On the numerical radius of operators on Lebesgue spaces

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Notation

Basic notation

X Banach space.

- $\mathbb{K}$ base field (it may be $\mathbb{R}$ or $\mathbb{C}$),
- S_X unit sphere, B_X unit ball,
- X^* dual space,
- $L(X)$ bounded linear operators,
- $T^* \in L(X^*)$ adjoint operator of $T \in L(X)$.
- For $z \in \mathbb{K}$,
  - $\overline{z}$ is the conjugate ($= z$ in the real case),
  - $\operatorname{Re} z$ is the real part ($= z$ in the real case).
- $L_p(\mu)$ (real or complex) Banach space of μ-measurable scalar functions with

$$\|x\|_p := \left(\int_\Omega |x|^p \, d\mu \right)^{\frac{1}{p}} < \infty$$

- $\ell_p^{(m)}$ m-dimensional L_p-space

The results

The result for the absolute numerical radius

For $1 < p < \infty$, let $\kappa_p = p^{-\frac{1}{p}} q^{-\frac{1}{q}}$. Then

$$\sup \left\{ \int |x|^{p-1} |Tx| \, d\mu \ : \ x \in L_p(\mu), \|x\|_p = 1 \right\} \geqslant \kappa_p \, \|T\|$$

for every $T \in L(L_p(\mu))$ (real and complex cases).

- This inequality is best possible when $\dim(L_p(\mu)) \geqslant 2$.

Positivity of the numerical index

For $1 < p < \infty$, $p \neq 2$, there is a **positive** constant $n(L_p)$ such that

$$\sup \left\{ \left| \int |x|^{p-1} \operatorname{sign}(\overline{x}) Tx \, d\mu \right| \ : \ x \in L_p(\mu), \|x\| = 1 \right\} \geqslant n(L_p) \, \|T\|$$

for every $T \in L(L_p(\mu))$ (real case).

- We do not know the best possibility for $n(L_p)$.

Schedule of the talk

Numerical index of Banach spaces

2 Numerical index of Banach spaces

- Numerical range
- Numerical index: definition and basic properties
- Examples
- Stability properties
- Duality
- Renorming
- Open problems

F. F. Bonsall and J. Duncan
Numerical Ranges. Vol I and II.
London Math. Soc. Lecture Note Series, 1971 & 1973.

V. Kadets, M. Martín, and R. Payá.
Recent progress and open questions on the numerical index of Banach spaces.
RACSAM (2006)

Numerical range: Hilbert spaces

Hilbert space numerical range (Toeplitz, 1918)

- A $n \times n$ real or complex matrix

$$W(A) = \left\{ (Ax \mid x) \ : \ x \in \mathbb{K}^n, \ (x \mid x) = 1 \right\}.$$

- H real or complex Hilbert space, $T \in L(H)$,

$$W(T) = \left\{ (Tx \mid x) \ : \ x \in H, \ \|x\| = 1 \right\}.$$

Some properties

H Hilbert space, $T \in L(H)$:

- $W(T)$ is convex.
- In the complex case, $\overline{W(T)}$ contains the spectrum of T.
- If T is normal, then $\overline{W(T)} = \overline{\mathrm{co}}\,\mathrm{Sp}(T)$.

Numerical range: Banach spaces

Banach spaces numerical range (Bauer 1962; Lumer, 1961)

X Banach space, $T \in L(X)$,

$$V(T) = \left\{ x^*(Tx) \ : \ x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1 \right\}$$

Some properties

X Banach space, $T \in L(X)$:

- $V(T)$ is connected (not necessarily convex).
- In the complex case, $\overline{V(T)}$ contains the spectrum of T.
- In fact,

$$\overline{\mathrm{co}}\,\mathrm{Sp}(T) = \bigcap \overline{V(T)},$$

  intersection taken over all numerical ranges $V(T)$ corresponding to equivalent norms on X.

Some motivations for the numerical range

For Hilbert spaces

- It is a comfortable way to study the spectrum.
- It is useful to work with some concepts like hermitian operator, skew-hermitian operator, dissipative operator...
- It is useful to estimate spectral radii of small perturbations of matrices.

For Banach spaces

- It allows to carry to the general case the concepts of hermitian operator, skew-hermitian operator, dissipative operators...
- It gives a description of the Lie algebra corresponding to the Lie group of all onto isometries on the space.
- It gives an easy and quantitative proof of the fact that Id is an strongly extreme point of $B_{L(X)}$ (MLUR point).

Numerical index of Banach spaces: definitions

Numerical radius

X Banach space, $T \in L(X)$. The numerical radius of T is

$$v(T) = \sup\left\{ |x^*(Tx)| \ : \ x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1 \right\}$$

Remark

The numerical radius is a continuous seminorm in $L(X)$. Actually, $v(\cdot) \leqslant \|\cdot\|$

Exponential formula

$\|\exp(\rho T)\| \leqslant \mathrm{e}^{|\rho| v(T)}$ $(\rho \in \mathbb{K})$ and $v(T)$ is best possible.

Numerical index (Lumer, 1968)

X Banach space, the numerical index of X is

$$\begin{aligned} n(X) &= \inf\left\{ v(T) \ : \ T \in L(X),\ \|T\| = 1 \right\} \\ &= \max\left\{ k \geqslant 0 \ : \ k\|T\| \leqslant v(T) \ \ \forall\, T \in L(X) \right\} \end{aligned}$$

Numerical index of Banach spaces: basic properties

Some basic properties

- $n(X) = 1$ iff v and $\|\cdot\|$ coincide.
- $n(X) = 0$ iff v is not an equivalent norm in $L(X)$

- X complex $\Rightarrow$ $n(X) \geqslant 1/\mathrm{e}$.

  (Bohnenblust–Karlin, 1955; Glickfeld, 1970)

- Actually,

$$\{n(X) \ : \ X \text{ complex}, \ \dim(X) = 2\} = [\mathrm{e}^{-1}, 1]$$
$$\{n(X) \ : \ X \text{ real}, \ \dim(X) = 2\} = [0, 1]$$

(Duncan–McGregor–Pryce–White, 1970)

Numerical index of Banach spaces: some examples

Examples

1. H Hilbert space, $\dim(H) > 1$,

$$n(H) = 0 \quad \text{if } H \text{ is real}$$
$$n(H) = 1/2 \quad \text{if } H \text{ is complex}$$

2. $n\big(L_1(\mu)\big) = 1$ $\quad \mu$ positive measure

   $n\big(C(K)\big) = 1$ $\quad K$ compact Hausdorff space

   (Duncan et al., 1970)

3. If A is a C^*-algebra $\Rightarrow \begin{cases} n(A) = 1 & A \text{ commutative} \\ n(A) = 1/2 & A \text{ not commutative} \end{cases}$

   (Huruya, 1977; Kaidi–Morales–Rodríguez, 2000)

4. If A is a function algebra $\Rightarrow n(A) = 1$

   (Werner, 1997)

Numerical index of Banach spaces: some examples II

More examples

5. For $n \geqslant 2$, the unit ball of X_n is a $2n$ regular polygon:

$$n(X_n) = \begin{cases} \tan\left(\frac{\pi}{2n}\right) & \text{if } n \text{ is even,} \\ \\ \sin\left(\frac{\pi}{2n}\right) & \text{if } n \text{ is odd.} \end{cases}$$

(M.–Merí, 2007)

6. Every finite-codimensional subspace of $C[0,1]$ has numerical index 1

(Boyko–Kadets–M.–Werner, 2007)

Stability properties

Direct sums of Banach spaces (M.–Payá, 2000)

$$n\Big([\oplus_{\lambda\in\Lambda} X_\lambda]_{c_0}\Big) = n\Big([\oplus_{\lambda\in\Lambda} X_\lambda]_{\ell_1}\Big) = n\Big([\oplus_{\lambda\in\Lambda} X_\lambda]_{\ell_\infty}\Big) = \inf_\lambda n(X_\lambda)$$

Vector-valued function spaces (López–M.–Merí–Payá–Villena, 200's)

E Banach space, μ positive measure, K compact space. Then

$$n\big(C(K,E)\big) = n\big(C_w(K,E)\big) = n\big(L_1(\mu,E)\big) = n\big(L_\infty(\mu,E)\big) = n(E),$$

and $n\big(C_{w^*}(K,E^*)\big) \leqslant n(E)$

Tensor products (Lima, 1980)

There is no general formula neither for $n\big(X\widetilde{\otimes}_\varepsilon Y\big)$ nor for $n\big(X\widetilde{\otimes}_\pi Y\big)$:

- $n\big(\ell_1^{(4)}\widetilde{\otimes}_\pi \ell_1^{(4)}\big) = n\big(\ell_\infty^{(4)}\widetilde{\otimes}_\varepsilon \ell_\infty^{(4)}\big) = 1.$
- $n\big(\ell_1^{(4)}\widetilde{\otimes}_\varepsilon \ell_1^{(4)}\big) = n\big(\ell_\infty^{(4)}\widetilde{\otimes}_\pi \ell_\infty^{(4)}\big) < 1.$

Numerical index and duality

Proposition

X Banach space, $T \in L(X)$. Then

- $\sup \operatorname{Re} V(T) = \lim_{\alpha \to 0^+} \dfrac{\|\mathrm{Id} + \alpha T\| - 1}{\alpha}$.
- $v(T^*) = v(T)$ for every $T \in L(X)$.
- Therefore, $n(X^*) \leqslant n(X)$.

(Duncan–McGregor–Pryce–White, 1970)

Question (From the 1970's)

Is $n(X) = n(X^*)$ **?**

Negative answer (Boyko–Kadets–M.–Werner, 2007)

Consider the space

$$X = \Big\{(x,y,z) \in c \oplus_\infty c \oplus_\infty c \ : \ \lim x + \lim y + \lim z = 0\Big\}.$$

Then, $n(X) = 1$ but $n(X^*) < 1$.

Numerical index and renorming

Proposition (Finet–M.–Payá, 2003)

X separable or reflexive, then

$$\{n(Y) : Y \simeq X\} \supseteq \begin{cases} [0,1[& \text{real case} \\ [e^{-1},1[& \text{complex case} \end{cases}$$

Theorem (Avilés–Kadets–M.–Merí–Shepelska, 2010)

X real Banach space, $n(X) = 1$, $\dim(X) = \infty$, $\implies X^* \supset \ell_1$.

Proposition (Boyko–Kadets–M.–Merí, 2009)

X separable, $X \supset c_0 \implies \exists Y \simeq X$ with $n(Y) = 1$.

Some interesting open problems

Open problems

1. Characterize (without operators) Banach spaces with numerical index 1.
2. X with $n(X) = 1$, $\dim(X) = \infty$ $X \supset c_0$ or $X \supset \ell_1$ **?**
3. Characterize those X admitting a renorming with numerical index 1.
4. For instance, if $X \supset c_0$ or $\supset \ell_1$ can X be renormed with numerical index 1 **?**
5. Find isomorphic or isometric conditions assuring that $n(X) = n(X^*)$.

The oldest open problem

Calculate the numerical index of "classical" spaces.

- In particular, calculate $n(L_p(\mu))$.

The 2000's results on the numerical index on L_p-spaces

3 The 2000's results on the numerical index on L_p-spaces

E. Ed-dari.
On the numerical index of Banach spaces.
Linear Algebra Appl. (2005)

E. Ed-dari and M. Khamsi.
The numerical index of the L_p space.
Proc. Amer. Math. Soc. (2006)

E. Ed-dari, M. Khamsi, and A. Aksoy.
On the numerical index of vector-valued function spaces.
Linear Mult. Algebra (2007)

M. Martín, and J. Merí.
A note on the numerical index of the L_p-space of dimension two.
Linear Mult. Algebra (2009)

M. Martín, J. Merí, M. Popov, and B. Randrianantoanina.
Numerical index of absolute sums of Banach spaces.
J. Math. Anal. Appl. (2011)

Numerical index of L_P-spaces

Known results on the numerical index of L_p-spaces

1. $n(\ell_p) \leqslant n\big(\ell_p^{(m+1)}\big) \leqslant n\big(\ell_p^{(m)}\big)$ for $m \in \mathbb{N}$.

   (M.–Payá, 2000)

2. $n\big(L_p[0,1]\big) = n(\ell_p) = \lim_{m\to\infty} n\big(\ell_p^{(m)}\big) = \inf_{m\in\mathbb{N}} n\big(\ell_p^{(m)}\big)$.

   (Ed-Dari, 2005 & Ed-Dari–Khamsi, 2006)

3. In the real case,

$$\max\left\{\frac{1}{2^{1/p}}, \frac{1}{2^{1/q}}\right\} v\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \leqslant n\big(\ell_p^{(2)}\big) \leqslant v\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $v\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \max_{t\in[0,1]} \dfrac{|t^{p-1} - t|}{1+t^p}$

(M.–Merí, 2009)

Ideas behind the proofs I

The numerical index decreases with the dimension

$$n(\ell_p) \leqslant n\left(\ell_p^{(m+1)}\right) \leqslant n\left(\ell_p^{(m)}\right) \text{ for } m \in \mathbb{N}.$$

Proposition (M.–Payá, 2000)

$Z = U \oplus V$ with absolute sum (i.e. $\|u+v\| = f(\|u\|, \|v\|)$ for $u \in U,\ v \in V$).
$\Longrightarrow\ n(Z) \leqslant \min\{n(U), n(V)\}$.

Proof of the decreasing

- $\ell_p^{(m)}$ is an absolute summand in both $\ell_p^{(m+1)}$ and in ℓ_p.

Ideas behind the proofs II

One inequality

$$n\Big(L_p[0,1]\Big) \leqslant \lim_{m\to\infty} n\Big(\ell_p^{(m)}\Big).$$

Proposition (M.–Merí–Popov–Randrianantoanina, 2011)

E order continuous Köthe space, X Banach space

$$\Longrightarrow n\Big(E(X)\Big) \leqslant n(X).$$

Proof of the inequality

- $E = L_p[0,1]$, $X = \ell_p^{(m)}$.
- $E \equiv E(X)$ so $n(E) \leqslant n(\ell_p^{(m)})$.

Ideas behind the proofs III

The reversed inequality

$$n\big(L_p[0,1]\big) \geqslant \lim_{m\to\infty} n\big(\ell_p^{(m)}\big) \qquad \text{and} \qquad n(\ell_p) \geqslant \lim_{m\to\infty} n\big(\ell_p^{(m)}\big).$$

Proposition (M.–Merí–Popov–Randrianantoanina, 2011)

Z Banach space, $\{Z_i\}_{i\in I}$ increasing family of one-complemented subspaces whose union is dense. Then, $\Longrightarrow n(Z) \geqslant \limsup_{i\in I} n(Z_i)$.

Corollary

Z Banach space with monotone basis (e_m), $Z_m = \text{span}\{e_k : 1 \leqslant k \leqslant m\}$.
$\Longrightarrow \quad n(Z) \geqslant \limsup_{m\to\infty} n(Z_m)$.

Proof of the inequality

- $Z = \ell_p$, (e_m) canonical basis $\quad\Longrightarrow\quad Z_m \equiv \ell_p^{(m)}$ for all $m \in \mathbb{N}$.
- $E = L_p[0,1]$, (e_m) Haar system $\quad\Longrightarrow\quad Z_m \equiv \ell_p^{(m)}$ for $m = 2^k$ $(k \in \mathbb{N})$.

Ideas behind the proofs IV

The two-dimensional case

In the real case,

$$\max\left\{\frac{1}{2^{1/p}},\ \frac{1}{2^{1/q}}\right\} M_p \leqslant n\left(\ell_p^{(2)}\right) \leqslant M_p \quad \text{where} \quad M_p = \max_{t\in[0,1]} \frac{|t^{p-1}-t|}{1+t^p}$$

Proposition (Duncan-McGregor-Pryce-White, 1970)

$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ operator in $\ell_p^{(2)}$. Then

$$v(T) = \max\left\{ \max_{t\in[0,1]} \frac{|a+dt^p| + \left|bt+ct^{p-1}\right|}{1+t^p},\ \max_{t\in[0,1]} \frac{|d+at^p| + \left|ct+bt^{p-1}\right|}{1+t^p} \right\}.$$

Proof of the result

- $n(\ell_p^{(2)}) \leqslant M_p$ since $\left\| \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\| = 1$ and $v\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = M_p$.
- We compare $v(T)$ with M_p, but we use $\|T\|_1$ and $\|T\|_\infty$ instead of $\|T\|_p$.

Questions

Questions

1. Is $n\left(\ell_p^{(m+1)}\right) = n\left(\ell_p^{(m)}\right)$ for $m \geqslant 2$ **?**
2. In the real case, is $n(L_p[0,1])$ positive **?**
3. We do not have results for the complex case, even for dimension two.

The 2010's results

- We left the finite-dimensional approach and introduce the absolute numerical radius.
- This allows to show that $n(L_p[0,1]) > 0$ in the real case.

The new results on the numerical index of L_p-spaces

4 The new results on the numerical index of L_p-spaces

M. Martín, J. Merí, M. Popov.
On the numerical index of real $L_p(\mu)$-spaces.
Isr. J. Math. (to appear)

M. Martín, J. Merí, M. Popov.
On the numerical radius of operators in Lebesgue spaces.
J. Funct. Anal. (to appear)

The absolute numerical radius in L_p

The numerical radius in L_p

- For $x \in L_p(\mu)$, write $x^{\#} = |x|^{p-1}\operatorname{sign}(\overline{x})$.
- It is the unique element in $L_q(\mu)$ such that

$$\|x\|_p^p = \|x^{\#}\|_q^q \quad \text{and} \quad \int x\,x^{\#}\ d\mu = \|x\|_p\,\|x^{\#}\|_q = \|x\|_p^p.$$

- Therefore, for $T \in L\big(L_p(\mu)\big)$ one has

$$v(T) = \sup\left\{\left|\int x^{\#}Tx\,d\mu\right| \ :\ x \in L_p(\mu),\ \|x\|_p = 1\right\}$$
$$= \sup\left\{\left|\int |x|^{p-1}\operatorname{sign}(\overline{x})\,Tx\,d\mu\right| \ :\ x \in L_p(\mu),\ \|x\|_p = 1\right\}$$

Absolute numerical radius

For $T \in L\big(L_p(\mu)\big)$,

$$|v|(T) := \sup\left\{\int |x^{\#}Tx|\,d\mu \ :\ x \in L_p(\mu), \|x\|_p = 1\right\}$$
$$= \sup\left\{\int |x|^{p-1}|Tx|\,d\mu \ :\ x \in L_p(\mu), \|x\|_p = 1\right\}.$$

The absolute numerical index of L_p

Obvious remark

$$v(T) \leqslant |v|(T) \leqslant \|T\| \text{ for every } T \in L\big(L_p(\mu)\big).$$

Absolute numerical index

$$\begin{aligned} |n|(L_p(\mu)) &= \inf \left\{ |v|(T) \ : \ T \in L(L_p(\mu)), \ \|T\| = 1 \right\} \\ &= \max \left\{ k \geqslant 0 \ : \ k\|T\| \leqslant |v|(T) \ \ \forall\, T \in L(L_p(\mu)) \right\}. \end{aligned}$$

- $n(L_p(\mu))$ is the greatest constant $M \geqslant 0$ such that

$$\sup \left\{ \left| \int |x|^{p-1} \operatorname{sign}(\overline{x})\, Tx \, d\mu \right| \ : \ x \in L_p(\mu), \ \|x\|_p = 1 \right\} \geqslant M \, \|T\|$$

  for every $T \in L(L_p(\mu))$.

- $|n|(L_p(\mu))$ is the greatest constant $K \geqslant 0$ such that

$$\sup \left\{ \left| \int |x|^{p-1} |Tx| \, d\mu \right| \ : \ x \in L_p(\mu), \ \|x\|_p = 1 \right\} \geqslant K \, \|T\|$$

  for every $T \in L(L_p(\mu))$.

The first results

Proposition 1

Write $M_p = \max\limits_{t\in[0,1]} \frac{|t^{p-1}-t|}{1+t^p}$. Then, in the real case,

$$v(T) \geqslant \frac{M_p}{6}\,|v|(T) \qquad \big(T \in L(L_p(\mu))\big).$$

Proposition 2

In the real case,

$$|v|(T) \geqslant \tfrac{1}{2} v(T_{\mathbb{C}}) \geqslant \frac{n(L_p^{\mathbb{C}}(\mu))}{2}\,\|T\| \qquad \big(T \in L(L_p(\mu))\big).$$

We do not know the vale of $n(L_p^{\mathbb{C}}(\mu))$, but $n(X) \geqslant 1/\mathrm{e}$ for complex spaces, so

Theorem

In the real case, $n(L_p(\mu)) \geqslant \dfrac{M_p}{12\mathrm{e}} > 0$ for $1 < p < \infty$, $p \neq 2$.

We improved Proposition 2 calculating $|n|(L_p(\mu))$ for real and complex spaces

Calculating $|n|(L_p(\mu))$ I

The constant

Set $\kappa_p := \max_{\tau>0} \frac{\tau^{p-1}}{1+\tau^p} = \max_{\lambda\in[0,1]} \lambda^{\frac{1}{q}}(1-\lambda)^{\frac{1}{p}} = \frac{1}{p^{1/p}q^{1/q}}$.

The best possibility for $|n|(L_p(\mu))$

If $\dim(L_p(\mu)) \geqslant 2$, then there is a (positive) operator $T \in L(L_p(\mu))$ with

$$\|T\| = 1, \qquad |v|(T) = \kappa_p.$$

The examples for ℓ_p and $L_p[0,1]$:

- For ℓ_p: consider the extension by zero of the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
- For $L_p[0,1]$:

$$Tf = 2\left[\int_0^{1/2} f(s)\,ds\right]\chi_{[\frac{1}{2},1]} \qquad \big(f \in L_p[0,1]\big).$$

Calculating $|n|(L_p(\mu))$ II

Theorem

$$|n|(L_p(\mu)) \geqslant \kappa_p$$

Proof for positive operators:

- Fix $T \in L(L_p(\mu))$ **positive** with $\|T\| = 1$, $\tau > 0$ and $\varepsilon > 0$.
- Find $x \geqslant 0$ with $\|x\| = 1$ and $\|Tx\|^p > 1 - \varepsilon$, set

$$y = x \vee \tau Tx \qquad \text{and} \qquad A = \{\omega \in \Omega : x(\omega) \geqslant \tau(Tx)(\omega)\},$$

  and observe that

$$\|y\|^p = \int_A x^p \, d\mu + \int_{\Omega \setminus A} (\tau Tx)^p \, d\mu \leqslant 1 + \tau^p \qquad \text{and} \qquad y^\# = x^{p-1} \vee (\tau Tx)^{p-1}.$$

- Now,

$$\begin{aligned} |v|(T) &\geqslant \frac{1}{\|y\|^p} \int_\Omega y^\# Ty \, d\mu \geqslant \frac{1}{1+\tau^p} \int_\Omega y^\# Ty \, d\mu \\ &\geqslant \frac{1}{1+\tau^p} \int_\Omega (\tau Tx)^{p-1} Tx \, d\mu = \frac{\tau^{p-1}}{1+\tau^p} \int_\Omega (Tx)^p \, d\mu \geqslant \frac{\tau^{p-1}}{1+\tau^p}(1-\varepsilon). \end{aligned}$$

- Taking supremum on $\tau > 0$ and $\varepsilon > 0$, we get $|v|(T) \geqslant \kappa_p$.

One consequence and further results

Corollary

$$n\big(L_p(\mu)\big) \geqslant \frac{M_p\,\kappa_p}{6} \text{ in the real case.}$$

More results

- If $T \in L(L_p[0,1])$ is rank-one $\implies v(T) \geqslant \kappa_p^2 \|T\|$.
- If $T \in L(L_p[0,1])$ is **compact**, then

$$v(T) \geqslant \kappa_p^2 \|T\| \text{ (complex case)}, \quad v(T) \geqslant \max_{\tau>0} \frac{\kappa_p \tau^{p-1} - \tau}{1+\tau^p} \|T\| \text{ (real case)}.$$

Open problems with conjectures

- Is $n(L_p(\mu)) = M_p$ $(\dim \geqslant 2)$ in the real case ?
  - It is enough to prove that $n(L_p[0,1]) \geqslant M_p$ or $n(\ell_p) \geqslant M_p$.
- Is $n(L_p(\mu)) = \kappa_p$ $(\dim \geqslant 2)$ in the complex case ?
  - It is enough to prove that $n(L_p[0,1]) \geqslant \kappa_p$ or $n(\ell_p) \geqslant \kappa_p$.