD $\Longrightarrow$ lushness

Characterization of ADP

X Banach space. TFAE:

- X has ADP (i.e. $\max_{\theta\in\mathbb{T}}\|\mathrm{Id}+\theta T\| = 1+\|T\|$ for all T rank-one).
- Given $x\in S_X$, a slice S of B_X and $\varepsilon>0$, there is $y\in S$ with
$$\max_{\theta\in\mathbb{T}}\|x+\theta y\| > 2-\varepsilon.$$
- Given $x\in S_X$, a sequence $\{S_n\}$ of slices of B_X, and $\varepsilon>0$, there is $y^*\in S_{X^*}$ such that $x\in S(B_X,y^*,\varepsilon)$ and
$$\overline{\mathrm{conv}}\big(\mathbb{T}\,S(B_X,y^*,\varepsilon)\big)\bigcap S_n\neq\emptyset \qquad (n\in\mathbb{N}).$$

Theorem

X ADP + B_X SCD $\Longrightarrow$ given $x\in S_X$ and $\varepsilon>0$, there is $y^*\in S_{X^*}$ such that
$$x\in S(B_X,y^*,\varepsilon) \quad\text{and}\quad B_X=\overline{\mathrm{conv}}\big(\mathbb{T}\,S(B_X,y^*,\varepsilon)\big).$$

- This clearly implies lushness, and so numerical index 1 (i.e. $\max_{\theta\in\mathbb{T}}\|\mathrm{Id}+\theta T\| = 1+\|T\|$ for all T).

Some consequences

Corollary

- ADP + strongly regular $\implies$ numerical index 1.
- ADP + $X \not\supseteq \ell_1$ $\implies$ numerical index 1.

Corollary

X real + $\dim(X) = \infty$ + ADP $\implies$ $X^* \supseteq \ell_1$.

In particular,

Corollary

X real + $\dim(X) = \infty$ + numerical index 1 $\implies$ $X^* \supseteq \ell_1$.

Some consequences II

Proposition (Kadets-M.-Merí-Werner, 2010)

- X with 1-unconditional basis $\implies$ B_X is SCD.
- X with 1-unconditional basis and ADP $\implies$ X is lush.

Theorem (Kadets-M.-Merí-Werner, 2010)

1. The unique Banach spaces with 1-symmetric basis and the ADP are c_0 and ℓ_1.
2. The unique r.i. Banach spaces over $\mathbb{N}$ with the ADP are c_0, ℓ_1 and ℓ_∞.
3. The unique separable r.i. Banach space on $[0, 1]$ with the Daugavet property is $L_1[0, 1]$.
4. The unique separable r.i. Banach space on $[0, 1]$ which is lush is $L_1[0, 1]$.

Question

Is it possible to prove the above results for the ADP **?**

SCD operators

SCD operators

SCD operator

$T \in L(X)$ is an SCD-operator if $T(B_X)$ is an SCD-set.

Examples

T is an SCD-operator when $T(B_X)$ is separable and

1. $T(B_X)$ is RNP,
2. $T(B_X)$ has no ℓ_1 sequences,
3. T does not fix copies of ℓ_1

Theorem

- X ADP + T SCD-operator $\implies \max_{\theta\in\mathbb{T}} \|\mathrm{Id} + \theta T\| = 1 + \|T\|$.
- X DPr + T SCD-operator $\implies \|\mathrm{Id} + T\| = 1 + \|T\|$.

Main corollary

X ADP + T does not fix copies of $\ell_1 \implies \max_{\theta\in\mathbb{T}} \|\mathrm{Id} + \theta T\| = 1 + \|T\|$.

HSCD-majorized operators (Kadets-Shepelska, 2010)

HSCD and HSDC-majorized operator

- $T \in L(X,Y)$ is an Hereditary-SCD-operator if every convex subset of $T(B_X)$ is an SCD-set.
- $T \in L(X,Y)$ is an HSCD-majorized operator if there is $S \in L(X,Z)$ HSCD-operator such that $\|Tx\| \leqslant \|Sx\|$ for every $x \in X$.

Proposition

The class of HSCD-majorized operators is a two-sided operator ideal.

Theorem

X DPr + $T \in L(X)$ HSCD-majorized operator $\implies \|\mathrm{Id} + T\| = 1 + \|T\|$.

Remark

The class of operators satisfying (DE) is not even a subspace.

Final remarks

Open questions

1. Find more sufficient conditions for a set to be SCD.
2. Is SCD equivalent to the existence of a countable π-base for the weak topology **?**
3. E with (1)-unconditional basis. Is E SCD **?**
4. E with 1-unconditional basis, $\{X_n\}$ a family of SCD spaces. Is $[\oplus X_n]_E$ SCD **?**
5. X, Y SCD. Are $X \otimes_\varepsilon Y$ and $X \otimes_\pi Y$ SCD **?**
6. Find a good extension of the SCD property to the nonseparable case.
7. Clarify the relationship between SCD and the Daugavet property.
8. X ADP, $T \in L(X)$ HSCD-majorized, does T satisfies (aDE) **?**

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