Slicely Countably Determined Banach spaces

(Espacios de Banach determinados numerablemente por rebanadas)

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SLICELY COUNTABLY DETERMINED BANACH SPACES

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Abstract. We introduce the class of slicely countably determined Banach spaces which contains in particular all spaces with the Radon-Nikodým property and all spaces without copies of ℓ_1. We present many examples and several properties of this class. We give some applications to Banach spaces with the Daugavet and the alternative Daugavet properties, lush spaces and Banach spaces with numerical index 1. In particular, we show that the dual of a real infinite-dimensional Banach space with the alternative Daugavet property contains ℓ_1 and that operators which do not fix copies of ℓ_1 on a space with the alternative Daugavet property satisfy the alternative Daugavet equation.

Basic notation and main objective

Basic notation

X real or complex Banach space.

- S_X unit sphere, B_X closed unit ball, $\mathbb{T}$ modulus-one scalars.
- X^* dual space, $L(X)$ bounded linear operators.
- $\mathrm{conv}(\cdot)$ convex hull, $\overline{\mathrm{conv}}(\cdot)$ closed convex hull
- A slice of $A \subset X$ is a subset of the form

$$S(A, x^*, \alpha) = \{x \in A : \mathrm{Re}\, x^*(x) > \sup \mathrm{Re}\, x^*(A) - \alpha\} \quad (x^* \in X^*,\ \alpha > 0)$$

Objective

- We introduce an isomorphic property for (separable) Banach spaces called

  Slicely Countable Determined (SCD)

  such that
  - it is satisfied by RNP spaces,
  - it is satisfied by spaces not containing ℓ_1.
- We present some stability results.
- We apply SCD to get results for the Daugavet property, the alternative Daugavet property and spaces with numerical index 1.

Outline

Slicely Countably Determined sets and spaces

SCD sets: Definitions and preliminary remarks

X Banach space, $A \subset X$ bounded and convex.

SCD sets

A is Slicely Countably Determined (SCD) if there is a sequence $\{S_n : n \in \mathbb{N}\}$ of slices of A satisfying one of the following equivalent conditions:

- every slice of A contains one of the S_n's,
- $A \subseteq \overline{\text{conv}}(B)$ if $B \subseteq A$ satisfies $B \cap S_n \neq \emptyset \ \forall n$,
- given $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in S_n \ \forall n \in \mathbb{N}$, $A \subseteq \overline{\text{conv}}\big(\{x_n : n \in \mathbb{N}\}\big)$.

Remarks

- A is SCD iff $\overline{A}$ is SCD.
- If A is SCD, then it is separable.

SCD sets: Elementary examples I

Example

A separable and $A = \overline{\text{conv}}(\text{dent}(A)) \implies A$ is SCD.

Proof.

- Take $\{a_n : n \in \mathbb{N}\}$ denting points with $A = \overline{\text{conv}}\big(\{a_n : n \in \mathbb{N}\}\big)$.
- For every $n, m \in \mathbb{N}$, take a slice $S_{n,m}$ containing a_n and of diameter $1/m$.
- If $B \cap S_{n,m} \neq \emptyset \implies a_n \in \overline{B}$.
- Therefore, $A = \overline{\text{conv}}\big(\{a_n : n \in \mathbb{N}\}\big) \subseteq \overline{\text{conv}}(\overline{B}) = \overline{\text{conv}}(B)$.

Example

In particular, A RNP separable $\implies A$ SCD.

Corollary

- If X is separable LUR $\implies B_X$ is SCD.
- So, every separable space can be renormed such that $B_{(X,|\cdot|)}$ is SCD.

SCD sets: Elementary examples II

Example

If X^* is separable $\Longrightarrow$ A is SCD.

Proof.

- Take $\{x_n^* : n \in \mathbb{N}\}$ dense in S_{X^*}.
- For every $n, m \in \mathbb{N}$, consider $S_{n,m} = S(A, x_n^*, 1/m)$.
- It is easy to show that any slice of A contains one of the $S_{n,m}$

Example

$B_{C[0,1]}$ and $B_{L_1[0,1]}$ are not SCD.

SCD sets: Further examples I

Convex combination of slices

$W = \sum_{k=1}^{m} \lambda_k S_k \subset A$ where $\lambda_k \geqslant 0$, $\sum \lambda_k = 1$, S_k slices.

Proposition

In the definition of SCD we can use a sequence $\{S_n : n \in \mathbb{N}\}$ of convex combination of slices.

Small combinations of slices

A has small combinations of slices iff every slice of A contains convex combinations of slices of A with arbitrary small diameter.

Example

If A has small combinations of slices + separable $\Longrightarrow$ A is SCD.

Particular case

A strongly regular (in particular, PCP) + separable $\Longrightarrow$ A is SCD.

SCD sets: Further examples II

Bourgain's lemma

Every relative weak open subset of A contains a convex combination of slices.

Corollary

In the definition of SCD we can use a sequence $\{S_n : n \in \mathbb{N}\}$ of relative weak open subsets.

π-bases

A π-base of the weak topology of A is a family $\{V_i : i \in I\}$ of weak open sets of A such that every weak open subset of A contains one of the V_i's.

Proposition

If $(A, \sigma(X, X^*))$ has a countable π-base $\Longrightarrow$ A is SCD.

SCD sets: Further examples III

Theorem

A separable without ℓ_1-sequences $\Longrightarrow$ $(A, \sigma(X, X^*))$ has a countable π-base.

Proof.

- We see $(A, \sigma(X, X^*)) \subset C(T)$ where $T = (B_{X^*}, \sigma(X^*, X))$.
- By Rosenthal ℓ_1 theorem, $(A, \sigma(X, X^*))$ is a relatively compact subset of the space of first Baire class functions on T.
- By a result of Todorčević, $(A, \sigma(X, X^*))$ has a σ-disjoint π-base.
- $\{V_i : i \in I\}$ is *σ-disjoint* if $I = \bigcup_{n\in\mathbb{N}} I_n$ and each $\{V_i : i \in I_n\}$ is pairwise disjoint.
- A σ-disjoint family of open subsets in a separable space is countable. ✓

Main example

A separable without ℓ_1-sequences $\Longrightarrow$ A is SCD.

SCD spaces: definition and examples

SCD space

X is Slicely Countably Determined (SCD) if so are its convex bounded subsets.

Examples of SCD spaces

1. X separable strongly regular. In particular, RNP, CPCP spaces.
2. X separable $X \not\supseteq \ell_1$. In particular, if X^* is separable.

Examples of NOT SCD spaces

1. $C[0,1]$, $L_1[0,1]$
2. Actually, every X containing (an isomorphic copy of) $C[0,1]$ or $L_1[0,1]$.
3. There is X with the Schur property which is not SCD.

Remark

- Every subspace of a SCD space is SCD.
- This is false for quotients.

SCD spaces: stability properties

Theorem

$Z \subset X$. If Z and X/Z are SCD $\Longrightarrow$ X is SCD.

Corollary

X separable NOT SCD $\Longrightarrow$ $X \supset \ell_1$ and

- If $\ell_1 \simeq Y \subset X \Longrightarrow X/Y$ contains a copy of ℓ_1.
- If $\ell_1 \simeq Y_1 \subset X \Longrightarrow$ there is $\ell_1 \simeq Y_2 \subset X$ with $Y_1 \cap Y_2 = 0$.

Corollary

$X_1, \ldots, X_m$ SCD $\Longrightarrow X_1 \oplus \cdots \oplus X_m$ SCD.

SCD spaces: stability properties II

Theorem

$X_1, X_2, \dots$ SCD, E with unconditional basis.

- $E \not\supseteq c_0 \Longrightarrow \left[\bigoplus_{n\in\mathbb{N}} X_n\right]_E$ SCD.
- $E \not\supseteq \ell_1 \Longrightarrow \left[\bigoplus_{n\in\mathbb{N}} X_n\right]_E$ SCD.

Examples

1. $c_0(\ell_1)$ and $\ell_1(c_0)$ are SCD.
2. $c_0 \otimes_\varepsilon c_0$, $c_0 \otimes_\pi c_0$, $c_0 \otimes_\varepsilon \ell_1$, $c_0 \otimes_\pi \ell_1$, $\ell_1 \otimes_\varepsilon \ell_1$, and $\ell_1 \otimes_\pi \ell_1$ are SCD.
3. $K(c_0)$ and $K(c_0, \ell_1)$ are SCD.
4. $\ell_2 \otimes_\varepsilon \ell_2 \equiv K(\ell_2)$ and $\ell_2 \oplus_\pi \ell_2 \equiv \mathcal{L}_1(\ell_2)$ are SCD

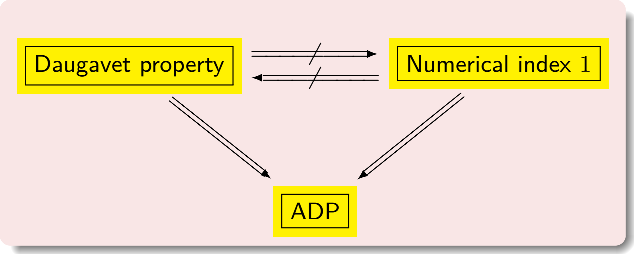
Applications

The DPr, the ADP and numerical index 1

Definition of the properties

1. **Kadets-Shvidkoy-Sirotkin-Werner, 1997:**
   X has the Daugavet property (DPr) if
   $$\|\mathrm{Id} + T\| = 1 + \|T\| \qquad \text{(DE)}$$
   for every rank-one $T \in L(X)$.
   - Then every T not fixing copies of ℓ_1 also satisfies (DE).
2. **Lumer, 1968:** X has numerical index 1 ($n(X) = 1$) if
   $$\max_{\theta \in \mathbb{T}} \|\mathrm{Id} + \theta T\| = 1 + \|T\| \qquad \text{(aDE)}$$
   for EVERY operator on X.
   - Equivalently,
     $$\|T\| = \sup\{|x^*(Tx)| : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}$$
     for EVERY $T \in L(X)$.
3. **M.-Oikhberg, 2004:** X has the alternative Daugavet property (ADP) if every rank-one $T \in L(X)$ satisfies (aDE).
   - Then every weakly compact T also satisfies (aDE).

Relations between these properties



Examples

- $C\big([0,1],K(\ell_2)\big)$ has DPr, but has not numerical index 1
- c_0 has numerical index 1, but has not DPr
- $c_0 \oplus_\infty C\big([0,1],K(\ell_2)\big)$ has ADP, neither DPr nor numerical index 1

Remarks

- For RNP or Asplund spaces, $\boxed{\text{ADP}} \Longrightarrow \boxed{\text{numerical index } 1}$.
- Every Banach space with the ADP can be renormed still having the ADP but failing the Daugavet property.

For C^*-algebras and preduals

Let V_* be the predual of the von Neumann algebra V.

The Daugavet property of V_* is equivalent to:

- V has no atomic projections, or
- the unit ball of V_* has no extreme points.

V_* has numerical index 1 iff:

- V is commutative, or
- $|v^*(v)| = 1$ for $v \in \operatorname{ext}(B_V)$ and $v^* \in \operatorname{ext}(B_{V^*})$.

The alternative Daugavet property of V_* is equivalent to:

- the atomic projections of V are central, or
- $|v(v_*)| = 1$ for $v \in \operatorname{ext}(B_V)$ and $v_* \in \operatorname{ext}(B_{V_*})$, or
- $V = C \oplus_\infty N$, where C is commutative and N has no atomic projections.

Let X be a C^*-algebra.

The Daugavet property of X is equivalent to:

- X does not have any atomic projection, or
- the unit ball of X^* does not have any w^*-strongly exposed point.

X has numerical index 1 iff:

- X is commutative, or
- $|x^{**}(x^*)| = 1$ for $x^{**} \in \operatorname{ext}(B_{X^{**}})$ and $x^* \in \operatorname{ext}(B_{X^*})$.

The alternative Daugavet property of X is equivalent to:

- the atomic projections of X are central, or
- $|x^{**}(x^*)| = 1$, for $x^{**} \in \operatorname{ext}(B_{X^{**}})$, and $x^* \in B_{X^*}$ w^*-strongly exposed, or
- $\exists$ a commutative ideal Y such that X/Y has the Daugavet property.

A sufficient condition for numerical index 1: lushness

Lushness (Boyko-Kadets-M.-Werner, 2007)

X is lush if given $x, y \in S_X$, $\varepsilon > 0$, there is $y^* \in S_{X^*}$ such that

$$x \in S = S(B_X, y^*, \varepsilon) \qquad \operatorname{dist}(y, \operatorname{conv}(\mathbb{T}S)) < \varepsilon.$$

Theorem (Boyko-Kadets-M.-Werner, 2007)

If X is lush, then X has numerical index 1

Example (Kadets-M.-Merí-Shepelska, 2009)

There is X with numerical index 1 which is not lush.

ADP + SCD $\Longrightarrow$ lushness

Characterization of ADP

X Banach space. TFAE:

- X has ADP (i.e. $\max_{\theta\in\mathbb{T}}\|\mathrm{Id}+\theta T\| = 1+\|T\|$ for all T rank-one).
- Given $x\in S_X$, a slice S of B_X and $\varepsilon>0$, there is $y\in S$ with

$$\max_{\theta\in\mathbb{T}}\|x+\theta y\|>2-\varepsilon.$$

- Given $x\in S_X$, a sequence $\{S_n\}$ of slices of B_X, and $\varepsilon>0$, there is $y^*\in S_{X^*}$ such that $x\in S(B_X,y^*,\varepsilon)$ and

$$\overline{\mathrm{conv}}\big(\mathbb{T}\,S(B_X,y^*,\varepsilon)\big)\bigcap S_n\neq\emptyset \qquad (n\in\mathbb{N}).$$

Theorem

X ADP + B_X SCD $\Longrightarrow$ given $x\in S_X$ and $\varepsilon>0$, there is $y^*\in S_{X^*}$ such that

$$x\in S(B_X,y^*,\varepsilon) \quad\text{and}\quad B_X=\overline{\mathrm{conv}}\big(\mathbb{T}\,S(B_X,y^*,\varepsilon)\big).$$

- This clearly implies lushness, and so numerical index 1 (i.e. $\max_{\theta\in\mathbb{T}}\|\mathrm{Id}+\theta T\|=1+\|T\|$ for all T).

Some consequences

Corollary

- ADP + strongly regular $\implies$ numerical index 1.
- ADP + $X \not\supseteq \ell_1$ $\implies$ numerical index 1.

Corollary

X real + $\dim(X) = \infty$ + ADP $\implies$ $X^* \supseteq \ell_1$.

In particular,

Corollary

X real + $\dim(X) = \infty$ + numerical index 1 $\implies$ $X^* \supseteq \ell_1$.

Some consequences II

Proposition (Kadets-M.-Merí-Werner, 2010)

- X with 1-unconditional basis $\Longrightarrow$ B_X is SCD.
- X with 1-unconditional basis and ADP $\Longrightarrow$ X is lush.

Theorem (Kadets-M.-Merí-Werner, 2010)

1. The unique Banach spaces with 1-symmetric basis and the ADP are c_0 and ℓ_1.
2. The unique r.i. Banach spaces over $\mathbb{N}$ with the ADP are c_0, ℓ_1 and ℓ_∞.
3. The unique separable r.i. Banach space on $[0,1]$ with the Daugavet property is $L_1[0,1]$.
4. The unique separable r.i. Banach space on $[0,1]$ which is lush is $L_1[0,1]$.

Question

Is it possible to prove the above results for the ADP **?**

SCD operators

SCD operator

$T \in L(X)$ is an SCD-operator if $T(B_X)$ is an SCD-set.

Examples

T is an SCD-operator when $T(B_X)$ is separable and

1. $T(B_X)$ is RPN,
2. $T(B_X)$ has no ℓ_1 sequences,
3. T does not fix copies of ℓ_1

Theorem

- X ADP + T SCD-operator $\implies \max_{\theta\in\mathbb{T}} \|\mathrm{Id} + \theta T\| = 1 + \|T\|$.
- X DPr + T SCD-operator $\implies \|\mathrm{Id} + T\| = 1 + \|T\|$.

Main corollary

X ADP + T does not fix copies of $\ell_1 \implies \max_{\theta\in\mathbb{T}} \|\mathrm{Id} + \theta T\| = 1 + \|T\|$.

Final remarks

Open questions

1. Find more sufficient conditions for a set to be SCD.
2. Is SCD equivalent to the existence of a countable π-base for the weak topology **?**
3. E with (1)-unconditional basis. Is E SCD **?**
4. E with 1-unconditional basis, $\{X_n\}$ a family of SCD spaces. Is $[\oplus X_n]_E$ SCD **?**
5. X, Y SCD. Are $X \otimes_\varepsilon Y$ and $X \otimes_\pi Y$ SCD **?**
6. $T : X \longrightarrow Y$ hereditary SCD, is there Z SCD-space such that T factor through Z **?**
7. Find a good extension of the SCD property to the nonseparable case.
8. Clarify the relationship between SCD and the Daugavet property.