

# Slicely Countably Determined Banach spaces

(Espacios de Banach determinados numerablemente por rebanadas)

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## SLICELY COUNTABLY DETERMINED BANACH SPACES

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**ABSTRACT.** We introduce the class of slicely countably determined Banach spaces which contains in particular all spaces with the RNP and all spaces without copies of  $\ell_1$ . We present many examples and several properties of this class. We give some applications to Banach spaces with the Daugavet and the alternative Daugavet properties, lush spaces and Banach spaces with numerical index 1. In particular, we show that the dual of a real infinite-dimensional Banach with the alternative Daugavet property contains  $\ell_1$  and that operators which do not fix copies of  $\ell_1$  on a space with the alternative Daugavet property satisfy the alternative Daugavet equation.

## Basic notation and main objective

### Basic notation

$X$  real or complex Banach space.

- $S_X$  unit sphere,  $B_X$  closed unit ball,  $\mathbb{T}$  modulus-one scalars.
- $X^*$  dual space,  $L(X)$  bounded linear operators.
- $\text{conv}(\cdot)$  convex hull,  $\overline{\text{conv}}(\cdot)$  closed convex hull
- A **slice** of  $A \subset X$  is a subset of the form

$$S(A, x^*, \alpha) = \{x \in A : \text{Re } x^*(x) > \sup \text{Re } x^*(A) - \alpha\} \quad (x^* \in X^*, \alpha > 0)$$

### Objective

- We introduce an isomorphic property for (separable) Banach spaces called **Slicely Countable Determined (SCD)** such that
  - it is satisfied by RNP spaces,
  - it is satisfied by spaces not containing  $\ell_1$ .
- We present some stability results.
- We apply SCD to get results for the Daugavet property, the alternative Daugavet property and spaces with numerical index 1.

# Outline

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## Slicely Countably Determined sets and spaces

## SCD sets: Definitions and preliminary remarks

$X$  Banach space,  $A \subset X$  bounded and convex.

### SCD sets

$A$  is **Slicely Countably Determined (SCD)** if there is a sequence  $\{S_n : n \in \mathbb{N}\}$  of slices of  $A$  satisfying one of the following equivalent conditions:

- every slice of  $A$  contains one of the  $S_n$ 's,
- $A \subseteq \overline{\text{conv}}(B)$  if  $B \subseteq A$  satisfies  $B \cap S_n \neq \emptyset \forall n$ ,
- given  $\{x_n\}_{n \in \mathbb{N}}$  with  $x_n \in S_n \forall n \in \mathbb{N}$ ,  $A \subseteq \overline{\text{conv}}(\{x_n : n \in \mathbb{N}\})$ .

### Remarks

- $A$  is SCD iff  $\overline{A}$  is SCD.
- If  $A$  is SCD, then it is separable.

## SCD sets: Elementary examples I

## Example

A separable and  $A = \overline{\text{conv}}(\text{dent}(A)) \implies A$  is SCD.

Proof.

- Take  $\{a_n : n \in \mathbb{N}\}$  denting points with  $A = \overline{\text{conv}}(\{a_n : n \in \mathbb{N}\})$ .
- For every  $n, m \in \mathbb{N}$ , take a slice  $S_{n,m}$  containing  $a_n$  and of diameter  $1/m$ .
- If  $B \cap S_{n,m} \neq \emptyset \implies a_n \in \overline{B}$ .
- Therefore,  $A = \overline{\text{conv}}(\{a_n : n \in \mathbb{N}\}) \subseteq \overline{\text{conv}}(\overline{B}) = \overline{\text{conv}}(B)$ .

## Example

In particular,  $A$  RNP separable  $\implies A$  SCD.

## Corollary

- If  $X$  is separable LUR  $\implies B_X$  is SCD.
- So, every separable space can be renormed such that  $B_{(X,|\cdot|)}$  is SCD.

## SCD sets: Elementary examples II

## Example

If  $X^*$  is separable  $\implies A$  is SCD.

Proof.

- Take  $\{x_n^* : n \in \mathbb{N}\}$  dense in  $S_{X^*}$ .
- For every  $n, m \in \mathbb{N}$ , consider  $S_{n,m} = S(A, x_n^*, 1/m)$ .
- It is easy to show that any slice of  $A$  contains one of the  $S_{n,m}$

## Example

$B_{C[0,1]}$  and  $B_{L_1[0,1]}$  are not SCD.



## SCD sets: Further examples I

### Convex combination of slices

$$W = \sum_{k=1}^m \lambda_k S_k \subset A \text{ where } \lambda_k \geq 0, \sum \lambda_k = 1, S_k \text{ slices.}$$

### Proposition

In the definition of SCD we can use a sequence  $\{S_n : n \in \mathbb{N}\}$  of convex combination of slices.

### Small combinations of slices

$A$  has **small combinations of slices** iff every slice of  $A$  contains convex combinations of slices of  $A$  with arbitrary small diameter.

### Example

If  $A$  has small combinations of slices + separable  $\implies A$  is SCD.

### Particular case

$A$  strongly regular (in particular, PCP) + separable  $\implies A$  is SCD.

## SCD sets: Further examples II

### Bourgain's lemma

Every relative weak open subset of  $A$  contains a convex combination of slices.

### Corollary

In the definition of SCD we can use a sequence  $\{S_n : n \in \mathbb{N}\}$  of relative weak open subsets.

### $\pi$ -bases

A  $\pi$ -base of the weak topology of  $A$  is a family  $\{V_i : i \in I\}$  of weak open sets of  $A$  such that every weak open subset of  $A$  contains one of the  $V_i$ 's.

### Proposition

If  $(A, \sigma(X, X^*))$  has a countable  $\pi$ -base  $\implies A$  is SCD.

## SCD sets: Further examples III

## Theorem

A separable without  $\ell_1$ -sequences  $\implies (A, \sigma(X, X^*))$  has a countable  $\pi$ -base.

Proof.

- We see  $(A, \sigma(X, X^*)) \subset C(T)$  where  $T = (B_{X^*}, \sigma(X^*, X))$ .
- By Rosenthal  $\ell_1$  theorem,  $(A, \sigma(X, X^*))$  is a relatively compact subset of the space of first Baire class functions on  $T$ .
- By a result of Todorčević,  $(A, \sigma(X, X^*))$  has a  $\sigma$ -disjoint  $\pi$ -base.
- $\{V_i : i \in I\}$  is  $\sigma$ -disjoint if  $I = \bigcup_{n \in \mathbb{N}} I_n$  and each  $\{V_i : i \in I_n\}$  is pairwise disjoint.
- A  $\sigma$ -disjoint family of open subsets in a separable space is countable. ✓

## Main example

A separable without  $\ell_1$ -sequences  $\implies A$  is SCD.

## SCD spaces: definition and examples

### SCD space

$X$  is **Slicely Countably Determined (SCD)** if so are its convex bounded subsets.

### Examples of SCD spaces

- 1  $X$  separable strongly regular. In particular, RNP, CPCP spaces.
- 2  $X$  separable  $X \not\cong \ell_1$ . In particular, if  $X^*$  is separable.

### Examples of NOT SCD spaces

- 1  $C[0,1], L_1[0,1]$
- 2 Actually, every  $X$  containing (an isomorphic copy of)  $C[0,1]$  or  $L_1[0,1]$ .
- 3 There is  $X$  with the Schur property which is not SCD.

### Remark

- Every subspace of a SCD space is SCD.
- This is false for quotients.

## SCD spaces: stability properties

## Theorem

$Z \subset X$ . If  $Z$  and  $X/Z$  are SCD  $\implies X$  is SCD.

## Corollary

$X$  separable NOT SCD  $\implies X \supset \ell_1$  and

- If  $\ell_1 \simeq Y \subset X \implies X/Y$  contains a copy of  $\ell_1$ .
- If  $\ell_1 \simeq Y_1 \subset X \implies$  there is  $\ell_1 \simeq Y_2 \subset X$  with  $Y_1 \cap Y_2 = 0$ .

## Corollary

$X_1, \dots, X_m$  SCD  $\implies X_1 \oplus \dots \oplus X_m$  SCD.

## SCD spaces: stability properties II

## Theorem

$X_1, X_2, \dots$  SCD,  $E$  with unconditional basis.

- $E \not\subseteq c_0 \implies \left[ \bigoplus_{n \in \mathbb{N}} X_n \right]_E$  SCD.
- $E \not\subseteq \ell_1 \implies \left[ \bigoplus_{n \in \mathbb{N}} X_n \right]_E$  SCD.

## Examples

- 1  $c_0(\ell_1)$  and  $\ell_1(c_0)$  are SCD.
- 2  $c_0 \otimes_{\varepsilon} c_0$ ,  $c_0 \otimes_{\pi} c_0$ ,  $c_0 \otimes_{\varepsilon} \ell_1$ ,  $c_0 \otimes_{\pi} \ell_1$ ,  $\ell_1 \otimes_{\varepsilon} \ell_1$ , and  $\ell_1 \otimes_{\pi} \ell_1$  are SCD.
- 3  $K(c_0)$  and  $K(c_0, \ell_1)$  are SCD.
- 4  $\ell_2 \otimes_{\varepsilon} \ell_2 \equiv K(\ell_2)$  and  $\ell_2 \oplus_{\pi} \ell_2 \equiv \mathcal{L}_1(\ell_2)$  are SCD

Introduction  
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SCD sets & spaces  
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**Applications**  
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Open questions  
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## Applications

# The DPr, the ADP and numerical index 1

## Definition of the properties

① **Kadets-Shvidkoy-Sirotkin-Werner, 1997:**

$X$  has the **Daugavet property (DPr)** if

$$\|\text{Id} + T\| = 1 + \|T\| \quad (\text{DE})$$

for every rank-one  $T \in L(X)$ .

- Then every  $T$  not fixing copies of  $\ell_1$  also satisfies (DE).

② **Lumer, 1968:**  $X$  has **numerical index 1** if EVERY operator on  $X$  satisfies

$$\max_{\theta \in \mathbb{T}} \|\text{Id} + \theta T\| = 1 + \|T\| \quad (\text{aDE})$$

- Equivalently,

$$\|T\| = \sup\{|x^*(Tx)| : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}$$

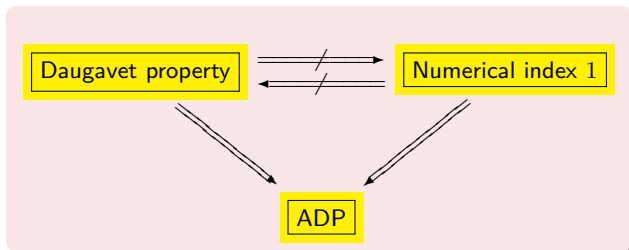
for EVERY  $T \in L(X)$ .

③ **M.-Oikhberg, 2004:**  $X$  has the **alternative Daugavet property (ADP)** if every rank-one  $T \in L(X)$  satisfies (aDE).

- Then every weakly compact  $T$  also satisfies (aDE).



## Relations between these properties



### Examples

- $C([0, 1], K(\ell_2))$  has DPr, but has not numerical index 1
- $c_0$  has numerical index 1, but has not DPr
- $c_0 \oplus_\infty C([0, 1], K(\ell_2))$  has ADP, neither DPr nor numerical index 1

### Remarks

- For RNP or Asplund spaces,  $\boxed{\text{ADP}} \implies \boxed{\text{numerical index 1}}$ .
- Every Banach space with the ADP can be renormed still having the ADP but failing the Daugavet property.

ADP + SCD  $\implies$  numerical index 1. Consequences

## Theorem

$X$  ADP +  $B_X$  SCD  $\implies$  numerical index 1 (actually, a little bit more).

## Corollary

- ADP + strongly regular  $\implies$  numerical index 1.
- ADP +  $X \not\supseteq \ell_1 \implies$  numerical index 1.

## Corollary

$X$  real +  $\dim(X) = \infty$  + ADP  $\implies X^* \supseteq \ell_1$ .

## Corollary

$X$  real +  $\dim(X) = \infty$  + numerical index 1  $\implies X^* \supseteq \ell_1$ .

## SCD operators

### SCD operator

$T \in L(X)$  is an **SCD-operator** if  $T(B_X)$  is an SCD-set.

### Examples

$T$  is an SCD-operator when  $T(B_X)$  is separable and

- 1  $T(B_X)$  is RPN,
- 2  $T(B_X)$  has no  $\ell_1$  sequences,
- 3  $T$  does not fix copies of  $\ell_1$

### Theorem

- $X$  ADP +  $T$  SCD-operator  $\implies \max_{\theta \in \mathbb{T}} \|\text{Id} + \theta T\| = 1 + \|T\|$ .
- $X$  DPr +  $T$  SCD-operator  $\implies \|\text{Id} + T\| = 1 + \|T\|$ .

### Main corollary

$X$  ADP +  $T$  does not fix copies of  $\ell_1 \implies \max_{\theta \in \mathbb{T}} \|\text{Id} + \theta T\| = 1 + \|T\|$ .

## Open questions

### On SCD-sets

- Find more sufficient conditions for a set to be SCD.
- For instance, if  $X$  has 1-symmetric basis, is  $B_X$  an SCD-set ?
- Is SCD equivalent to the existence of a countable  $\pi$ -base for the weak topology ?

### On SCD-spaces

- $E$  with unconditional basis. Is  $E$  SCD ?
- $X, Y$  SCD. Are  $X \otimes_\varepsilon Y$  and  $X \otimes_\pi Y$  SCD ?

### On SCD-operators

- $T_1, T_2$  SCD-operators, is  $T_1 + T_2$  an SCD-operator ?
- $T : X \rightarrow Y$  hereditary SCD, is there  $Z$  SCD-space such that  $T$  factor through  $Z$  ?