The Daugavet property

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The talk is mainly based on these papers

J. Becerra Guerrero and M. Martín,
The Daugavet Property of C^*-algebras, JB^*-triples, and of their isometric preduals.
Journal of Functional Analysis (2005)

J. Becerra Guerrero and M. Martín,
The Daugavet property for Lindenstrauss spaces
in: Methods in Banach space theory (Jesus M.F. Castillo and William B. Johnson Editors), London Mathematical Society Lecture Note Series 337, 2006.

V. Kadets, M. Martín, and J. Merí.
Norm equalities for operators on Banach spaces.
Indiana U. Math. J. (2007).

M. Martín,
The alternative Daugavet property of C^*-algebras and JB^*-triples.
Mathematische Nachrichten (2008)

M. Martín and T. Oikhberg,
An alternative Daugavet property.
Journal of Mathematical Analysis and Applications (2004)

Outline

Introduction

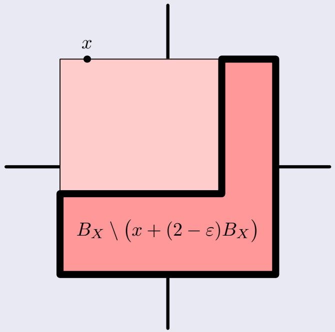
- In a Banach space X with the Radon-Nikodým property the unit ball has many denting points.
- $x \in S_X$ is a denting point of B_X if for every $\varepsilon > 0$ one has

$$x \notin \overline{\text{co}}\left(B_X \setminus (x + \varepsilon\, B_X)\right).$$

- $C[0,1]$ and $L_1[0,1]$ have an extremely opposite property: for every $x \in S_X$ and every $\varepsilon > 0$

$$\overline{\text{co}}\,\Big(B_X \setminus \big(x + (2-\varepsilon)B_X\big)\Big) = B_X\,.$$

- This geometric property is equivalent to a property of operators on the space.



The Daugavet equation

X Banach space, $T \in L(X)$

$$\|\mathrm{Id} + T\| = 1 + \|T\| \qquad \text{(DE)}$$

Classical examples

1. **Daugavet, 1963:**
   Every compact operator on $C[0,1]$ satisfies (DE).
2. **Lozanoskii, 1966:**
   Every compact operator on $L_1[0,1]$ satisfies (DE).
3. **Abramovich, Holub, and more, 80's:**
   $X = C(K)$, K perfect compact space
   or $X = L_1(\mu)$, μ atomless measure
   $\Longrightarrow$ every weakly compact $T \in L(X)$ satisfies (DE).

The Daugavet property

- A Banach space X is said to have the Daugavet property if every rank-one operator on X satisfies (DE).
- If X^* has the Daugavet property, so does X. The converse is not true:

  $C[0,1]$ has it but $C[0,1]^*$ not.

(Kadets–Shvidkoy–Sirotkin–Werner, 1997 & 2000)

Prior versions of: *Chauveheid, 1982; Abramovich–Aliprantis–Burkinshaw, 1991*

Some examples...

1. K perfect, μ atomeless, E arbitrary Banach space
   $\Longrightarrow C(K,E)$, $L_1(\mu,E)$, and $L_\infty(\mu,E)$ have the Daugavet property.

   (Kadets, 1996; Nazarenko, –; Shvidkoy, 2001)
2. $A(\mathbb{D})$ and H^∞ have the Daugavet property.

   (Wojtaszczyk, 1992)

More examples...

3. A function algebra whose Choquet boundary is perfect has the Daugavet property.

   (Werner, 1997)

4. "Large" subspaces of $C[0,1]$ and $L_1[0,1]$ have the Daugavet property (in particular, this happends for finite-codimensional subspaces).

   (Kadets–Popov, 1997)

5. A C^*-algebra has the Daugavet property if and only if it is non-atomic.
6. The predual of a von Neumann algebra has the Daugavet property if and only if the algebra is non-atomic.

   (Oikhberg, 2002)

7. Lip(K) when $K \subseteq \mathbb{R}^n$ is compact and convex.

   (Ivankhno, Kadets, Werner, 2007)

Some *propaganda*...

Let X be a Banach space with the Daugavet property. Then

- X does not have the Radon-Nikodým property.

  (Wojtaszczyk, 1992)

- Every slice of B_X and every w^*-slice of B_{X^*} have diameter 2.

  (Kadets–Shvidkoy–Sirotkin–Werner, 2000)

- Actually, every weakly-open subset of B_X has diameter 2.

  (Shvidkoy, 2000)

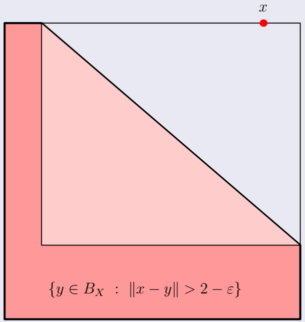
- X contains a copy of ℓ_1. X^* contains a copy or $L_1[0,1]$.
- Actually, given $x_0 \in S_X$ and slices $\{S_n : n \geqslant 1\}$, one may take $x_n \in S_n$ $\forall n \geqslant 1$ such that $\{x_n : n \geqslant 0\}$ is equivalent to the ℓ_1-basis.

  (Kadets–Shvidkoy–Sirotkin–Werner, 2000)

Geometric characterizations

Theorem [KSSW]. TFAE:

- X has the Daugavet property.
- For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y \in S_X$ such that
$$\operatorname{Re} x^*(y) > 1 - \varepsilon \quad \text{and} \quad \|x - y\| \geqslant 2 - \varepsilon.$$
- For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y^* \in S_{X^*}$ such that
$$\operatorname{Re} y^*(x) > 1 - \varepsilon \quad \text{and} \quad \|x^* - y^*\| \geqslant 2 - \varepsilon.$$
- For every $x \in S_X$ and every $\varepsilon > 0$, we have
$$B_X = \overline{\text{co}}\left(\{y \in B_X \ : \ \|x - y\| \geqslant 2 - \varepsilon\}\right).$$



From rank-one to other classes of operators

Theorem

Let X be a Banach space with the Daugavet property.

- Every weakly compact operator on X satisfies (DE).

  (Kadets–Shvidkoy–Sirotkin–Werner, 2000)

- Actually, every operator on X not fixing a copy of ℓ_1 satisfies (DE).

  (Sirotkin, 2000)

Consequences

1. X does not have unconditional basis.

   (Kadets, 1996)

2. Moreover, X does not embed into any space with unconditional basis.

   (Kadets–Shvidkoy–Sirotkin–Werner, 2000)

3. Actually, X does not embed into an unconditional sum of Banach spaces without a copy of ℓ_1.

   (Shvidkoy, 2000)

A sufficient condition

2 A sufficient condition

A sufficient condition

Theorem

Let X be a Banach space such that

$$X^* = Y \oplus_1 Z$$

with Y and Z norming subspaces. Then, X has the Daugavet property.

A closed subspace $W \subseteq X^*$ is norming if

$$\|x\| = \sup\{|w^*(x)| \ : \ w^* \in W, \ \|w^*\| = 1\}$$

or, equivalently, if B_W is w^*-dense in B_{X^*}.

Proof of the theorem

We have...

$X^* = Y \oplus_1 Z,$
$B_Y,\ B_Z$ w^*-dense in B_{X^*}.

$\overset{?}{\Longrightarrow}$

We need...

fixed $x_0 \in S_X$, $x_0^* \in S_{X^*}$, $\varepsilon > 0$, find $y^* \in S_{X^*}$ such that

$\|x_0^* + y^*\| > 2 - \varepsilon$ and $\operatorname{Re} y^*(x_0) > 1 - \varepsilon$.

- Write $x_0^* = y_0^* + z_0^*$ with $y_0^* \in Y$, $z_0^* \in Z$, $\|x_0^*\| = \|y_0^*\| + \|z_0^*\|$, and write

$$U = \{x^* \in B_{X^*} \ :\ \operatorname{Re} x^*(x_0) > 1 - \varepsilon/2\}.$$

- Take $z^* \in B_Z \cap U$ and a net (y_λ^*) in $B_Y \cap U$, such that $(y_\lambda^*) \xrightarrow{w^*} z^*$.
- $(y_\lambda^* + y_0^*) \longrightarrow z^* + y_0^*$ and the norm is w^*-lower semi-continuous, so

$$\liminf \|y_\lambda^* + y_0^*\| \geqslant \|z^* + y_0^*\| = \|z^*\| + \|y_0^*\| > 1 + \|y_0^*\| - \varepsilon/2.$$

- Then we may find μ such that $\|y_\mu^* + y_0^*\| \geqslant 1 + \|y_0^*\| - \varepsilon/2$.
- Finally, observe that

$$\|x_0^* + y_\mu^*\| = \|(y_0^* + y_\mu^*) + z_0^*\| =$$
$$= \|y_0^* + y_\mu^*\| + \|z_0^*\| > 1 + \|y_0^*\| - \varepsilon + \|z_0^*\| = 2 - \varepsilon,$$

and that $\operatorname{Re} y_\mu^*(x_0) > 1 - \varepsilon$ (since $y_\mu^* \in U$). ✓

Some immediate consequences

Corollary

Let X be an L-embedded space with $\operatorname{ext}(B_X) = \emptyset$. Then, X^* (and hence X) has the Daugavet property.

Corollary

If Y is an L-embedded space which is a subspace of $L_1 \equiv L_1[0,1]$, then $(L_1/Y)^*$ has the Daugavet property.

It was already known that...

- If $Y \subset L_1$ is reflexive, then L_1/Y has the Daugavet property.
  (Kadets–Shvidkoy–Sirotkin–Werner, 2000)
- If $Y \subset L_1$ is L-embedded, then L_1/Y does not have the RNP.
  (Harmand–Werner–Werner, 1993)

Application:

The Daugavet property of

C^*-algebras and von Neumann preduals

3. Application: C^*-algebras and von Neumann preduals
   - von Neumann preduals
   - C^*-algebras

von Neumann preduals

von Neumann preduals

- A C^*-algebra X is a von Neumann algebra if it is a dual space.
- In such a case, X has a unique predual X_*.
- X_* is always L-embedded.
- Therefore, if $\operatorname{ext}(B_{X_*})$ is empty, then X and X_* have the Daugavet property.
  Example: $L_\infty[0,1]$ and $L_1[0,1]$.

Actually, much more can be proved:

Theorem

Let X_* be the predual of the von Neumann algebra X. Then, TFAE:

- X has the Daugavet property.
- X_* has the Daugavet property.
- Every weakly open subset of B_{X_*} has diameter 2.
- B_{X_*} has no strongly exposed points.
- B_{X_*} has no extreme points.
- X is non-atomic (i.e. it has no atomic projections).

An atomic projection is an element $p \in X$ such that

$$p^2 = p^* = p \qquad \text{and} \qquad p\,X\,p = \mathbb{C}p.$$

C^*-algebras

Let X be a C^*-algebra. Then, X^{**} is a von Neumann algebra.
Write $X^* = (X^{**})_* = A \oplus_1 N$, where

- A is the atomic part,
- N is the non-atomic part.

- Every extreme point of B_{X^*} is in B_A.
- Therefore, A is norming.
- What's about N **?**

Theorem

If X is non-atomic, then N is norming. So, X has the Daugavet property.
Example: $C[0,1]$

sketch of the proof of the theorem

We have...

X non-atomic C^*-algebra,
$X^* = A \oplus_1 N$.

$\overset{?}{\Longrightarrow}$

We need...

N to be norming for X, i.e.
$\|x\| = \sup\{|f(x)| \ : \ f \in B_N\} \qquad (x \in X)$.

- Write $X^{**} = \mathcal{A} \oplus_\infty \mathcal{N}$ and $Y = \mathcal{A} \cap X$.
- Y is an ideal of X, so Y has no atomic projections.
- Therefore, the norm of Y has no point of Fréchet-smoothness.
- But Y is an Asplund space, so $Y = 0$.
- Now, the mapping
$$X \hookrightarrow X^{**} = \mathcal{A} \oplus_\infty \mathcal{N} \twoheadrightarrow \mathcal{N}$$
in injective. Since it is an homomorphism, it is an isometry.
- But $N^* \equiv \mathcal{N}$, so N is norming for $\mathcal{N}$ and now, also for X. $\checkmark$

Theorem

Let X be a C^*-algebra. Then, TFAE:

- X has the Daugavet property.
- The norm of X is extremely rough, i.e.

$$\limsup_{\|h\|\to 0} \frac{\|x+h\| + \|x-h\| - 2}{\|h\|} = 2$$

  for every $x \in S_X$ (equivalently, every w^*-slice of B_{X^*} has diameter 2).
- The norm of X is not Fréchet-smooth at any point.
- X is non-atomic.

The Daugavet property of Lindenstrauss spaces

4 Lindenstrauss spaces

Lindenstrauss spaces

Definition

X is a Lindenstrauss space if $X^* \equiv L_1(\mu)$ (isometrically) for some measure μ.

Examples

1. $C(K)$ spaces, $C_0(L)$ spaces.
2. $A(K)$ (affine continuous functions on a Choquet simplex).
3. G-spaces due to Grothendieck...

An equivalent relation

X Lindenstrauss space, $f, g \in \operatorname{ext}(B_{X^*})$,

$$f \sim g \qquad \Longleftrightarrow \qquad \dim_{\mathbb{K}}(\{f, g\}) = 1.$$

Werner, 1997

X Lindenstrauss space.
$\big(\operatorname{ext}(B_{X^*}) / \sim, w^*\big)$ perfect $\implies$ X has the Daugavet property.

But more can be said...

The result

Theorem (Becerra-M., 2006)

X Lindenstrauss space. TFAE:

1. X has the Daugavet property.
2. The norm of X is extremely rough.
3. The norm of X is not Fréchet-smooth at any point.
4. $\operatorname{ext}(B_{X^*})/\sim$ does not have any isolated point.

★ Only $(3) \Longrightarrow (4)$ needs proof.

★ We use the following result which is of independent interest:

Proposition

X Banach space, $f \in \operatorname{ext}(B_{X^*})$ which $[f]$ isolated point of $\operatorname{ext}(B_{X^*})/\sim$
$\Longrightarrow$ f w^*-strongly exposed point of B_{X^*}.

★ It is a consequence of Choquet's lemma.

The alternative Daugavet equation

5. The alternative Daugavet equation
   - Definitions and basic results
   - Geometric characterizations
   - C^*-algebras and preduals

The alternative Daugavet equation

The alternative Daugavet equation

X Banach space, $T \in L(X)$

$$\max_{|\omega|=1} \|\mathrm{Id} + \omega\, T\| = 1 + \|T\| \qquad \text{(aDE)}$$

(Duncan–McGregor–Pryce–White, 1970; Holub, Abramovich..., 80's)

Two equivalent formulations

- There exists $\omega \in \mathbb{T}$ such that $\omega\, T$ satisfies (DE).
- The numerical radius of T, $v(T)$, coincides with $\|T\|$, where

$$v(T) := \sup\{|x^*(Tx)| \ : \ x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1\}.$$

Two possible properties

Let X be a Banach space.

- X is said to have the alternative Daugavet property (ADP) iff every rank-one operator on X satisfies (aDE).
  - Then, every weakly compact operator also satisfies (aDE).
  - If X^* has the ADP, so does X. The converse is not true: $C([0,1],\ell_2)$.

(M.–Oikhberg, 2004; briefly appearance: Abramovich, 1991)

- X is said to have numerical index 1 iff $v(T) = \|T\|$ for every operator on X. Equivalently, if EVERY operator on X satisfies (aDE).

(Lumer, 1968; Duncan–McGregor–Pryce–White, 1970)

The numerical index of a Banach space X is the greater constant k such that

$$v(T) \geqslant k\,\|T\|$$

for every operator $T \in L(X)$.

Numerical index 1

- $C(K)$ and $L_1(\mu)$ have numerical index 1.

  (Duncan–McGregor–Pryce–White, 1970)
- All function algebras have numerical index 1.

  (Werner, 1997)
- A C^*-algebra has numerical index 1 iff it is commutative.

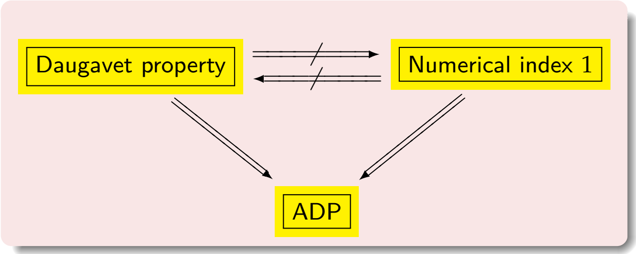
  (Huruya, 1977; Kaidi–Morales–Rodríguez-Palacios, 2000)
- In case $\dim(X) < \infty$, X has numerical index 1 iff

$$|x^*(x)| = 1 \qquad x^* \in \operatorname{ext}(B_{X^*}),\ x \in \operatorname{ext}(B_X).$$

  (McGregor, 1971)
- If $\dim(X) = \infty$ and X has numerical index 1, then $X^* \supseteq \ell_1$.

  (Avilés–Kadets–M.–Merí–Shepelska, 2009)

The alternative Daugavet property

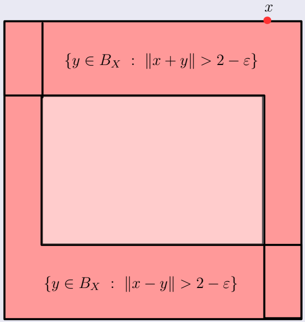


- $c_0 \oplus_\infty C([0,1],\ell_2)$ has the ADP, but neither the Daugavet property, nor numerical index 1.
- For RNP or Asplund spaces, the ADP implies numerical index 1.
- Every Banach space with the ADP can be renormed still having the ADP but failing the Daugavet property.

Geometric characterizations

Theorem. TFAE:

- X has the ADP.
- For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y \in S_X$ such that
$$|x^*(y)| > 1 - \varepsilon \quad \text{and} \quad \|x - y\| \geqslant 2 - \varepsilon.$$
- For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y^* \in S_{X^*}$ such that
$$|y^*(x)| > 1 - \varepsilon \quad \text{and} \quad \|x^* - y^*\| \geqslant 2 - \varepsilon.$$
- For every $x \in S_X$ and every $\varepsilon > 0$, we have
$$B_X = \overline{\text{co}}\left(\mathbb{T}\left\{y \in B_X \ : \ \|x - y\| \geqslant 2 - \varepsilon\right\}\right).$$



Let V_* be the predual of the von Neumann algebra V.

The Daugavet property of V_* is equivalent to:

- V has no atomic projections, or
- the unit ball of V_* has no extreme points.

V_* has numerical index 1 iff:

- V is commutative, or
- $|v^*(v)| = 1$ for $v \in \operatorname{ext}(B_V)$ and $v^* \in \operatorname{ext}(B_{V^*})$.

The alternative Daugavet property of V_* is equivalent to:

- the atomic projections of V are central, or
- $|v(v_*)| = 1$ for $v \in \operatorname{ext}(B_V)$ and $v_* \in \operatorname{ext}(B_{V_*})$, or
- $V = C \oplus_\infty N$, where C is commutative and N has no atomic projections.

Let X be a C^*-algebra.

The Daugavet property of X is equivalent to:

- X does not have any atomic projection, or
- the unit ball of X^* does not have any w^*-strongly exposed point.

X has numerical index 1 iff:

- X is commutative, or
- $|x^{**}(x^*)| = 1$ for $x^{**} \in \operatorname{ext}(B_{X^{**}})$ and $x^* \in \operatorname{ext}(B_{X^*})$.

The alternative Daugavet property of X is equivalent to:

- the atomic projections of X are central, or
- $|x^{**}(x^*)| = 1$, for $x^{**} \in \operatorname{ext}(B_{X^{**}})$, and $x^* \in B_{X^*}$ w^*-strongly exposed, or
- $\exists$ a commutative ideal Y such that X/Y has the Daugavet property.

Norm equalities for operators

6 Norm equalities for operators
- The equations
- Extremely non-complex Banach spaces

Norm equalities for operators

Motivating question

Are there other norm equalities like Daugavet equation which could define interesting properties of Banach spaces **?**

Concretely

We looked for non-trivial norm equalities of the forms

$$\|g(T)\| = f(\|T\|) \quad \text{or} \quad \|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$$

(g analytic, f arbitrary) satisfied by all rank-one operators on a Banach space.

Solution

We proved that there are few possibilities. . .

Equalities of the form $\|g(T)\| = f(\|T\|)$

Theorem

X real or complex with $\dim(X) \geqslant 2$.
Suppose that the norm equality

$$\|g(T)\| = f(\|T\|)$$

holds for every rank-one operator $T \in L(X)$, where

- $g : \mathbb{K} \longrightarrow \mathbb{K}$ is analytic,
- $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ is arbitrary.

Then, there are $a, b \in \mathbb{K}$ such that

$$g(\zeta) = a + b\,\zeta \qquad (\zeta \in \mathbb{K}).$$

Corollary

Only three norm equalities of the form

$$\|g(T)\| = f(\|T\|)$$

are possible:

- $b = 0$: $\|a\,\mathrm{Id}\| = |a|$,
- $a = 0$: $\|b\,T\| = |b|\,\|T\|$,

  (trivial cases)
- $a \neq 0$, $b \neq 0$:
  $\|a\,\mathrm{Id} + b\,T\| = |a| + |b|\,\|T\|$,

  (Daugavet property)

Equalities of the form $\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$

Remark

If X has the Daugavet property and g is analytic, then

$$\|\mathrm{Id} + g(T)\| = |1 + g(0)| - |g(0)| + \|g(T)\|$$

for every rank-one $T \in L(X)$.

- Our aim here is not to show that g has a suitable form,
- but it is to see that for every g another simpler equation can be found.
- From now on, we have to separate the complex and the real case.

Equalities of the form $\|\mathrm{Id}+g(T)\| = f(\|g(T)\|)$

- COMPLEX CASE:

Proposition

X complex, $\dim(X) \geqslant 2$. Suppose that

$$\|\mathrm{Id}+g(T)\| = f(\|g(T)\|)$$

for every rank-one $T \in L(X)$, where

- $g : \mathbb{C} \longrightarrow \mathbb{C}$ analytic non-constant,
- $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ *continuous.*

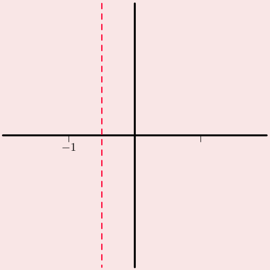
Then

$$\big\|(1+g(0))\mathrm{Id}+T\big\| = |1+g(0)| - |g(0)| + \big\|g(0)\mathrm{Id}+T\big\|$$

for every rank-one $T \in L(X)$.

We obtain two different cases:

- $|1+g(0)| - |g(0)| \neq 0$ or
- $|1+g(0)| - |g(0)| = 0$.



Equalities of the form $\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$. Complex case

Theorem

If $\operatorname{Re} g(0) \neq -1/2$ and

$$\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$$

for every rank-one T, then X has the Daugavet property.

Theorem

If $\operatorname{Re} g(0) = -1/2$ and

$$\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$$

for every rank-one T, then exists $\theta_0 \in \mathbb{R}$ s.t.

$$\left\|\mathrm{Id} + e^{i\theta_0}\, T\right\| = \|\mathrm{Id} + T\|$$

for every rank-one $T \in L(X)$.

Example

If $X = C[0,1] \oplus_2 C[0,1]$, then

- $\left\|\mathrm{Id} + e^{i\theta}\, T\right\| = \|\mathrm{Id} + T\|$ for every $\theta \in \mathbb{R}$, rank-one $T \in L(X)$.
- X does not have the Daugavet property.

Equalities of the form $\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$. Real case

- REAL CASE:

Remarks

- The proofs are not valid (we use Picard's Theorem).
- They work when g is onto.
- But we do not know what is the situation when g is not onto, even in the easiest examples:
  - $\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$,
  - $\|\mathrm{Id} - T^2\| = 1 + \|T^2\|$.

$g(0) = -1/2$:

Example

If $X = C[0,1] \oplus_2 C[0,1]$, then

- $\|\mathrm{Id} - T\| = \|\mathrm{Id} + T\|$ for every rank-one $T \in L(X)$.
- X does not have the Daugavet property.

Extremely non-complex Banach spaces: the motivating question

Godefroy, private communication

Is there any real Banach space X (with $\dim(X) > 1$) such that

$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$$

for every operator $T \in L(X)$ **?**

Definition

X is extremely non-complex if $\mathrm{dist}(T^2, -\mathrm{Id})$ is the maximum possible, i.e.

$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\| \qquad (\forall T \in L(X))$$

The solution (Koszmider–M.–Merí, 2009)

Existence

There are infinitely many nonisomorphic extremely non-complex Banach spaces.

Examples

1. $C(K)$ spaces with "few operators".
2. A $C(K_1)$ containing a (1-complemented) isometric copy of ℓ_∞.
3. A $C(K_2)$ containing a complemented copy of c_0.
4. Many spaces nonisomorphic to $C(K)$ spaces

Main application

There exists a Banach space $\mathcal{X}$ such that

$$\operatorname{Iso}(\mathcal{X}) = \{-\mathrm{Id}, +\mathrm{Id}\} \qquad \text{and} \qquad \operatorname{Iso}(\mathcal{X}^*) \supset \operatorname{Iso}(\ell_2).$$