Numerical index theory

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Schedule of the talk

1. Basic notation
2. Numerical range of operators
3. Two results on surjective isometries
4. Numerical index of Banach spaces
5. The alternative Daugavet property
6. Lush spaces
7. Slicely countably determined spaces
8. Remarks on two recent results
9. Extremely non-complex Banach spaces

Notation

Basic notation I

- $\mathbb{K}$ base field ($\mathbb{R}$ or $\mathbb{C}$):
  - $\mathbb{T}$ modulus-one scalars,
  - $\operatorname{Re} z$ real part of z ($\operatorname{Re} z = z$ if $\mathbb{K} = \mathbb{R}$).
- H Hilbert space: $(\cdot \mid \cdot)$ denotes the inner product.
- X Banach space:
  - S_X unit sphere, B_X unit ball,
  - X^* dual space,
  - $L(X)$ bounded linear operators,
  - $W(X)$ weakly compact linear operators,
  - $\operatorname{Iso}(X)$ surjective linear isometries,
- X Banach space, $T \in L(X)$:
  - $\operatorname{Sp}(T)$ spectrum of T.
  - $T^* \in L(X^*)$ adjoint operator of T.

Notation

Basic notation (II)

X Banach space, $B \subset X$, C convex subset of X:

- B is *rounded* if $\mathbb{T}B = B$,
- $\mathrm{co}(B)$ convex hull of B,
- $\overline{\mathrm{co}}(B)$ closed convex hull of B,
- $\mathrm{aconv}(B) = \mathrm{co}(\mathbb{T}\, B)$ absolutely convex hull of B,
- $\mathrm{ext}(C)$ extreme points of C,
- *slice* of C:

$$S(C, x^*, \alpha) = \big\{x \in C \ : \ \mathrm{Re}\, x^*(x) > \sup \mathrm{Re}\, x^*(C) - \alpha\big\}$$

where $x^* \in X^*$ and $0 < \alpha < \sup \mathrm{Re}\, x^*(C)$.

Numerical range of operators

2. Numerical range of operators
   - Definitions and first properties
   - The exponential function
   - Numerical ranges and isometries

F. F. Bonsall and J. Duncan
Numerical Ranges. Vol I and II.
London Math. Soc. Lecture Note Series, 1971 & 1973.

Numerical range: Hilbert spaces

Hilbert space numerical range (Toeplitz, 1918)

- A $n \times n$ real or complex matrix

$$W(A) = \{(Ax \mid x) \ : \ x \in \mathbb{K}^n, \ (x \mid x) = 1\}.$$

- H real or complex Hilbert space, $T \in L(H)$,

$$W(T) = \{(Tx \mid x) \ : \ x \in H, \ \|x\| = 1\}.$$

Remark

★ Given $T \in L(H)$ we associate

- a sesquilinear form $\varphi_T(x,y) = (Tx \mid y) \qquad (x,y \in H)$,
- a quadratic form $\widehat{\varphi_T}(x) = \varphi_T(x,x) = (Tx \mid x) \qquad (x \in H)$.

★ Then, $W(T) = \widehat{\varphi_T}(S_H)$. Therefore:

- $\widehat{\varphi_T}(B_H) = [0,1]\, W(T)$,
- $\widehat{\varphi_T}(H) = \mathbb{R}^+\, W(T)$.
- But we cannot get $W(T)$ from $\widehat{\varphi_T}(B_H)$!

Numerical range: Hilbert spaces. Properties.

Some properties

H Hilbert space, $T \in L(H)$:

- (Toeplitz-Hausdorff) $W(T)$ is convex.
- $T, S \in L(H)$, $\alpha, \beta \in \mathbb{K}$:
  - $W(\alpha T + \beta S) \subseteq \alpha W(T) + \beta W(S)$;
  - $W(\alpha \mathrm{Id} + S) = \alpha + W(S)$.
- $W(U^*TU) = W(T)$ for every $T \in L(H)$ and every U unitary.
- $\mathrm{Sp}(T) \subseteq \overline{W(T)}$.
- If T is normal, then $\overline{W(T)} = \overline{\mathrm{co}}\,\mathrm{Sp}(T)$.
- In the real case ($\dim(H) > 1$), there is $T \in L(H)$, $T \neq 0$ with $W(T) = \{0\}$.
- In the complex case,
$$\sup\{|(Tx \mid x)| \ : \ x \in S_H\} \geqslant \frac{1}{2}\,\|T\|.$$
  If T is actually self-adjoint, then
$$\sup\{|(Tx \mid x)| \ : \ x \in S_H\} = \|T\|.$$

Proving a result

H complex Hilbert space, $T \in L(H)$, then

$$M := \sup\{|(Tx \mid x)| \ : \ x \in S_H\} \geqslant \frac{1}{2}\,\|T\|.$$

- For $x, y \in S_H$ fixed, use the polarization formula:

$$(Tx \mid y) = \frac{1}{4}\Big[(T(x+y) \mid x+y) \ - \ (T(x-y) \mid x-y) \\ + \ i\,(T(x+iy) \mid x+iy) \ - i\,(T(x-iy) \mid x-iy)\Big].$$

- $|(Tx \mid y)| \leqslant \dfrac{1}{4}\,M\big[\|x+y\|^2 + \|x-y\|^2 + \|x+iy\|^2 + \|x-iy\|^2\big].$
- By the parallelogram's law:

$$|(Tx \mid y)| \leqslant \frac{1}{4}\,M\big[2\|x\|^2 + 2\|y\|^2 + 2\|x\|^2 + 2\|iy\|^2\big] = 2M.$$

- We just take supremum on $x, y \in S_H$ ✓

Numerical range: Hilbert spaces. Motivation.

Some reasons to study numerical ranges

- It gives a "picture" of the matrix/operator which allows to "see" many properties (algebraic or geometrical) of the matrix/operator.
- It is a comfortable way to study the spectrum.
- It is useful to estimate spectral radii of small perturbations of matrices.
- It is useful to work with some concepts like hermitian operator, skew-hermitian operator, dissipative operator. . .

Example

Consider $A = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ \varepsilon & 0 \end{pmatrix}$.

- $\mathrm{Sp}(A) = \{0\}$, $\mathrm{Sp}(B) = \{0\}$.
- $\mathrm{Sp}(A+B) = \{\pm\sqrt{M\varepsilon}\} \subseteq W(A+B) \subseteq W(A) + W(B)$,
- so the spectral radius of $A+B$ is bounded above by $\frac{1}{2}(|M| + |\varepsilon|)$.

Numerical range: Banach spaces (I)

Banach spaces numerical range (Bauer 1962; Lumer, 1961)

X Banach space, $T \in L(X)$,

$$V(T) = \{x^*(Tx) \ : \ x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1\}$$

Some properties

X Banach space, $T \in L(X)$.

- $V(T)$ is connected but not necessarily convex.
- $T, S \in L(X)$, $\alpha, \beta \in \mathbb{K}$:
  - $V(\alpha T + \beta S) \subseteq \alpha V(T) + \beta V(S)$;
  - $V(\alpha \mathrm{Id} + S) = \alpha + V(S)$.
- $\mathrm{Sp}(T) \subseteq \overline{V(T)}$.
- (Zenger–Crabb) Actually, $\overline{\mathrm{co}}(\mathrm{Sp}(T)) \subseteq \overline{V(T)}$.
- $\overline{\mathrm{co}}\,\mathrm{Sp}(T) = \bigcap\{V_p(T) \ : \ p$ equivalent norm$\}$
  where $V_p(T)$ is the numerical range of T in the Banach space (X, p).
- $V(U^{-1}TU) = V(T)$ for every $T \in L(X)$ and every $U \in \mathrm{Iso}(X)$.
- $V(T) \subseteq V(T^*) \subseteq \overline{V(T)}$.

Numerical range: Banach spaces (II)

Observation

The numerical range depends on the base field:

- X complex Banach space $\Longrightarrow$ $X_{\mathbb{R}}$ real space underlying X.
- $T \in L(X)$ $\Longrightarrow$ $T_{\mathbb{R}} \in L(X_{\mathbb{R}})$ is T view as a real operator.
- Then $V(T_{\mathbb{R}}) = \operatorname{Re} V(T)$.
- Consequence:
  X complex, then there is $S \in L(X_{\mathbb{R}})$ with $\|S\| = 1$ and $V(S) = \{0\}$.

Some motivation for the numerical range

- It allows to carry to the general case the concepts of hermitian operator, skew-hermitian operator, dissipative operators...
- It gives a description of the Lie algebra corresponding to the Lie group of all onto isometries on the space.
- It gives an easy and quantitative proof of the fact that Id is an strongly extreme point of $B_{L(X)}$ (MLUR point).

Numerical radius: definition and properties

Numerical radius

X real or complex Banach space, $T \in L(X)$,

$$\begin{aligned} v(T) &= \sup \big\{ |\lambda| \ : \ \lambda \in V(T) \big\} \\ &= \sup \big\{ |x^*(Tx)| \ : \ x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1 \big\} \end{aligned}$$

Elementary properties

X Banach space, $T \in L(X)$

- $v(\cdot)$ is a seminorm, i.e.
  - $v(T+S) \leqslant v(T) + v(S)$ for every $T, S \in L(X)$.
  - $v(\lambda T) = |\lambda|\, v(T)$ for every $\lambda \in \mathbb{K}$, $T \in L(X)$.
- $\sup |\mathrm{Sp}(T)| \leqslant v(T)$.
- $v(U^{-1}TU) = v(T)$ for every $U \in \mathrm{Iso}(X)$.
- $v(T^*) = v(T)$.

Numerical radius: examples

Some examples

1. H real Hilbert space $\dim(H) > 1$
   $\implies$ exist $T \in L(X)$ with $v(T) = 0$ and $\|T\| = 1$.
2. H complex Hilbert space $\dim(H) > 1$
   - $v(T) \geqslant \frac{1}{2}\|T\|$,
   - the constant $\dfrac{1}{2}$ is optimal.
3. $X = L_1(\mu) \implies v(T) = \|T\|$ for every $T \in L(X)$.
4. $X^* \equiv L_1(\mu) \implies v(T) = \|T\|$ for every $T \in L(X)$.
5. In particular, this is the case for $X = C(K)$.

Proving a result

$$X = C(K) \implies v(T) = \|T\| \text{ for every } T \in L(X).$$

- Fix $T \in L\big(C(K)\big)$. Find $f_0 \in X(E)$ and $\xi_0 \in K$ such that $\big|[Tf_0](\xi_0)\big| \sim \|T\|$.
- Consider the non-empty open set
$$V = \big\{\xi \in]0,1] \times [0,1] \ : \ f_0(\xi) \sim f_0(\xi_0)\big\}$$
and find $\varphi : [0,1] \times [0,1] \longrightarrow [0,1]$ continuous with $\operatorname{supp}(\varphi) \subset V$ and $\varphi(\xi_0) = 1$.
- Write $f_0(\xi_0) = \lambda\omega_1 + (1-\lambda)\omega_2$ with $|\omega_i| = 1$, and consider the functions
$$f_i = (1-\varphi)f_0 + \varphi\omega_i \text{ for } i = 1,2.$$
- Then, $f_i \in C(K)$, $\|f_i\| \leqslant 1$, and
$$\big\|f_0 - \big(\lambda f_1 + (1-\lambda)f_2\big)\big\| = \|\varphi f_0 - \varphi f_0(\xi_0)\| \sim 0.$$
- Therefore, there is $i \in \{1,2\}$ such that $\big|[T(f_i)](\xi_0)\big| \sim \|T\|$, but now $|f_i(\xi_0)| = 1$.
- Equivalently,
$$\big|\delta_{\xi_0}\big(T(f_i)\big)\big| \sim \|T\| \qquad \text{and} \qquad |\delta_{\xi_0}(f_i)| = 1,$$
meaning that $v(T) \sim \|T\|$. ✓

If $X = L_1(\mu)$, then $X^* \equiv C(K_\mu)$. Therefore, $v(T) = v(T^*) = \|T^*\| = \|T\|$ ✓

Numerical radius: real and complex spaces

Example

X complex Banach space, define $T \in L(X_{\mathbb{R}})$ by

$$T(x) = i\,x \qquad (x \in X).$$

- $\|T\| = 1$ and $v(T) = 0$ if viewed in $X_{\mathbb{R}}$.
- $\|T\| = 1$ and $V(T) = \{i\}$, so $v(T) = 1$ if viewed in (complex) X.

Theorem (Bohnenblust-Karlin; Glickfeld)

X complex Banach space, $T \in L(X)$:

$$v(T) \geqslant \frac{1}{\mathrm{e}}\|T\|.$$

The constant $\frac{1}{\mathrm{e}}$ is optimal:

$\exists\ X$ two-dimensional complex, $\exists\ T \in L(X)$ with $\|T\| = \mathrm{e}$ and $v(T) = 1$.

Numerical index: definition and properties

Numerical index

X real or complex Banach space

$$n(X) = \max\{k \geqslant 0 \ : \ K\,\|T\| \leqslant v(T)\ \forall T \in L(X)\}$$
$$= \inf\big\{v(T) \ : \ T \in L(X),\ \|T\| = 1\big\}.$$

Elementary properties

X Banach space.

- In the real case, $0 \leqslant n(X) \leqslant 1$.
- In the complex case, $1/\mathrm{e} \leqslant n(X) \leqslant 1$.
- Actually, the above inequalities are best possible:

$$\{n(X) \ : \ X \text{ complex Banach space }\} = [\mathrm{e}^{-1}, 1],$$
$$\{n(X) \ : \ X \text{ real Banach space }\} = [0, 1].$$

- v norm on $L(X)$ equivalent to the given norm $\iff n(X) > 0$.
- $v(T) = \|T\|$ for every $T \in L(X) \iff n(X) = 1$.
- $n(X^*) \leqslant n(X)$.

Numerical index: examples

Some examples

1. H Hilbert, $\dim(H) > 1$:
$$n(H) = \begin{cases} 0 & \text{real case,} \\ \frac{1}{2} & \text{complex case.} \end{cases}$$
2. X complex space $\implies n\big(X_{\mathbb{R}}\big) = 0$.
3. $n\big(L_1(\mu)\big) = 1$, μ positive measure.
4. $X^* \equiv L_1(\mu) \implies n(X) = 1$.
5. In particular,
$$n\big(C(K)\big) = 1, \quad n\big(C_0(L)\big) = 1, \quad n\big(L_\infty(\mu)\big) = 1.$$
6. $n(A(\mathbb{D})) = 1$ and $n(H^\infty) = 1$.

The exponential function. Definition

The exponential function

X Banach space, $T \in L(X)$:

$$\exp(T) = \sum_{n=0}^{\infty} \frac{1}{n!}\, T^n$$

where $T^0 = \mathrm{Id}$ and $T^n = T \circ \overset{n)}{\cdots} \circ T$.

- It is well-defined since the series is absolutely convergent.
- $\|\exp(T)\| \leqslant \mathrm{e}^{\|T\|}$.
- **We will improve this inequality in the sequel**

The exponential function: properties

Properties

X Banach space, $T, S \in L(X)$.

- $TS = ST \implies \exp(T+S) = \exp(T)\exp(S)$.
- $\exp(T)\exp(-T) = \exp(0) = \mathrm{Id} \implies \exp(T)$ surjective isomorphism.
- $\{\exp(\rho T) : \rho \in \mathbb{R}_0^+\}$ exponential one-parameter semigroup generated by T.

The exponential formula

X Banach space, $T \in L(X)$:

$$\sup \operatorname{Re} V(T) = \sup_{\alpha>0} \frac{\log \|\exp(\alpha T)\|}{\alpha} = \lim_{\alpha\downarrow 0} \frac{\log \|\exp(\alpha T)\|}{\alpha}.$$

Consequence

X Banach space, $T \in L(X)$:

- $\|\exp(\lambda T)\| \leqslant \mathrm{e}^{|\lambda|\, v(T)}$ $(\lambda \in \mathbb{K})$.
- $v(T)$ is the best possible constant.

Semigroups of isometries: motivating example

A motivating example

A real or complex $n \times n$ matrix. TFAE:

- A is skew-adjoint (i.e. $A^* = -A$).
- $\mathrm{Re}(Ax \mid x) = 0$ for every $x \in H$.
- $B = \exp(\rho A)$ is unitary for every $\rho \in \mathbb{R}$ (i.e. $B^*B = BB^* = \mathrm{Id}$).

In term of Hilbert spaces

H (n-dimensional) Hilbert space, $T \in L(H)$. TFAE:

- $\mathrm{Re}\, W(T) = \{0\}$.
- $\exp(\rho T) \in \mathrm{Iso}(H)$ for every $\rho \in \mathbb{R}$.

For general Banach spaces

X Banach space, $T \in L(X)$. TFAE:

- $\mathrm{Re}\, V(T) = \{0\}$.
- $\exp(\rho T) \in \mathrm{Iso}(X)$ for every $\rho \in \mathbb{R}$.

Semigroups of isometries: characterization

Theorem (Bonsall-Duncan, 1970's; Rosenthal, 1984)

X real or complex Banach space, $T \in L(X)$. TFAE:

- $\operatorname{Re} V(T) = \{0\}$ (T is skew-hermitian).
- $\|\exp(\rho T)\| \leqslant 1$ for every $\rho \in \mathbb{R}$.
- $\{\exp(\rho T) \ : \ \rho \in \mathbb{R}_0^+\} \subset \operatorname{Iso}(X)$.
- T belongs to the tangent space to $\operatorname{Iso}(X)$ at Id.
- $\displaystyle\lim_{\rho\to 0} \frac{\|\operatorname{Id} + \rho\, T\| - 1}{\rho} = 0$.

Main consequence

If X is a real Banach space such that

$$V(T) = \{0\} \quad \Longrightarrow \quad T = 0,$$

then $\operatorname{Iso}(X)$ is "small":

- it does not contain any exponential one-parameter semigroup,
- the tangent space of $\operatorname{Iso}(X)$ at Id is zero.

Surjective isometries

3. Two results on surjective isometries
   - Isometries on finite-dimensional spaces
   - Isometries and duality

M. Martín
The group of isometries of a Banach space and duality.
J. Funct. Anal. (2008).

M. Martín, J. Merí, and A. Rodríguez-Palacios.
Finite-dimensional spaces with numerical index zero.
Indiana U. Math. J. (2004).

H. P. Rosenthal
The Lie algebra of a Banach space.
in: *Banach spaces* (Columbia, Mo., 1984), LNM, Springer, 1985.

Isometries in finite-dimensional spaces

Theorem

X finite-dimensional real space. TFAE:

- $\mathrm{Iso}(X)$ is infinite.
- $n(X) = 0$.
- There is $T \in L(X)$, $T \neq 0$, with $v(T) = 0$.

Examples of spaces of this kind

1. Hilbert spaces.
2. $X_{\mathbb{R}}$, the real space subjacent to any complex space X.
3. An absolute sum of any real space and one of the above.
4. Moreover, if $X = X_0 \oplus X_1$ where X_1 is complex and

$$\left\| x_0 + e^{i\theta} x_1 \right\| = \|x_0 + x_1\| \qquad \big(x_0 \in X_0,\ x_1 \in X_1,\ \theta \in \mathbb{R}\big).$$

(Note that the other 3 cases are included here)

Question

Can every Banach space X with $n(X) = 0$ be decomposed as in 4 ?

Negative answer

Infinite-dimensional case

There is an infinite-dimensional real Banach space X with $n(X) = 0$ but X is polyhedral. In particular, X does not contain $\mathbb{C}$ isometrically.

An easy example is

$$X = \left[\bigoplus_{n \geqslant 2} X_n \right]_{c_0}$$

X_n is the two-dimensional space whose unit ball is the regular polygon of $2n$ vertices.

Note

Such an example is not possible in the finite-dimensional case.

Quasi affirmative answer

Finite-dimensional case

X finite-dimensional real space. TFAE:

- $n(X) = 0$.
- $X = X_0 \oplus X_1 \oplus \cdots \oplus X_n$ such that
  - X_0 is a (possible null) real space,
  - $X_1, \ldots, X_n$ are non-null complex spaces,

  there are $\rho_1, \ldots, \rho_n$ **rational** numbers, such that

$$\left\| x_0 + \mathrm{e}^{i\,\rho_1\,\theta}\, x_1 + \cdots + \mathrm{e}^{i\,\rho_n\,\theta}\, x_n \right\| = \left\| x_0 + x_1 + \cdots + x_n \right\|$$

  for every $x_i \in X_i$ and every $\theta \in \mathbb{R}$.

Remark

- The theorem is due to Rosenthal, but with real ρ's.
- The fact that the ρ's may be chosen as rational numbers is due to M.–Merí–Rodríguez-Palacios.

Sketch of the proof

- Fix $T \in L(X)$ with $\|T\| = 1$ and $v(T) = 0$.
- We get that $\big\| \exp(\rho T) \big\| = 1$ for every $\rho \in \mathbb{R}$.
- A Theorem by Auerbach: there exists a Hilbert space H with $\dim(H) = \dim(X)$ such that every surjective isometry in $L(X)$ remains isometry in $L(H)$.
- Apply the above to $\exp(\rho T)$ for every $\rho \in \mathbb{R}$.
- You get that T is skew-hermitian in $L(H)$, so $T^* = -T$ and T^2 is self-adjoint. The X_j's are the eigenspaces of T^2.
- Use Kronecker's Approximation Theorem to change the eigenvalues of T^2 by rational numbers.✓

A simple case of getting rational numbers

- Let $X = X_0 \oplus X_1 \oplus X_2$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ s.t.

$$\left\| x_0 + e^{i\rho} x_1 + e^{i\alpha\rho} x_2 \right\| = \|x_0 + x_1 + x_2\| \quad \forall \rho,\ \forall x_0, x_1, x_2.$$

- Then $\|x_0 + x_1 + x_2\| = \left\| x_0 + e^{i\rho} \left(x_1 + e^{i(\alpha-1)\rho} x_2 \right) \right\| \quad \forall \rho.$
- Take $\rho = \dfrac{2\pi k}{\alpha - 1}$ with $k \in \mathbb{Z}$.
- Then $\|x_0 + (x_1 + x_2)\| = \left\| x_0 + e^{i\,\frac{2\pi k}{\alpha-1}} (x_1 + x_2) \right\| \quad \forall k \in \mathbb{Z}$
- But $\left\{ \dfrac{2\pi k}{\alpha - 1} \ : \ k \in \mathbb{Z} \right\}$ is dense in $\mathbb{T}$, so

$$\|x_0 + (x_1 + x_2)\| = \left\| x_0 + e^{i\rho} (x_1 + x_2) \right\| \quad \forall \rho \in \mathbb{R}$$

and $X = X_0 \oplus Z$ where $Z = X_1 \oplus X_2$ is a complex space

Consequences

Corollary

X real space with $n(X)=0$.

- If $\dim(X)=2$, then $X \equiv \mathbb{C}$.
- If $\dim(X)=3$, then $X \equiv \mathbb{R} \oplus \mathbb{C}$ (absolute sum).

Natural question

Are all finite-dimensional X's with $n(X)=0$ of the form $X = X_0 \oplus X_1$ **?**

Answer

No.

Example

$X = (\mathbb{R}^4, \|\cdot\|)$, $\|(a,b,c,d)\| = \dfrac{1}{4}\displaystyle\int_0^{2\pi} \left|\mathrm{Re}\left(\mathrm{e}^{2it}(a+ib)+\mathrm{e}^{it}(c+id)\right)\right| \, dt.$

Then $n(X)=0$ but the unique possible decomposition is $X = \mathbb{C} \oplus \mathbb{C}$ with

$$\left\|\mathrm{e}^{it}x_1 + \mathrm{e}^{2it}x_2\right\| = \|x_1 + x_2\|.$$

The Lie-algebra of a Banach space

Lie-algebra

X real Banach space, $\mathcal{Z}(X) = \big\{T \in L(X) \ : \ v(T) = 0\big\}$.

- When X is finite-dimensional, $\operatorname{Iso}(X)$ is a Lie-group and $\mathcal{Z}(X)$ is the tangent space (i.e. its Lie-algebra).

Remark

- $\dim(X) = n \quad \Longrightarrow \quad \dim(\mathcal{Z}(X)) \leqslant \dfrac{n(n-1)}{2}$.
- Equality holds $\quad \Longleftrightarrow \quad H$ Hilbert space.

An open problem

Given $n \geqslant 3$, which are the possible $\dim\big(\mathcal{Z}(X)\big)$ over all n-dimensional X's?

Observation (Javier Merí, PhD)

When $\dim(X) = 3$, $\dim(\mathcal{Z}(X))$ cannot be 2.

Semigroups of surjective isometries and duality

Remark

X Banach space.

- $T \in \mathrm{Iso}(X) \implies T^* \in \mathrm{Iso}(X^*)$.
- $\mathrm{Iso}(X^*)$ can be bigger than $\mathrm{Iso}(X)$.

The problem

- How much bigger can be $\mathrm{Iso}(X^*)$ than $\mathrm{Iso}(X)$?
- Is it possible that $\mathcal{Z}\big(\mathrm{Iso}(X^*)\big)$ is big while $\mathcal{Z}\big(\mathrm{Iso}(X)\big)$ is trivial?

The answer is yes. This is what we are going to present next.

Semigroups of surjective isometries and duality

Spaces $C_E(K\|L)$

K compact, $L \subset K$ closed nowhere dense, $E \subset C(L)$.

$$C_E(K\|L) = \{f \in C(K) \,:\, f|_L \in E\}.$$

Theorem

$$C_E(K\|L)^* \equiv E^* \oplus_1 C_0(K\|L)^* \quad \& \quad n\big(C_E(K\|L)\big) = 1.$$

Consequence: the example

Take $K = [0,1]$, $L = \Delta$ (Cantor set), $E = \ell_2 \subset C(\Delta)$.

- $\mathrm{Iso}\big(C_{\ell_2}([0,1]\|\Delta)\big)$ has no exponential one-parameter semigroups.
- $C_{\ell_2}([0,1]\|\Delta)^* \equiv \ell_2 \oplus_1 C_0([0,1]\|\Delta)^*$, so taken $S \in \mathrm{Iso}(\ell_2)$

$$\Longrightarrow \quad T = \begin{pmatrix} S & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \in \mathrm{Iso}\big(C_{\ell_2}([0,1]\|\Delta)^*\big)$$

Then, $\mathrm{Iso}\big(C_{\ell_2}([0,1]\|\Delta)^*\big)$ contains infinitely many exponential one-parameter semigroups.

Some comments

In terms of linear dynamical systems

- In $C_{\ell_2}([0,1]\|\Delta)$ there is no $A \in L(X)$ such that the solution to the linear dynamical system

$$x' = A\,x \qquad \left(x : \mathbb{R}_0^+ \longrightarrow C_{\ell_2}([0,1]\|\Delta)\right)$$

  (which is $x(t) = \exp(t\,A)(x(0))$) is given by a semigroup of isometries.
- There are infinitely many such A's in $C_{\ell_2}([0,1]\|\Delta)^*$, in $C_{\ell_2}([0,1]\|\Delta)^{**}\ldots$

Further results (Koszmider–M.–Merí., 2009)

- There are unbounded As on $C_{\ell_2}([0,1]\|\Delta)$ such that the solution to the linear dynamical system

$$x'(t) = A\,x(t)$$

  is a one-parameter C_0 semigroup of isometries.
- There is X such that $\operatorname{Iso}(X) = \{-\mathrm{Id}, \mathrm{Id}\}$ and $X^* = \ell_2 \oplus_1 L_1(\nu)$.
- Therefore, there is no semigroups in $\operatorname{Iso}(X)$, but there are infinitely many exponential one-parameter semigroups in $\operatorname{Iso}(X^*)$.

Numerical index of Banach spaces

4. Numerical index of Banach spaces
   - Basic definitions and examples
   - Stability properties
   - Duality
   - The isomorphic point of view
   - Banach spaces with numerical index one
     - Isomorphic properties
     - Isometric properties
     - Asymptotic behavior
   - How to deal with numerical index 1 property?

V. Kadets, M. Martín, and R. Payá.
Recent progress and open questions on the numerical index of Banach spaces.
RACSAM (2006)

Numerical index of Banach spaces: definitions

Numerical radius

X Banach space, $T \in L(X)$. The numerical radius of T is

$$v(T) = \sup\left\{|x^*(Tx)| \ : \ x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1\right\}$$

Remark

The numerical radius is a continuous seminorm in $L(X)$. Actually, $v(\cdot) \leqslant \|\cdot\|$

Numerical index (Lumer, 1968)

X Banach space, the numerical index of X is

$$\begin{aligned} n(X) &= \inf\left\{ v(T) \ : \ T \in L(X),\ \|T\| = 1\right\} \\ &= \max\left\{ k \geqslant 0 \ : \ k\,\|T\| \leqslant v(T) \ \ \forall\ T \in L(X)\right\} \\ &= \inf\left\{ M \geqslant 0 \ : \ \exists T \in L(X),\, \|T\| = 1,\ \|\exp(\rho T)\| \leqslant e^{\rho M}\ \forall \rho \in \mathbb{R}\right\} \end{aligned}$$

Numerical index of Banach spaces: basic properties

Recalling some basic properties

- $n(X) = 1$ iff v and $\|\cdot\|$ coincide.
- $n(X) = 0$ iff v is not an equivalent norm in $L(X)$
- X complex $\Rightarrow$ $n(X) \geqslant 1/\mathrm{e}$.

  (Bohnenblust–Karlin, 1955; Glickfeld, 1970)
- Actually,

$$\{n(X) \ : \ X \text{ complex}, \ \dim(X) = 2\} = [\mathrm{e}^{-1}, 1]$$
$$\{n(X) \ : \ X \text{ real}, \ \dim(X) = 2\} = [0, 1]$$

(Duncan–McGregor–Pryce–White, 1970)

Numerical index of Banach spaces: examples (I)

Some examples

1. H Hilbert space, $\dim(H) > 1$,

$$\begin{array}{ll} n(H) = 0 & \text{if } H \text{ is real} \\ n(H) = 1/2 & \text{if } H \text{ is complex} \end{array}$$

2. $n\big(L_1(\mu)\big) = 1 \qquad \mu$ positive measure

   $n\big(C(K)\big) = 1 \qquad K$ compact Hausdorff space

   (Duncan et al., 1970)

3. If A is a C^*-algebra $\Rightarrow \begin{cases} n(A) = 1 & A \text{ commutative} \\ n(A) = 1/2 & A \text{ not commutative} \end{cases}$

   (Huruya, 1977; Kaidi–Morales–Rodríguez, 2000)

4. If A is a function algebra $\Rightarrow n(A) = 1$

   (Werner, 1997)

Numerical index of Banach spaces: some examples (II)

More examples

5. For $n \geqslant 2$, the unit ball of X_n is a $2n$ regular polygon:

$$n(X_n) = \begin{cases} \tan\left(\frac{\pi}{2n}\right) & \text{if } n \text{ is even,} \\ \\ \sin\left(\frac{\pi}{2n}\right) & \text{if } n \text{ is odd.} \end{cases}$$

(M.–Merí, 2007)

6. Every finite-codimensional subspace of $C[0,1]$ has numerical index 1

(Boyko–Kadets–M.–Werner, 2007)

Numerical index of Banach spaces: some examples (III)

Even more examples

7. Numerical index of L_p-spaces, $1 < p < \infty$:
   - $n\big(L_p[0,1]\big) = n(\ell_p) = \lim\limits_{m\to\infty} n\big(\ell_p^{(m)}\big)$.

     (Ed-Dari, 2005 & Ed-Dari-Khamsi, 2006)
   - $n\big(\ell_p^{(2)}\big)$ **?**
   - In the real case,

$$\max\left\{\frac{1}{2^{1/p}}, \frac{1}{2^{1/q}}\right\} M_p \leqslant n\big(\ell_p^{(2)}\big) \leqslant M_p$$

and $M_p = v\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \max\limits_{t\in[0,1]} \dfrac{|t^{p-1} - t|}{1+t^p}$

(M.–Merí, 2009)

   - In the real case, $n\big(L_p(\mu)\big) \geqslant \dfrac{M_p}{8e}$.
   - In particular, $n\big(L_p(\mu)\big) > 0$ for $p \neq 2$.

(M.–Merí–Popov, 2009)

Numerical index: open problems on computing

Open problems

1. Compute $n\big(L_p[0,1]\big)$ for $1 < p < \infty$, $p \neq 2$.
2. Is $n\big(\ell_p^{(2)}\big) = M_p$ (real case) **?**
3. Is $n\big(\ell_p^{(2)}\big) = \Big(p^{\frac{1}{p}}\, q^{\frac{1}{q}}\Big)^{-1}$ (complex case) **?**
4. Compute the numerical index of real C^*-algebras.
5. Compute the numerical index of more classical Banach spaces: $C^m[0,1]$, Lip(K), Lorentz spaces, Orlicz spaces...

Stability properties

Direct sums of Banach spaces (M.–Payá, 2000)

$$n\Big([\oplus_{\lambda\in\Lambda} X_\lambda]_{c_0}\Big) = n\Big([\oplus_{\lambda\in\Lambda} X_\lambda]_{\ell_1}\Big) = n\Big([\oplus_{\lambda\in\Lambda} X_\lambda]_{\ell_\infty}\Big) = \inf_\lambda n(X_\lambda)$$

Consequences

- There is a real Banach space X such that
$$v(T) > 0 \quad \text{when } T \neq 0,$$
but $n(X) = 0$
(i.e. $v(\cdot)$ is a norm on $L(X)$ which is not equivalent to the operator norm).
- For every $t \in [0,1]$, there exist a real X_t isomorphic to c_0 (or ℓ_1 or ℓ_∞) with $n(X_t) = t$.
- For every $t \in [e^{-1},1]$, there exist a complex Y_t isomorphic to c_0 (or ℓ_1 or ℓ_∞) with $n(Y_t) = t$.

Stability properties (II)

Vector-valued function spaces (López-M.-Merí-Payá-Villena, 2000's)

E Banach space, μ positive σ-finite measure, K compact space. Then

$$n\big(C(K,E)\big) = n\big(C_w(K,E)\big) = n\big(L_1(\mu,E)\big) = n\big(L_\infty(\mu,E)\big) = n(E),$$

and $n\big(C_{w^*}(K,E^*)\big) \leqslant n(E)$

Tensor products (Lima, 1980)

There is no general formula for $n\big(X\tilde{\otimes}_\varepsilon Y\big)$ nor for $n\big(X\tilde{\otimes}_\pi Y\big)$:

- $n\big(\ell_1^{(4)}\tilde{\otimes}_\pi \ell_1^{(4)}\big) = n\big(\ell_\infty^{(4)}\tilde{\otimes}_\varepsilon \ell_\infty^{(4)}\big) = 1.$
- $n\big(\ell_1^{(4)}\tilde{\otimes}_\varepsilon \ell_1^{(4)}\big) = n\big(\ell_\infty^{(4)}\tilde{\otimes}_\pi \ell_\infty^{(4)}\big) < 1.$

L_p-spaces (Askoy–Ed-Dari–Khamsi, 2007)

$$n\big(L_p([0,1],E)\big) = n\big(\ell_p(E)\big) = \lim_{m\to\infty} n\big(E \oplus_p \overset{m}{\cdots} \oplus_p E\big).$$

Numerical index and duality

Proposition

X Banach space, $T \in L(X)$. Then

- $\sup \operatorname{Re} V(T) = \lim_{\alpha \to 0^+} \dfrac{\|\mathrm{Id} + \alpha T\| - 1}{\alpha}$.
- Then, $v(T^*) = v(T)$ for every $T \in L(X)$.
- Therefore, $n(X^*) \leqslant n(X)$.

(Duncan–McGregor–Pryce–White, 1970)

Question (From the 1970's)

Is $n(X) = n(X^*)$ **?**

Negative answer (Boyko–Kadets–M.–Werner, 2007)

Consider the space

$$X = \{(x, y, z) \in c \oplus_\infty c \oplus_\infty c \ : \ \lim x + \lim y + \lim z = 0\}.$$

Then, $n(X) = 1$ but $n(X^*) < 1$.

Numerical index and duality (II)

The above example can be squeezed to get more counterexamples.

Example 1

- Exists X real with $n(X) = 1$ and $n(X^*) = 0$.
- Exists X complex with $n(X) = 1$ and $n(X^*) = 1/\mathrm{e}$.

Example 2

- Given $t \in]0, 1]$, exists X real with $n(X) = t$ and $n(X^*) = 0$.
- Given $t \in]1/\mathrm{e}, 1]$, exists X complex with $n(X) = 1$ and $n(X^*) = 1/\mathrm{e}$.

Numerical index and duality (III)

Some positive partial answers

One has $n(X) = n(X^*)$ when

- X is reflexive (evident).
- X is a C^*-algebra or a von Neumann predual (1970's – 2000's).
- X is L-embedded in X^{**} (M., 2009).
- If X has RNP and $n(X) = 1$, then $n(X^*) = 1$ (M., 2002).
- If X is M-embedded in X^{**} and $n(X) = 1$
  $\Longrightarrow\ n(Y) = 1$ for $X \subseteq Y \subseteq X^{**}$.

Example

$X = C_{K(\ell_2)}([0,1]\|\Delta)$. Then $n(X) = 1$ and

$$X^* \equiv K(\ell_2)^* \oplus_1 C_0(K\|\Delta)^* \qquad \text{and} \qquad X^{**} \equiv L(\ell_2) \oplus_\infty C_0(K\|\Delta)^{**}.$$

Therefore, X^{**} is a C^*-algebra, but $n(X^*) = 1/2 < n(X) = 1$.

Numerical index and duality: open problems

Main question

Find isometric or isomorphic properties assuring that $n(X) = n(X^*)$.

Question 1

If Z has a unique predual X, does $n(X) = n(X^*)$ **?**

Question 2

Z dual space, does there exists a predual X such that $n(X) = n(X^*)$ **?**

Question 4

If X has the RNP, does $n(X) = n(X^*)$ **?**

The isomorphic point of view

Renorming and numerical index (Finet–M.–Payá, 2003)

$(X, \|\cdot\|)$ (separable or reflexive) Banach space. Then

- Real case:
$$[0,1[\subseteq \{n(X,|\cdot|) \ : \ |\cdot| \simeq \|\cdot\|\}$$
- Complex case:
$$[\mathrm{e}^{-1},1[\subseteq \{n(X,|\cdot|) \ : \ |\cdot| \simeq \|\cdot\|\}$$

Open question

The result is known to be true when X has a long biorthogonal system.
Is it true in general **?**

Remark

In some sense, any other value of $n(X)$ but $\mathbf{1}$ is isomorphically trivial.
★ What about the value $\mathbf{1}$ **?**

Banach spaces with numerical index one

Numerical index 1

Recall that X has numerical index one ($n(X) = 1$) iff

$$\|T\| = \sup\{|x^*(Tx)| \ : \ x \in S_X,\ x^* \in S_{X^*},\ x^*(x) = 1\}$$

(i.e. $v(T) = \|T\|$) for every $T \in L(X)$.

Observation

For Hilbert spaces, the above formula is equivalent to

$$\|T\| = \sup\{|\langle Tx, x\rangle| \ : \ x \in S_X\}$$

which is known to be valid for every self-adjoint operator T.

Examples

$C(K)$, $L_1(\mu)$, $A(\mathbb{D})$, H^∞, finite-codimensional subspaces of $C[0,1]$...

Isomorphic properties (prohibitive results)

Question

Does every Banach space admit an equivalent norm with numerical index 1 **?**

Negative answer (López–M.–Payá, 1999)

Not every real Banach space can be renormed to have numerical index 1.
Concretely:

- If X is real, reflexive, and $\dim(X) = \infty$, then $n(X) < 1$.
- Actually, if X is real, X^{**}/X separable and $n(X) = 1$, then X is finite-dimensional.
- Moreover, if X is real, RNP, $\dim(X) = \infty$, and $n(X) = 1$, then $X \supset \ell_1$.

A very recent result (Avilés–Kadets–M.–Merí–Shepelska)

If X is real, $\dim(X) = \infty$ and $n(X) = 1$, then $X^* \supset \ell_1$.

More details on this later on.

Proving the 1999 results (I)

Lemma

X Banach space, $n(X) = 1$
$\Longrightarrow$ $|x_0^*(x_0)| = 1$ for all $x_0^* \in \operatorname{ext}(B_{X^*})$ and all denting point x_0 of B_X.

Proof:

- Fix $\varepsilon > 0$. AS x_0 denting point, $\exists y^* \in S_{X^*}$ and $\alpha > 0$ such that
$$\|z - x_0\| < \varepsilon \quad \text{whenever } z \in B_{X^*} \text{ satisfies } \operatorname{Re} y^*(z) > 1 - \alpha.$$
- (Choquet's lemma): $x_0^* \in \operatorname{ext}(B_{X^*})$, $\exists y \in S_X$ and $\beta > 0$ such that
$$|z^*(x_0) - x_0^*(x_0)| < \varepsilon \quad \text{whenever } z^* \in B_{X^*} \text{ satisfies } \operatorname{Re} z^*(y) > 1 - \beta.$$
- Let $T = y^* \otimes y \in L(X)$. $\|T\| = 1 \implies v(T) = 1$.
- We may find $x \in S_X$, $x^* \in S_{X^*}$, such that
$$x^*(x) = 1 \qquad \text{and} \qquad |x^*(Tx)| = |y^*(x)||x^*(y)| > 1 - \min\{\alpha, \beta\}.$$
- By choosing suitable $s, t \in \mathbb{T}$ we have
$$\operatorname{Re} y^*(sx) = |y^*(x)| > 1 - \alpha \qquad \& \qquad \operatorname{Re} tx^*(y) = |x^*(y)| > 1 - \beta.$$
- It follows that $\|sx - x_0\| < \varepsilon$ and $|tx^*(x_0) - x_0^*(x_0)| < \varepsilon$, and so
$$\begin{aligned} 1 - |x_0^*(x_0)| &\leqslant |tx^*(sx) - x_0^*(x_0)| \leqslant \\ &\leqslant |tx^*(sx) - tx^*(x_0)| + |tx^*(x_0) - x_0^*(x_0)| < 2\varepsilon. \checkmark \end{aligned}$$

Proving the 1999 results (II)

Proposition

X real, $A \subset S_X$ infinite with $|x^*(a)| = 1\ \forall x^* \in \operatorname{ext}(B_{X^*})$, $\forall a \in A$.
$\Longrightarrow\ X \supseteq c_0$ or $X \supseteq \ell_1$.

Proof:

- $X \supseteq \ell_1$ ✓
- (Rosenthal ℓ_1-theorem): Otherwise, $\exists\ \{a_n\} \subseteq A$ non-trivial weak Cauchy.
- Consider Y the closed linear span of $\{a_n\ :\ n \in \mathbb{N}\}$.
- $\|a_n - a_m\| = 2$ if $n \neq m \implies \dim(Y) = \infty$.
- (Krein-Milman theorem): every $y^* \in \operatorname{ext}(B_{Y^*})$ has an extension which belongs to $\operatorname{ext}(B_{X^*})$.
- So, $|y^*(a_n)| = 1\ \forall y^* \in \operatorname{ext}(B_{Y^*})$, $\forall n \in \mathbb{N}$.
- $\{a_n\}$ weak Cauchy $\implies$ $\{y^*(a_n)\}$ is eventually 1 or -1.
- Then $\operatorname{ext}(B_{Y^*}) = \bigcup\limits_{k\in\mathbb{N}} (E_k \cup -E_k)$ where
$$E_k = \{y^* \in \operatorname{ext}(B_{Y^*})\ :\ y^*(a_n) = 1 \text{ for } n \geqslant k\}.$$
- $\{a_n\}$ separates points of Y^* $\implies$ E_k finite, so $\operatorname{ext}(B_{Y^*})$ countable.
- (Fonf): $Y \supseteq c_0$. So, $X \supseteq c_0$. ✓

Proving the 1999 results (III)

Lemma

X Banach space, $n(X) = 1$
$\Longrightarrow \ |x_0^*(x_0)| = 1$ for all $x_0^* \in \operatorname{ext}(B_{X^*})$ and all denting point x_0 of B_X.

Proposition

X real, $A \subset S_X$ infinite with $|x^*(a)| = 1 \ \forall x^* \in \operatorname{ext}(B_{X^*})$, $\forall a \in A$.
$\Longrightarrow \ X \supseteq c_0$ or $X \supseteq \ell_1$.

Main consequence

X real, RNP, $\dim(X) = \infty$, and $n(X) = 1 \implies X \supseteq \ell_1$.

Corollary

X real, $\dim(X) = \infty$, $n(X) = 1$.

- X is not reflexive.
- X^{**}/X is non-separable.

Isomorphic properties (positive results)

A renorming result (Boyko–Kadets–M.–Merí, 2009)

If X is separable, $X \supset c_0$, then X can be renormed to have numerical index 1.

Consequence

X separable containing $c_0 \implies$ there is $Z \simeq X$ such that

$$n(Z) = 1 \qquad \text{and} \qquad \begin{cases} n(Z^*) = 0 & \text{real case} \\ n(Z^*) = \mathrm{e}^{-1} & \text{complex case} \end{cases}$$

Open questions

- Find isomorphic properties which assures renorming with numerical index 1
- In particular, if $X \supset \ell_1$, can X be renormed to have numerical index 1 **?**

Negative result (Bourgain–Delbaen, 1980)

There is X such that $X^* \simeq \ell_1$ and X has the RNP. Then, X can not be renormed with numerical index 1 (in such a case, $X \supset \ell_1$ **!**)

Isometric properties: finite-dimensional spaces

Finite-dimensional spaces (McGregor, 1971; Lima, 1978)

X real or complex finite-dimensional space. TFAE:

- $n(X) = 1$.
- $|x^*(x)| = 1$ for every $x^* \in \operatorname{ext}(B_{X^*})$, $x \in \operatorname{ext}(B_X)$.
- $B_X = \operatorname{aconv}(F)$ for every maximal convex subset F of S_X (X is a CL-space).

Remark

This shows a rough behavior of the norm of a finite-dimensional space with numerical index 1:

- The space is not smooth.
- The space is not strictly convex.

Question

What is the situation in the infinite-dimensional case **?**

Isometric properties: infinite-dimensional spaces

Theorem (Kadets–M.–Merí–Payá, 2009)

X infinite-dimensional Banach space, $n(X) = 1$. Then

- X^* is neither smooth nor strictly convex.
- The norm of X cannot be Fréchet-smooth.
- There is no WLUR points in S_X.

Corollary

$X = C(\mathbb{T})/A(\mathbb{D})$. $X^* = H^1$ is smooth $\implies$ $n(X) < 1$ & $n(H^1) < 1$.

Example without completeness

- There is X (non-complete) strictly convex with $X^* \equiv L_1(\mu)$, so $n(X) = 1$.
- $\widetilde{X}$ completion of X. For $F \subseteq S_{\widetilde{X}}$ maximal face, $B_{\widetilde{X}} = \overline{\operatorname{aconv}}(F)$.

Open question

Is there X with $n(X) = 1$ which is smooth or strictly convex **?**

Asymptotic behavior of the set of spaces with numerical index one

Theorem (Oikhberg, 2005)

There is a universal constant c such that

$$\operatorname{dist}\big(X, \ell_2^{(m)}\big) \geqslant c\, m^{\frac{1}{4}}$$

for every $m \in \mathbb{N}$ and every m-dimensional X with $n(X) = 1$.

Old examples

$$\operatorname{dist}\big(\ell_1^{(m)}, \ell_2^{(m)}\big) = \operatorname{dist}\big(\ell_\infty^{(m)}, \ell_2^{(m)}\big) = m^{\frac{1}{2}}$$

Open questions

- Is there a universal constant $\widetilde{c}$ such that

  $$\operatorname{dist}\big(X, \ell_2^{(m)}\big) \geqslant \widetilde{c}\, m^{\frac{1}{2}}$$

  for every $m \in \mathbb{N}$ and every m-dimensional X's with $n(X) = 1$ **?**
- What is the diameter of the set of all m-dimensional X's with $n(X) = 1$ **?**

How to deal with numerical index 1 property?

One the one hand: weaker properties

- In a general Banach space, we only can construct compact (actually, finite-rank) operators.
- Actually, we only may easily calculate the norm of rank-one operators.
- All the results given before for Banach spaces in which we use numerical index 1 only need

$$v(T) = \|T\| \text{ for every rank-one operator } T.$$

- This is called the alternative Daugavet property (ADP) and we will present it in the next section.

One the other hand: stronger properties

- We do not know any operator-free characterization of Banach spaces with numerical index 1.
- When we know that a Banach space has numerical index 1 (or that it can be renormed with numerical index 1), we actually prove more.
- Later we will study sufficient geometrical conditions.
- The weakest property is called lushness.

How to deal with numerical index 1 property?

Relationship between the properties

- One of the key ideas to get interesting results for Banach spaces with numerical index 1 is to study when the three properties below are equivalent.
- A very interesting property appears: the slicely countably determination.
- We will study this property later on.

lushness $\begin{smallmatrix}\Longrightarrow \\ \not\Longleftarrow\end{smallmatrix}$ Numerical index 1 $\begin{smallmatrix}\Longrightarrow \\ \not\Longleftarrow\end{smallmatrix}$ ADP

The alternative Daugavet property

M. Martín and T. Oikberg
An alternative Daugavet property
J. Math. Anal. Appl. (2004)

M. Martín
The alternative Daugavet property of C^*-algebras and JB^*-triples
Math. Nachr. (2008)

The Daugavet property: motivation

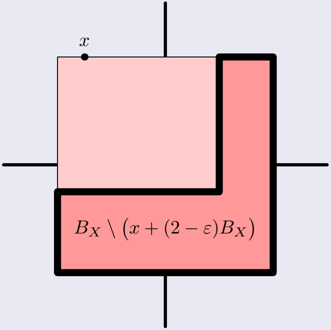
- In a Banach space X with the Radon-Nikodým property the unit ball has many denting points.
- $x \in S_X$ is a denting point of B_X if for every $\varepsilon > 0$ one has

$$x \notin \overline{\text{co}}\big(B_X \setminus (x + \varepsilon B_X)\big).$$

- $C[0,1]$ and $L_1[0,1]$ have an extremely opposite property: for every $x \in S_X$ and every $\varepsilon > 0$

$$\overline{\text{co}}\,\Big(B_X \setminus (x + (2-\varepsilon)B_X)\Big) = B_X\,.$$

- This geometric property is equivalent to a property of operators on the space.



The Daugavet property: definition

The Daugavet equation

X Banach space, $T \in L(X)$

$$\|\mathrm{Id} + T\| = 1 + \|T\| \qquad \text{(DE)}$$

The Daugavet property

A Banach space X is said to have the Daugavet property iff every rank-one operator on X satisfies (DE).

★ Then, every weakly compact operator on X satisfies (DE).

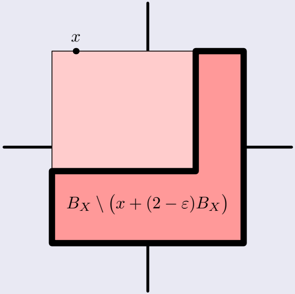
(Kadets–Shvidkoy–Sirotkin–Werner, 1997 & 2000)

The Daugavet property: geometric characterizations

Theorem [KSSW]

X Banach space. TFAE:

- X has the Daugavet property.
- For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y \in S_X$ such that
$$\operatorname{Re} x^*(y) > 1 - \varepsilon \quad \text{and} \quad \|x - y\| \geqslant 2 - \varepsilon.$$
- For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y^* \in S_{X^*}$ such that
$$\operatorname{Re} y^*(x) > 1 - \varepsilon \quad \text{and} \quad \|x^* - y^*\| \geqslant 2 - \varepsilon.$$
- For every $x \in S_X$ and every $\varepsilon > 0$, we have
$$\overline{\text{co}}\Big(B_X \setminus \big(x + (2-\varepsilon)B_X\big)\Big) = B_X\,.$$



The Daugavet property: some results

Some propaganda

X with the Daugavet property. Then:

- X does not have the Radon-Nikodým property.

  (Wojtaszczyk, 1992)

- Every weakly-open subset of B_X has diameter 2.

  (Shvidkoy, 2000)

- X contains a copy of ℓ_1. X^* contains a copy of $L_1[0,1]$.

  (Kadets–Shvidkoy–Sirotkin–Werner, 2000)

- X does not have unconditional basis.

  (Kadets, 1996)

- X does not embed into a unconditional sum of Banach spaces without a copy of ℓ_1.

  (Shvidkoy, 2000)

The DPr, the ADP and numerical index 1

Observation (Duncan-McGregor-Price-White, 1970)

X Banach space, $T \in L(X)$:

- $\sup \operatorname{Re} V(T) = \|T\| \iff \|\mathrm{Id} + T\| = 1 + \|T\|$.
- $v(T) = \|T\| \iff \max_{\theta \in \mathbb{T}} \|\mathrm{Id} + \theta\, T\| = 1 + \|T\|$.

X Banach space:

- Daugavet property (DPr): every rank-one T satisfies

$$\|\mathrm{Id} + T\| = 1 + \|T\| \qquad \text{(DE)}$$

- numerical index 1: EVERY T satisfies

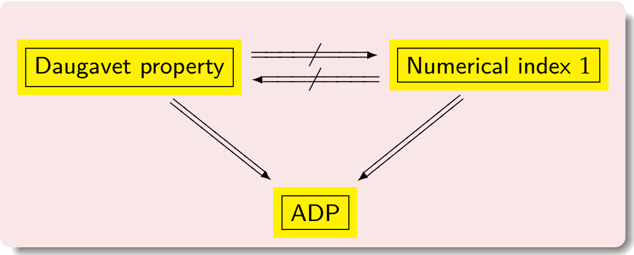
$$\max_{\theta \in \mathbb{T}} \|\mathrm{Id} + \theta\, T\| = 1 + \|T\| \qquad \text{(aDE)}$$

The alternative Daugavet property (M.–Oikhberg, 2004)

alternative Daugavet property (ADP): every rank-one $T \in L(X)$ satisfies (aDE).

★ Then, every weakly compact operator satisfies (aDE).

Relations between the properties



Examples

- $C([0,1], K(\ell_2))$ has DPr, but has not numerical index 1
- c_0 has numerical index 1, but has not DPr
- $c_0 \oplus_\infty C([0,1], K(\ell_2))$ has ADP, neither DPr nor numerical index 1

Remarks

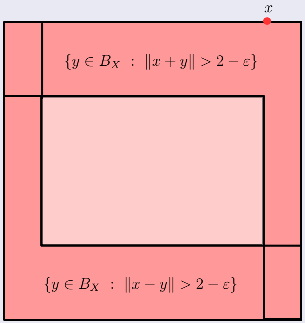
- For RNP or Asplund spaces, $\boxed{\text{ADP}} \Longrightarrow \boxed{\text{numerical index 1}}$.
- Every Banach space with the ADP can be renormed still having the ADP but failing the Daugavet property.

Geometric characterizations of the ADP

Theorem

X Banach space. TFAE:

- X has the ADP.
- For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y \in S_X$ such that
$$|x^*(y)| > 1 - \varepsilon \quad \text{and} \quad \|x - y\| \geqslant 2 - \varepsilon.$$
- For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y^* \in S_{X^*}$ such that
$$|y^*(x)| > 1 - \varepsilon \quad \text{and} \quad \|x^* - y^*\| \geqslant 2 - \varepsilon.$$
- For every $x \in S_X$ and every $\varepsilon > 0$, we have
$$B_X = \overline{\text{co}}\left(\mathbb{T}\,\{y \in B_X \ : \ \|x - y\| \geqslant 2 - \varepsilon\}\right).$$



C^*-algebras and preduals (I)

Let V_* be the predual of the von Neumann algebra V.

The Daugavet property of V_* is equivalent to:

- V has no atomic projections, or
- the unit ball of V_* has no extreme points.

V_* has numerical index 1 iff:

- V is commutative, or
- $|v^*(v)| = 1$ for $v \in \operatorname{ext}(B_V)$ and $v^* \in \operatorname{ext}(B_{V^*})$.

The alternative Daugavet property of V_* is equivalent to:

- the atomic projections of V are central, or
- $|v(v_*)| = 1$ for $v \in \operatorname{ext}(B_V)$ and $v_* \in \operatorname{ext}(B_{V_*})$, or
- $V = C \oplus_\infty N$, where C is commutative and N has no atomic projections.

C^*-algebras and preduals (II)

Let X be a C^*-algebra.

The Daugavet property of X is equivalent to:

- X does not have any atomic projection, or
- the unit ball of X^* does not have any w^*-strongly exposed point.

X has numerical index 1 iff:

- X is commutative, or
- $|x^{**}(x^*)| = 1$ for $x^{**} \in \operatorname{ext}(B_{X^{**}})$ and $x^* \in \operatorname{ext}(B_{X^*})$.

The alternative Daugavet property of X is equivalent to:

- the atomic projections of X are central, or
- $|x^{**}(x^*)| = 1$, for $x^{**} \in \operatorname{ext}(B_{X^{**}})$, and $x^* \in B_{X^*}$ w^*-strongly exposed, or
- $\exists$ a commutative ideal Y such that X/Y has the Daugavet property.

Some results on the ADP: isomorphic properties

Remark

Since when we use the numerical index 1 only rank-one operators may be used, most of the known results are valid for the ADP.

Theorem (López–M.–Payá, 1999)

Not every real Banach space can be renormed with the ADP.

- X real reflexive with ADP $\implies$ X finite-dimensional.
- Moreover, X real, RNP, $\dim(X) = \infty$, and ADP, then $X \supset \ell_1$.

A very recent result (Avilés–Kadets–M.–Merí–Shepelska)

If X is real, $\dim(X) = \infty$ and X has the ADP, then $X^* \supset \ell_1$.

A renorming result (Boyko–Kadets–M.–Merí, 2009)

If X is separable, $X \supset c_0$, then X can be renormed with the ADP.

Some results on the ADP: isometric properties

Remark

Also some isometric properties of Banach spaces with numerical index 1 are actually true for ADP.

Theorem (Kadets–M.–Merí–Payá, 2009)

X infinite-dimensional with the ADP. Then

- X^* is neither smooth nor strictly convex.
- The norm of X cannot be Fréchet-smooth.
- There is no WLUR points in S_X.

Corollary

$X = C(\mathbb{T})/A(\mathbb{D})$. Since $X^* = H^1$ is smooth $\Longrightarrow$ nor X nor H^1 have the ADP.

Open question

Is there X with the ADP which is smooth or strictly convex **?**

Lush spaces

6 Lush spaces
- Definition and examples
- Lush renorming
- Reformulations of lushness and applications
- Lushness is not equivalent to numerical index one

K. Boyko, V. Kadets, M. Martín, and J. Merí.
Properties of lush spaces and applications to Banach spaces with numerical index 1.
Studia Math. (2009).

K. Boyko, V. Kadets, M. Martín, and D. Werner.
Numerical index of Banach spaces and duality.
Math. Proc. Cambridge Philos. Soc. (2007).

V. Kadets, M. Martín, J. Merí, and R. Payá.
Convexity and smoothnes of Banach spaces with numerical index one.
Illinois J. Math. (to appear).

V. Kadets, M. Martín, J. Merí, and V. Shepelska.
Lushness, numerical index one and duality.
J. Math. Anal. Appl. (2009).

Motivation

Remark

- Usually, when we show that a Banach space has numerical index 1, we actually prove more.
- We do not have an operator-free characterization of the spaces with numerical index 1.
- Hence, it makes sense to study geometrical sufficient conditions.

Some sufficient conditions

Let X be a Banach space. Consider:

(a) **Lindenstrauss, 1964:** X has the 3.2.I.P. if the intersection of every family of three mutually intersecting balls is not empty.

(b) **Fullerton, 1961:** X is a CL-space if B_X is the absolutely convex hull of every maximal face of S_X.

(c) **Lima, 1978:** X is an almost-CL-space if B_X is the closed absolutely convex hull of every maximal face of S_X.

$$\boxed{\text{(a)}} \underset{\not\Longleftarrow}{\Longrightarrow} \boxed{\text{(b)}} \underset{\not\Longleftarrow}{\Longrightarrow} \boxed{\text{(c)}} \underset{\not\Longleftarrow}{\Longrightarrow} \boxed{n(X) = 1}$$

Motivation

Some sufficient conditions

Let X be a Banach space. Consider:

(a) **Lindenstrauss, 1964:** X has the 3.2.I.P. if the intersection of every family of three mutually intersecting balls is not empty.

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$$\boxed{(a)} \underset{\not\Longleftarrow}{\Longrightarrow} \boxed{(b)} \underset{\not\Longleftarrow}{\Longrightarrow} \boxed{(c)} \underset{\not\Longleftarrow}{\Longrightarrow} \boxed{n(X)=1}$$

Observation

Showing that (c) $\Longrightarrow n(X)=1$, one realizes that (c) is too much.

Lushness (Boyko–Kadets–M.–Werner, 2007)

X is lush if given $x,y \in S_X$, $\varepsilon > 0$, there is $x^* \in S_{X^*}$ such that

$$x \in S(B_X, x^*, \varepsilon) \quad \text{and} \quad \operatorname{dist}\big(y, \operatorname{aconv}\big(S(B_X, x^*, \varepsilon)\big)\big) < \varepsilon.$$

Definition and first property

Lushness (Boyko–Kadets–M.–Werner, 2007)

X is lush if given $x, y \in S_X$, $\varepsilon > 0$, there is $x^* \in S_{X^*}$ such that

$$x \in S(B_X, x^*, \varepsilon) \quad \text{and} \quad \operatorname{dist}\big(y, \operatorname{aconv}\big(S(B_X, x^*, \varepsilon)\big)\big) < \varepsilon.$$

Theorem

X lush $\implies$ $n(X) = 1$.

Proof.

- $T \in L(X)$ with $\|T\| = 1$, $\varepsilon > 0$. Find $y_0 \in S_X$ which $\|Ty_0\| > 1 - \varepsilon$.
- Use lushness for $x_0 = Ty_0 / \|Ty_0\|$ and y_0 to get $x^* \in S_{X^*}$ and

$$v = \sum_{i=1}^{n} \lambda_i \theta_i x_i \quad \text{where} \quad x_i \in S(B_X, x^*, \varepsilon),\ \lambda_i \in [0,1],\ \sum \lambda_i = 1,\ \theta_i \in \mathbb{T},$$

$$\text{with} \qquad \operatorname{Re} x^*(x_0) > 1 - \varepsilon \qquad \text{and} \qquad \|v - y_0\| < \varepsilon.$$

- Then $|x^*(Tv)| = \left| x^*(x_0) - x^*\left(T\left(\frac{y_0}{\|Ty_0\|} - v\right)\right)\right| \sim \|T\|$.
- By a convexity argument, $\exists\, i$ such that $|x^*(Tx_i)| \sim \|T\|$ and $\operatorname{Re} x^*(x_i) \sim 1$.
- Then $\max_{\omega \in \mathbb{T}} \|\mathrm{Id} + \omega T\| \sim 1 + \|T\| \implies v(T) \sim \|T\|$. ✓

Examples of lush spaces

Examples of lush spaces

1. Almost-CL-spaces.
2. In particular, $C(K)$, $L_1(\mu)$, $C_0(L)\ldots$
3. Preduals of $L_1(\mu)$-spaces.

C-rich subspaces

K compact, X subspace of $C(K)$ is C-rich iff $\forall U$ open nonempty and $\forall \varepsilon > 0$ exists $h : K \longrightarrow [0,1]$ continuous, $\text{supp}(h) \subseteq U$ such that $\text{dist}\big(h, X\big) < \varepsilon$.

More examples of lush spaces

4. C-rich subspaces of $C(K)$.
5. In particular, finite-codimensional subspaces of $C[0,1]$.
6. $C_E(K\|L)$, where L nowhere dense in K and $E \subseteq C(L)$.
7. Y if $c_0 \subseteq Y \subseteq \ell_\infty$ (canonical copies).

Lush rernoming

The goal

When we may get a lush equivalent norm?

Proposition

X separable, $X \supseteq c_0 \implies$ exists $|||\cdot||| \simeq \|\cdot\|$ and $T : (X, |||\cdot|||) \longrightarrow \ell_\infty$ with T isometric embedding & $c_0 \subseteq T(X)$ (canonical copy).

Recall this family of examples of lush spaces

7. Y if $c_0 \subseteq Y \subseteq \ell_\infty$ (canonical copies).

Theorem

X separable, $X \supseteq c_0 \implies X$ admits an equivalent lush norm.

Corollary

Every closed subspace of c_0 admits an equivalent lush norm.

Even more examples of lush spaces

Observation

X Banach space. Consider the following assertions.

(a) Exists $A \subset B_{X^*}$ norming, $|x^{**}(a^*)| = 1$ $\forall a^* \in A$ and $\forall x^{**} \in \text{ext}(B_{X^{**}})$.

(b) For $x \in S_X$ and $\varepsilon > 0$, exists $x^* \in S_{X^*}$ such that

$$x \in S(B_X, x^*, \varepsilon) \qquad \text{and} \qquad B_X = \overline{\text{aconv}}\big(S(B_X, x^*, \varepsilon)\big).$$

(a) $\Longrightarrow$ (b) $\Longrightarrow$ lushness

Definition (Werner, 1997)

X is nicely embedded in $C_b(\Omega)$ if exists $J : X \longrightarrow C_b(\Omega)$ linear isometry with

(N1) $\|J^*\delta_s\| = 1$ $\forall s \in \Omega$,

(N2) $\text{span}(J^*\delta_s)$ L-summand in X^* $\forall s \in \Omega$.

Even more examples of lush spaces

8. Nicely embedded Banach spaces (they fulfil (a)).
9. In particular, function algebras (as $A(\mathbb{D})$ and H^∞).

Some reformulations of lushness

Proposition

X Banach space. TFAE:

- X is lush,
- Every separable $E \subset X$ is contained in a separable lush Y with $E \subset Y \subset X$.

Separable lush spaces (real case)

X real separable. TFAE:

- X is lush.
- There is $G \subseteq S_{X^*}$ norming such that

$$B_X = \overline{\text{aconv}}\left(\{x \in B_X \ : \ x^*(x) = 1\}\right) \qquad (x^* \in G).$$

Therefore, $|x^{**}(x^*)| = 1 \ \forall x^{**} \in \text{ext}\,(B_{X^{**}}) \ \forall x^* \in G$.

Consequence (real case)

$X \subseteq C[0,1]$ strictly convex or smooth $\implies$ $C[0,1]/X$ contains $C[0,1]$.

An important consequence

Remark

X lush separable, $\dim(X) = \infty \implies$ there is $G \in S_{X^*}$ infinite such that

$$|x^{**}(x^*)| = 1 \qquad \big(x^{**} \in \operatorname{ext}(B_{X^{**}}),\ x^* \in G\big).$$

Proposition (López–M.–Payá, 1999)

X real, $A \subset S_X$ infinite such that

$$|x^*(a)| = 1 \qquad \big(x^* \in \operatorname{ext}(B_{X^*}),\ a \in A\big).$$

Then, $X \supseteq c_0$ or $X \supseteq \ell_1$.

Main consequence

X real lush, $\dim(X) = \infty \implies X^* \supseteq \ell_1$.

Question

What happens if just $n(X) = 1$ **?** The same, we will prove later.

Lushness is not equivalent to numerical index one

Example

There is a separable Banach space $\mathcal{X}$ such that

- $\mathcal{X}^*$ is lush but $\mathcal{X}$ is not lush.
- Since $n(\mathcal{X}^*) = 1$, also $n(\mathcal{X}) = 1$.
- The set

$$\{x^* \in S_{\mathcal{X}^*} \ : \ |x^{**}(x^*)| = 1 \text{ for every } x^{**} \in \operatorname{ext}(B_{\mathcal{X}^{**}})\}$$

is empty.

Consequence

$$\boxed{X \text{ lush}} \quad \overset{\not\Longrightarrow}{\underset{\not\Longleftarrow}{}} \quad \boxed{X^* \text{ lush}}$$

Proposition

$$\boxed{X^{**} \text{ lush}} \quad \overset{\Longrightarrow}{\underset{\not\Longleftarrow}{}} \quad \boxed{X \text{ lush}}$$

Slicely countably determined spaces

7. Slicely countably determined spaces
   - Slicely Countably Determined sets and spaces
   - Applications to numerical index 1 spaces
   - SCD operators
   - Open questions

A. Avilés, V. Kadets, M. Martín, J. Merí, and V. Shepelska
Slicely Countably Determined Banach spaces
Trans. Amer. Math. Soc. (to appear)

SCD sets: Definitions and preliminary remarks

X Banach space, $A \subset X$ bounded and convex.

SCD sets

A is Slicely Countably Determined (SCD) if there is a sequence $\{S_n : n \in \mathbb{N}\}$ of slices of A satisfying one of the following equivalent conditions:

- every slice of A contains one of the S_n's,
- $A \subseteq \overline{\text{conv}}(B)$ if $B \subseteq A$ satisfies $B \cap S_n \neq \emptyset\ \forall n$,
- given $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in S_n\ \forall n \in \mathbb{N}$, $A \subseteq \overline{\text{conv}}\big(\{x_n : n \in \mathbb{N}\}\big)$.

Remarks

- A is SCD iff $\overline{A}$ is SCD.
- If A is SCD, then it is separable.

SCD sets: Elementary examples I

Example

A separable and $A = \overline{\text{conv}}(\text{dent}(A)) \implies A$ is SCD.

Proof.

- Take $\{a_n : n \in \mathbb{N}\}$ denting points with $A = \overline{\text{conv}}\big(\{a_n : n \in \mathbb{N}\}\big)$.
- For every $n, m \in \mathbb{N}$, take a slice $S_{n,m}$ containing a_n and of diameter $1/m$.
- If $B \cap S_{n,m} \neq \emptyset \ \forall n, m \in \mathbb{N} \implies a_n \in \overline{B} \ \forall n \in \mathbb{N}$.
- Therefore, $A = \overline{\text{conv}}\big(\{a_n : n \in \mathbb{N}\}\big) \subseteq \overline{\text{conv}}(\overline{B}) = \overline{\text{conv}}(B)$. ✓

Example

In particular, A RNP separable $\implies A$ SCD.

Corollary

- If X is separable LUR $\implies B_X$ is SCD.
- So, every separable space can be renormed such that $B_{(X,|\cdot|)}$ is SCD.

SCD sets: Elementary examples II

Example

If X^* is separable $\Longrightarrow$ A is SCD.

Proof.

- Take $\{x_n^* : n \in \mathbb{N}\}$ dense in S_{X^*}.
- For every $n, m \in \mathbb{N}$, consider $S_{n,m} = S(A, x_n^*, 1/m)$.
- It is easy to show that any slice of A contains one of the $S_{n,m}$. ✓

Negative example

If X has the Daugavet property $\Longrightarrow$ B_X is not SCD.
Therefore, $B_{C[0,1]}$, $B_{L_1[0,1]}$ are not SCD.

Proof.

- Fix $x_0 \in B_X$ and $\{S_n\}$ sequence of slices of B_X.
- By [KSSW] there is a sequence $(x_n) \subset B_X$ such that
  - $x_n \in S_n$ for every $n \in \mathbb{N}$,
  - $(x_n)_{n\geqslant 0}$ is equivalent to the basis of ℓ_1,
  - so $x_0 \notin \overline{\text{lin}}\{x_n : n \in \mathbb{N}\}$. ✓

SCD sets: Further examples I

Convex combination of slices

$$W = \sum_{k=1}^{m} \lambda_k S_k \subset A \text{ where } \lambda_k \geqslant 0,\ \sum \lambda_k = 1,\ S_k \text{ slices.}$$

Proposition

In the definition of SCD we can use a sequence $\{S_n : n \in \mathbb{N}\}$ of convex combination of slices.

Small combinations of slices

A has small combinations of slices iff every slice of A contains convex combinations of slices of A with arbitrary small diameter.

Example

If A has small combinations of slices + separable $\Longrightarrow$ A is SCD.

Particular case

A strongly regular + separable $\Longrightarrow$ A is SCD.

SCD sets: Further examples II

Bourgain's lemma

Every relative weak open subset of A contains a convex combination of slices.

Corollary

In the definition of SCD we can use a sequence $\{S_n : n \in \mathbb{N}\}$ of relative weak open subsets.

π-bases

A π-base of the weak topology of A is a family $\{V_i : i \in I\}$ of weak open sets of A such that every weak open subset of A contains one of the V_i's.

Proposition

If $(A, \sigma(X, X^*))$ has a countable π-base $\Longrightarrow A$ is SCD.

SCD sets: Further examples III

Theorem

A separable without ℓ_1-sequences $\Longrightarrow$ $(A,\sigma(X,X^*))$ has a countable π-base.

Proof.

- We see $(A,\sigma(X,X^*)) \subset C(T)$ where $T = (B_{X^*},\sigma(X^*,X))$.
- By Rosenthal ℓ_1 theorem, $(A,\sigma(X,X^*))$ is a relatively compact subset of the space of first Baire class functions on T.
- By a result of Todorčević, $(A,\sigma(X,X^*))$ has a σ-disjoint π-base.
- $\{V_i : i \in I\}$ is σ-*disjoint* if $I = \bigcup_{n\in\mathbb{N}} I_n$ and each $\{V_i : i \in I_n\}$ is pairwise disjoint.
- A σ-disjoint family of open subsets in a separable space is countable. ✓

Example

A separable without ℓ_1-sequences $\Longrightarrow$ A is SCD.

SCD spaces: definition and examples

SCD space

X is Slicely Countably Determined (SCD) if so are its convex bounded subsets.

Examples of SCD spaces

1. X separable strongly regular. In particular, RNP, CPCP spaces.
2. X separable $X \not\supseteq \ell_1$. In particular, if X^* is separable.

Examples of NOT SCD spaces

1. X having the Daugavet property.
2. In particular, $C[0,1]$, $L_1[0,1]$
3. There is X with the Schur property which is not SCD.

Remark

- Every subspace of a SCD space is SCD.
- This is false for quotients.

SCD spaces: stability properties

Theorem

$Z \subset X$. If Z and X/Z are SCD $\Longrightarrow$ X is SCD.

Corollary

X separable NOT SCD

- If $\ell_1 \simeq Y \subset X \Longrightarrow X/Y$ contains a copy of ℓ_1.
- If $\ell_1 \simeq Y_1 \subset X \Longrightarrow$ there is $\ell_1 \simeq Y_2 \subset X$ with $Y_1 \cap Y_2 = 0$.

Corollary

$X_1, \ldots, X_m$ SCD $\Longrightarrow$ $X_1 \oplus \cdots \oplus X_m$ SCD.

SCD spaces: stability properties II

Theorem

$X_1, X_2, \dots$ SCD, E with unconditional basis.

- $E \not\supseteq c_0 \Longrightarrow [\bigoplus_{n\in\mathbb{N}} X_n]_E$ SCD.
- $E \not\supseteq \ell_1 \Longrightarrow [\bigoplus_{n\in\mathbb{N}} X_n]_E$ SCD.

Examples

1. $c_0(\ell_1)$ and $\ell_1(c_0)$ are SCD.
2. $c_0 \otimes_\varepsilon c_0$, $c_0 \otimes_\pi c_0$, $c_0 \otimes_\varepsilon \ell_1$, $c_0 \otimes_\pi \ell_1$, $\ell_1 \otimes_\varepsilon \ell_1$, and $\ell_1 \otimes_\pi \ell_1$ are SCD.
3. $K(c_0)$ and $K(c_0, \ell_1)$ are SCD.
4. $\ell_2 \otimes_\varepsilon \ell_2 \equiv K(\ell_2)$ and $\ell_2 \oplus_\pi \ell_2 \equiv \mathcal{L}_1(\ell_2)$ are SCD

The DPr, the ADP and numerical index 1

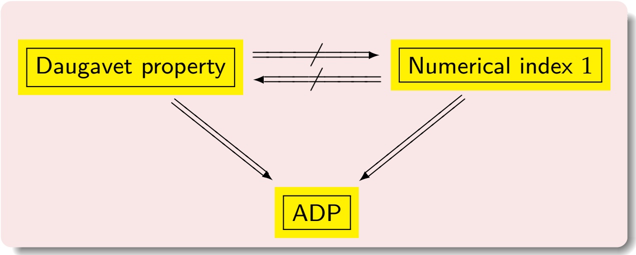
Recalling the properties

1. **Kadets-Shvidkoy-Sirotkin-Werner, 1997:**
   X has the Daugavet property (DPr) if
   $$\|\mathrm{Id} + T\| = 1 + \|T\| \qquad \text{(DE)}$$
   for every rank-one $T \in L(X)$.
   ★Then every weakly compact T also satisfies (DE).

2. **Lumer, 1968:** X has numerical index 1 if EVERY operator on X satisfies
   $$\max_{\theta\in\mathbb{T}} \|\mathrm{Id} + \theta\, T\| = 1 + \|T\| \qquad \text{(aDE)}$$
   ★Equivalently, $v(T) = \|T\|$ for EVERY $T \in L(X)$.

3. **M.-Oikhberg, 2004:** X has the alternative Daugavet property (ADP) if every rank-one $T \in L(X)$ satisfies (aDE).
   ★Then every weakly compact T also satisfies (aDE).

Relations between these properties



Examples

- $C([0,1], K(\ell_2))$ has DPr, but has not numerical index 1
- c_0 has numerical index 1, but has not DPr
- $c_0 \oplus_\infty C([0,1], K(\ell_2))$ has ADP, neither DPr nor numerical index 1

Remarks

- For RNP or Asplund spaces, $\boxed{\text{ADP}} \Longrightarrow \boxed{\text{numerical index 1}}$.
- Every Banach space with the ADP can be renormed still having the ADP but failing the Daugavet property.

ADP + SCD $\Longrightarrow$ numerical index 1

Characterizations of the ADP

X Banach space. TFAE:

- X has ADP (i.e. $\max_{\theta\in\mathbb{T}} \|\mathrm{Id}+\theta\, T\| = 1+\|T\|$ for all T rank-one).
- Given $x\in S_X$, a slice S of B_X and $\varepsilon>0$, there is $y\in S$ with

$$\max_{\theta\in\mathbb{T}} \|x+\theta\, y\| > 2-\varepsilon.$$

- Given $x\in S_X$, a sequence $\{S_n\}$ of slices of B_X, and $\varepsilon>0$, there is $y^*\in S_{X^*}$ such that $x\in S(B_X,y^*,\varepsilon)$ and

$$\overline{\mathrm{conv}}\big(\mathbb{T}\, S(B_X,y^*,\varepsilon)\big) \bigcap S_n \neq \emptyset \qquad (n\in\mathbb{N}).$$

Theorem

X ADP + B_X SCD $\Longrightarrow$ given $x\in S_X$ and $\varepsilon>0$, there is $y^*\in S_{X^*}$ such that

$$x\in S(B_X,y^*,\varepsilon) \quad \text{and} \quad B_X = \overline{\mathrm{conv}}\big(\mathbb{T}\, S(B_X,y^*,\varepsilon)\big).$$

★ This implies lushness and so, numerical index 1.

Some consequences

Corollary

- ADP + strongly regular $\Longrightarrow$ numerical index 1 (actually, lushness).
- ADP + $X \not\supseteq \ell_1$ $\Longrightarrow$ numerical index 1 (actually, lushness).

Corollary

X real + $\dim(X) = \infty$ + ADP $\Longrightarrow$ $X^* \supseteq \ell_1$.

In particular,

Corollary

X real + $\dim(X) = \infty$ + numerical index 1 $\Longrightarrow$ $X^* \supseteq \ell_1$.

Open question

X real, $\dim(X) = \infty$, $n(X) = 1$ $\Longrightarrow$ $X \supset c_0$ or $X \supset \ell_1$?

SCD operators

SCD operator

$T \in L(X)$ is an SCD-operator if $T(B_X)$ is an SCD-set.

Examples

T is an SCD-operator when $T(B_X)$ is separable and

1. $T(B_X)$ is RPN,
2. $T(B_X)$ has no ℓ_1 sequences,
3. T does not fix copies of ℓ_1

Theorem

- X ADP + T SCD-operator $\Longrightarrow$ $\max_{\theta \in \mathbb{T}} \|\mathrm{Id} + \theta\, T\| = 1 + \|T\|$.
- X DPr + T SCD-operator $\Longrightarrow$ $\|\mathrm{Id} + T\| = 1 + \|T\|$.

Main corollary

X ADP + T does not fix copies of ℓ_1 $\Longrightarrow$ $\max_{\theta \in \mathbb{T}} \|\mathrm{Id} + \theta\, T\| = 1 + \|T\|$.

Open questions

On SCD-sets

- Find more sufficient conditions for a set to be SCD.
- For instance, if X has 1-symmetric basis, is B_X an SCD-set **?**
- Is SCD equivalent to the existence of a countable π-base for the weak topology **?**

On SCD-spaces

- E with unconditional basis. Is E SCD **?**
- X, Y SCD. Are $X \otimes_\varepsilon Y$ and $X \otimes_\pi Y$ SCD **?**

On SCD-operators

- T_1, T_2 SCD-operators, is $T_1 + T_2$ an SCD-operator **?**
- $T : X \longrightarrow Y$ hereditary SCD, is there Z SCD-space such that T factor through Z **?**

Remarks on two recent results

8 Remarks on two recent results
- Containment of c_0 or ℓ_1
- On the numerical index of $L_p(\mu)$

A. Avilés, V. Kadets, M. Martín, J. Merí, and V. Shepelska.
Slicely countably determined Banach spaces.
Trans. Amer. Math. Soc. (to appear).

V. Kadets, M. Martín, J. Merí, and R. Payá.
Smoothness and convexity for Banach spaces with numerical index 1.
Illinois J. Math. (to appear).

M. Martín, J. Merí, and M. Popov.
On the numerical index of real $L_p(\mu)$-spaces.
Preprint.

Containment of c_0 or ℓ_1

Open question (Godefroy, private communication)

X real, $\dim(X)=\infty$, $n(X)=1 \implies X\supset c_0$ or $X\supset \ell_1$ **?**

★ Old approaches to this problem:

- López–M.–Payá, 1999:
  X real, RNP, $\dim(X)=\infty$, $n(X)=1 \implies X\supset \ell_1$.
- Kadets–M.–Merí–Payá, 2009:
  X real lush, $\dim(X)=\infty \implies X^*\supset \ell_1$.
- Avilés–Kadets–M.–Merí–Shepelska, 2010:
  X real, $\dim(X)=\infty \implies X^*\supset \ell_1$.

★ Equivalent reformulation of the problem:

Equivalent open problem

X real separable, $X\not\supseteq \ell_1$, exists $G\subseteq S_{X^*}$ norming with

$$B_X=\overline{\operatorname{aconv}}\left(\{x\in B_X \ :\ x^*(x)=1\}\right) \qquad (x^*\in G).$$

Does $X\supseteq c_0$ **?**

On the numerical index of $L_p(\mu)$. I

The numerical radius for $L_p(\mu)$

For $T \in L\big(L_p(\mu)\big)$, $1 < p < \infty$, one has

$$v(T) = \sup\left\{ \left| \int_\Omega x^{\#} T x \, d\mu \right| \ : \ x \in L_p(\mu),\ \|x\|_p = 1 \right\}.$$

where for $x \in L_p(\mu)$, $x^{\#} = |x|^{p-1}\,\mathrm{sign}(x) \in L_q(\mu)$ satisfies (unique)

$$\|x\|_p^p = \|x^{\#}\|_q^q \qquad \text{and} \qquad \int_\Omega x\, x^{\#}\, d\mu = \|x\|_p \, \|x^{\#}\|_q = \|x\|_p^p.$$

The absolute numerical radius

For $T \in L\big(L_p(\mu)\big)$ we write

$$\begin{aligned} |v|(T) &:= \sup\left\{ \int_\Omega |x^{\#} T x| \, d\mu \ : \ x \in L_p(\mu),\ \|x\|_p = 1 \right\} \\ &= \sup\left\{ \int_\Omega |x|^{p-1} |Tx| \, d\mu \ : \ x \in L_p(\mu),\ \|x\|_p = 1 \right\} \end{aligned}$$

On the numerical index of $L_p(\mu)$ (II)

Theorem

For $T \in L\big(L_p(\mu)\big)$, $1 < p < \infty$, one has

$$v(T) \geqslant \frac{M_p}{4}\,|v|(T), \qquad \text{where} \quad M_p = \max_{t\in[0,1]} \frac{|t^{p-1} - t|}{1 + t^p}.$$

Theorem

For $T \in L\big(L_p(\mu)\big)$, $1 < p < \infty$, one has

$$2\,|v|(T) \geqslant v(T_{\mathbb{C}}) \geqslant n\big(L_p^{\mathbb{C}}(\mu)\big)\,\|T\|,$$

- $T_{\mathbb{C}}$ complexification of T, $n\big(L_p^{\mathbb{C}}(\mu)\big)$ numerical index *complex case*.

Consequence

For $1 < p < \infty$, $n\big(L_p(\mu)\big) \geqslant \dfrac{M_p}{8e}$.

- If $p \neq 2$, then $n\big(L_p(\mu)\big) > 0$, so v and $\|\cdot\|$ are equivalent in $L\big(L_p(\mu)\big)$.

Extremely non-complex Banach spaces

9 Extremely non-complex Banach spaces
- Motivation
- Extremely non-complex Banach spaces
- Surjective isometries

V. Kadets, M. Martín, and J. Merí.
Norm equalities for operators on Banach spaces.
Indiana U. Math. J. (2007).

P. Koszmider, M. Martín, and J. Merí.
Extremely non-complex $C(K)$ spaces.
J. Math. Anal. Appl. (2009).

P. Koszmider, M. Martín, and J. Merí.
Isometries on extremely non-complex Banach spaces.
Preprint (2008).

Isometries and duality. Reminder

Example (produced with numerical ranges)

There is a Banach space X such that

- $\mathrm{Iso}(X)$ has no exponential one-parameter semigroups.
- $\mathrm{Iso}(X^*)$ contains infinitely many exponential one-parameter semigroups.

★ In terms of linear dynamical systems:

- There is no $A \in L(X)$ such that

$$x' = A\,x \qquad \left(x : \mathbb{R}_0^+ \longrightarrow X\right)$$

  is given by a semigroup of isometries.
- There are infinitely many such A's on X^*
- But there are unbounded As on X such that the solution of the linear dynamical system is a one-parameter C_0 semigroup of isometries.

We would like to find $\mathcal{X}$ such that

- $\mathrm{Iso}(\mathcal{X})$ has no C_0 semigroup of isometries.
- $\mathrm{Iso}(\mathcal{X}^*)$ has exponential semigroup of isometries

Numerical range of unbounded operators

Numerical range of unbounded operators (1960's)

X Banach space, $T : D(T) \longrightarrow X$ linear,

$$V(T) = \{x^*(Tx) \ : \ x^* \in X^*, \ x \in D(T), \ x^*(x) = \|x^*\| = \|x\| = 1\}.$$

Teorema (Stone, 1932)

H Hilbert space, A densely defined operator. TFAE:

- A generates an strongly continuous one-parameter semigroup of unitary operators (onto isometries).
- $A^* = -A$.
- $\mathrm{Re}(Ax \mid x) = 0$ for every $x \in D(A)$.

Numerical range of unbounded operators. II

Difficulty

Which Banach spaces have unbounded operators with numerical range zero?

Examples

- In $C_0(\mathbb{R})$, $\Phi(t)(f)(s) = f(t+s)$ is an strongly continuous one-parameter semigroup of isometries (generated by the derivative).
- In $C_E([0,1]\|\Delta)$ there are also strongly continuous one-parameter semigroup of isometries.

Consequence

We have to completely change our approach to the problem.

Complex structures

Definition

X has complex structure if there is $T \in L(X)$ such that $T^2 = -\mathrm{Id}$.

Some remarks

- This gives a structure of vector space over $\mathbb{C}$:

$$(\alpha + i\,\beta)\,x = \alpha\,x + \beta\,T(x) \qquad \big(\alpha + i\,\beta \in \mathbb{C},\ x \in X\big)$$

- Defining

$$|||x||| = \max\{\|e^{i\theta}x\| \ : \ \theta \in [0, 2\pi]\} \qquad (x \in X)$$

  one gets that $(X, |||\cdot|||)$ is a complex Banach space.
- If T is an isometry, then actually the given norm of X is complex.
- Conversely, if X is a complex Banach space, then

$$T(x) = i\,x \qquad (x \in X)$$

  satisfies $T^2 = -\mathrm{Id}$ and T is an isometry.

Complex structures II

Some examples

1. If $\dim(X) < \infty$, X has complex structure iff $\dim(X)$ is even.
2. If $X \simeq Z \oplus Z$ (in particular, $X \simeq X^2$), then X has complex structure.
3. There are infinite-dimensional Banach spaces without complex structure:
   - **Dieudonné, 1952:** the James' space $\mathcal{J}$ (since $\mathcal{J}^{**} \equiv \mathcal{J} \oplus \mathbb{R}$).
   - **Szarek, 1986:** uniformly convex examples.
   - **Gowers-Maurey, 1993:** their H.I. space.
   - **Ferenczi-Medina Galego, 2007:** there are odd and even infinite-dimensional spaces X.
     - X is even if admits a complex structure but its hyperplanes does not.
     - X is odd if its hyperplanes are even (and so X does not admit a complex structure).

Definition

X is extremely non-complex if $\operatorname{dist}(T^2, -\mathrm{Id})$ is the maximum possible, i.e.

$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\| \qquad (T \in L(X))$$

The Daugavet equation

What Daugavet did in 1963

The norm equality

$$\|\mathrm{Id} + T\| = 1 + \|T\|$$

holds for every compact $T \in L\big(C[0,1]\big)$.

The Daugavet equation

X Banach space, $T \in L(X)$, $\|\mathrm{Id} + T\| = 1 + \|T\|$ (DE).

Classical examples

1. **Daugavet, 1963:**
   Every compact operator on $C[0,1]$ satisfies (DE).
2. **Lozanoskii, 1966:**
   Every compact operator on $L_1[0,1]$ satisfies (DE).
3. **Abramovich, Holub, and more, 80's:**
   $X = C(K)$, K perfect compact space
   or $X = L_1(\mu)$, μ atomless measure
   $\Longrightarrow$ every weakly compact $T \in L(X)$ satisfies (DE).

The Daugavet property

The Daugavet property (Kadets–Shvidkoy–Sirotkin–Werner, 1997)

A Banach space X is said to have the Daugavet property iff every rank-one operator on X satisfies (DE).

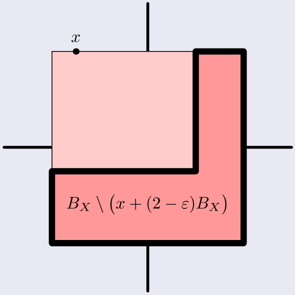
Some results

Let X be a Banach space with the Daugavet property. Then

- Every weakly compact operator on X satisfies (DE).
- X contains ℓ_1.
- X does not embed into a Banach space with unconditional basis.
- **Geometric characterization**: X has the Daugavet property iff for each $x \in S_X$

$$\overline{\text{co}}\left(B_X \setminus \big(x + (2-\varepsilon)B_X\big)\right) = B_X\,.$$

(Kadets–Shvidkoy–Sirotkin–Werner, 1997 & 2000)



The Daugavet property II

More examples

The following spaces have the Daugavet property.

- **Wojtaszczyk, 1992:**
  The disk algebra and H^∞.
- **Werner, 1997:**
  "Nonatomic" function algebras.
- **Oikhberg, 2005:**
  Non-atomic C^*-algebras and preduals of non-atomic von Neumann algebras.
- **Becerra–M., 2005**:
  Non-atomic JB^*-triples and their preduals.
- **Becerra–M., 2006:**
  Preduals of $L_1(\mu)$ without Fréchet-smooth points.
- **Ivankhno, Kadets, Werner, 2007:**
  Lip(K) when $K \subseteq \mathbb{R}^n$ is compact and convex.

Daugavet–type inequalities

Some examples

- **Benyamini–Lin, 1985:**
  For every $1 < p < \infty$, $p \neq 2$, there exists $\psi_p : (0,\infty) \longrightarrow (0,\infty)$ such that
  $$\|\mathrm{Id} + T\| \geqslant 1 + \psi_p(\|T\|)$$
  for every compact operator T on $L_p[0,1]$.
  - If $p = 2$, then there is a non-null compact T on $L_2[0,1]$ such that
    $$\|\mathrm{Id} + T\| = 1.$$
- **Boyko–Kadets, 2004:**
  If ψ_p is the best possible function above, then
  $$\lim_{p\to 1^+} \psi_p(t) = t \qquad (t > 0).$$
- **Oikhberg, 2005:**
  If $K(\ell_2) \subseteq X \subseteq L(\ell_2)$, then
  $$\|\mathrm{Id} + T\| \geqslant 1 + \frac{1}{8\sqrt{2}}\|T\|$$
  for every compact T on X.

Norm equalities for operators

Motivating question

Are there other norm equalities which could define interesting properties of Banach spaces **?**

Concretely

We looked for non-trivial norm equalities of the forms

$$\|\mathrm{Id}+T\| = f(\|T\|) \quad \text{or} \quad \|g(T)\| = f(\|T\|) \quad \text{or} \quad \|\mathrm{Id}+g(T)\| = f(\|g(T)\|)$$

(g analytic, f arbitrary) satisfied by all rank-one operators on a Banach space.

Solution

We proved that there are few possibilities. . .

Equalities of the form $\|\mathrm{Id} + T\| = f(\|T\|)$

Proposition

X real or complex, $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ arbitrary, $a, b \in \mathbb{K}$. If the norm equality

$$\|a\,\mathrm{Id} + b\,T\| = f(\|T\|)$$

holds for every rank-one operator $T \in L(X)$, then

$$f(t) = |a| + |b|\,t \qquad \left(t \in \mathbb{R}_0^+\right).$$

If $a \neq 0$, $b \neq 0$, then X has the Daugavet property.

Then, we have to look for Daugavet-type equalities in which $\mathrm{Id} + T$ is replaced by something different.

Proof

We have...

$\|a\,\mathrm{Id} + b\,T\| = f(\|T\|)\ \forall T \in L(X)$ rank-one

$\overset{?}{\Longrightarrow}$

We want...

$f(t) = |a| + |b|\,t \qquad (t \in \mathbb{R}_0^+).$

- Trivial if $a \cdot b = 0$. Suppose $a \neq 0$ and $b \neq 0$ and write $\omega_0 = \frac{\overline{b}}{|b|}\frac{a}{|a|} \in \mathbb{T}$.
- Fix $x_0 \in S_X$, $x_0^* \in S_{X^*}$ with $x_0^*(x_0) = \omega_0$ and consider

$$T_t = t\,x_0^* \otimes x_0 \in L(X) \qquad (t \in \mathbb{R}_0^+).$$

- Since $\|T_t\| = t$, we have

$$f(t) = \|a\mathrm{Id} + b\,T_t\| \qquad (t \in \mathbb{R}_0^+).$$

- It follows that

$$|a| + |b|\,t \geqslant f(t) = \|a\mathrm{Id} + b\,T_t\| \geqslant \big\|[a\mathrm{Id} + b\,T_t](x_0)\big\|$$

$$= \|a\,x_0 + b\,\omega_0 t\,x_0\| = |a + b\,\omega_0 t|\,\|x_0\| = \left|a + b\frac{\overline{b}}{|b|}\frac{a}{|a|}\,t\right| = |a| + |$$

- Finally, for rank-one $T \in L(X)$, write $S = \frac{a}{b}\,T$ and observe

$$|a|\big(1 + \|T\|\big) = |a| + |b|\,\|S\| = \|a\mathrm{Id} + b\,S\| = |a|\,\|\mathrm{Id} + T\|.\checkmark$$

Equalities of the form $\|g(T)\| = f(\|T\|)$

Theorem

X real or complex with $\dim(X) \geqslant 2$.
Suppose that the norm equality

$$\|g(T)\| = f(\|T\|)$$

holds for every rank-one operator $T \in L(X)$, where

- $g : \mathbb{K} \longrightarrow \mathbb{K}$ is analytic,
- $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ is arbitrary.

Then, there are $a, b \in \mathbb{K}$ such that

$$g(\zeta) = a + b\,\zeta \qquad (\zeta \in \mathbb{K}).$$

Corollary

Only three norm equalities of the form

$$\|g(T)\| = f(\|T\|)$$

are possible:

- $b = 0$: $\|a\,\mathrm{Id}\| = |a|$,
- $a = 0$: $\|b\,T\| = |b|\,\|T\|$,

  (trivial cases)
- $a \neq 0,\ b \neq 0$:
  $\|a\,\mathrm{Id} + b\,T\| = |a| + |b|\,\|T\|$,

  (Daugavet property)

Proof (complex case)

We have...

$\|g(T)\| = f(\|T\|)\ \forall T \in L(X)$ rank-one

$\overset{?}{\Longrightarrow}$

We want...

g is affine

- Write $g(\zeta) = \sum_{k=0}^{\infty} a_k \zeta^k$ y $\widetilde{g} = g - a_0$.
- Take $x_0, x_1 \in S_X$ and $x_0^*, x_1^* \in S_{X^*}$ such that
$$x_0^*(x_0) = 0 \qquad \text{and} \qquad x_1^*(x_1) = 1,$$
and define the operators $T_0 = x_0^* \otimes x_0$ and $T_1 = x_1^* \otimes x_1$.
- Then $g(\lambda T_0) = a_0 \mathrm{Id} + a_1 \lambda\, T_0$ and $g(\lambda T_1) = a_0 \mathrm{Id} + \widetilde{g}(\lambda)\, T_1$ $(\lambda \in \mathbb{C})$.
- Therefore, for $\lambda \in \mathbb{C}$ we have
$$\|a_0 \mathrm{Id} + \widetilde{g}(\lambda) T_1\| = \|g(\lambda T_1)\| = f(|\lambda|) = \|g(\lambda T_0)\| = \|a_0 \mathrm{Id} + a_1 \lambda T_0\|.$$
- We use the triangle inequality to get
$$|\widetilde{g}(\lambda)| \leqslant 2|a_0| + |a_1||\lambda| \qquad (\lambda \in \mathbb{C}),$$
- and so $\widetilde{g}$ is a degree-one polynomial by Cauchy inequalities. ✓

Equalities of the form $\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$

Remark

If X has the Daugavet property and g is analytic, then

$$\|\mathrm{Id} + g(T)\| = |1 + g(0)| - |g(0)| + \|g(T)\|$$

for every rank-one $T \in L(X)$.

- Our aim here is not to show that g has a suitable form,
- but it is to see that for every g another simpler equation can be found.
- From now on, we have to separate the complex and the real case.

Equalities of the form $\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$

- **Complex case:**

Proposition

X complex, $\dim(X) \geqslant 2$. Suppose that

$$\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$$

for every rank-one $T \in L(X)$, where

- $g : \mathbb{C} \longrightarrow \mathbb{C}$ analytic non-constant,
- $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ *continuous*.

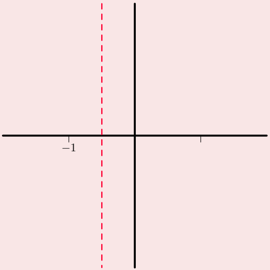
Then

$$\big\|(1 + g(0))\mathrm{Id} + T\big\| = |1 + g(0)| - |g(0)| + \big\|g(0)\mathrm{Id} + T\big\|$$

for every rank-one $T \in L(X)$.

We obtain two different cases:

- $|1 + g(0)| - |g(0)| \neq 0$ or
- $|1 + g(0)| - |g(0)| = 0$.



Equalities of the form $\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$. Complex case

Theorem

If $\operatorname{Re} g(0) \neq -1/2$ and

$$\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$$

for every rank-one T, then X has the Daugavet property.

Theorem

If $\operatorname{Re} g(0) = -1/2$ and

$$\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$$

for every rank-one T, then exists $\theta_0 \in \mathbb{R}$ s.t.

$$\left\|\mathrm{Id} + e^{i\theta_0}\, T\right\| = \|\mathrm{Id} + T\|$$

for every rank-one $T \in L(X)$.

Example

If $X = C[0,1] \oplus_2 C[0,1]$, then

- $\left\|\mathrm{Id} + e^{i\theta}\, T\right\| = \|\mathrm{Id} + T\|$
  for every $\theta \in \mathbb{R}$, rank-one $T \in L(X)$.
- X does not have the Daugavet property.

Equalities of the form $\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$. Real case

- Real case:

Remarks

- The proofs are not valid (we use Picard's Theorem).
- They work when g is onto.
- But we do not know what is the situation when g is not onto, even in the easiest examples:
  - $\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$,
  - $\|\mathrm{Id} - T^2\| = 1 + \|T^2\|$.

$g(0) = -1/2$:

Example

If $X = C[0,1] \oplus_2 C[0,1]$, then

- $\|\mathrm{Id} - T\| = \|\mathrm{Id} + T\|$ for every rank-one $T \in L(X)$.
- X does not have the Daugavet property.

The question

Godefroy, private communication

Is there any real Banach space X (with $\dim(X) > 1$) such that

$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$$

for every operator $T \in L(X)$ **?**

In other words, are there extremely non-complex spaces other than $\mathbb{R}$ **?**

The first attempts

The first idea

We may try to check whether the known spaces without complex structure are actually extremely non-complex.

Some examples

1. If $\dim(X) < \infty$, X has complex structure iff $\dim(X)$ is even.
2. **Dieudonné, 1952:** the James' space $\mathcal{J}$ (since $\mathcal{J}^{**} \equiv \mathcal{J} \oplus \mathbb{R}$).
3. **Szarek, 1986:** uniformly convex examples.
4. **Gowers-Maurey, 1993:** their H.I. space.
5. **Ferenczi-Medina Galego, 2007:** there are odd and even infinite-dimensional spaces X.
    - X is even if admits a complex structure but its hyperplanes does not.
    - X is odd if its hyperplanes are even (and so X does not admit a complex structure).

(Un)fortunately...

This did not work and we moved to $C(K)$ spaces.

The first example: weak multiplications

Weak multiplication

Let K be a compact space. $T \in L\big(C(K)\big)$ is a weak multiplication if

$$T = g\,\mathrm{Id} + S$$

where $g \in C(K)$ and S is weakly compact.

Theorem

K perfect, $T = g\,\mathrm{Id} + S \in L\big(C(K)\big)$ weak multiplication
$\Longrightarrow \quad \|\mathrm{Id} + T^2\| = 1 + \|T^2\|$

Proof of the theorem

We have $X = C(K)$, K perfect, $T = g\mathrm{Id} + S$

- $\max \|\mathrm{Id} \pm T\| = 1 + \|T\|$ (true for every K and every T)
- $\|\mathrm{Id} + S\| = 1 + \|S\|$ (if $S \in W(X)$, K perfect)

We need

$\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$

- If $T = g\mathrm{Id} + S$, then $T^2 = g^2\,\mathrm{Id} + S'$ with S' weakly compact.
- We will prove that $\|\mathrm{Id} + g^2\,\mathrm{Id} + S\| = 1 + \|g^2\,\mathrm{Id} + S\|$ for $g \in C(K)$ and S weakly compact.
- **Step 1:** We assume $\|g^2\| \leqslant 1$ and $\min g^2(K) > 0$.
- **Step 2:** We can avoid the assumption that $\min g^2(K) > 0$.
- **Step 3:** Finally, for every g the above gives

$$\left\|\mathrm{Id} + \frac{1}{\|g^2\|}\left(g^2\,\mathrm{Id} + S\right)\right\| = 1 + \frac{1}{\|g^2\|}\|g^2\,\mathrm{Id} + S\|$$

which gives us the result. ✓

The first example: weak multiplications. II

Weak multiplication

Let K be a compact space. $T \in L\big(C(K)\big)$ is a weak multiplication if

$$T = g\,\mathrm{Id} + S$$

where $g \in C(K)$ and S is weakly compact.

Theorem

K perfect, $T = g\,\mathrm{Id} + S \in L\big(C(K)\big)$ weak multiplication
$\Longrightarrow \ \|\mathrm{Id} + T^2\| = 1 + \|T^2\|$

Example (Koszmider, 2004; Plebanek, 2004)

There are perfect compact spaces K such that all operators on $C(K)$ are weak multiplications.

Consequence

Therefore, there are extremely non-complex $C(K)$ spaces.

More examples: weak multipliers

Weak multiplier

Let K be a compact space. $T \in L\big(C(K)\big)$ is a weak multiplier if

$$T^* = g\,\mathrm{Id} + S$$

where g is a Borel function and S is weakly compact.

Theorem

If K is perfect and all operators on $C(K)$ are weak multipliers, then $C(K)$ is extremely non-complex.

Example (Koszmider, 2004)

There are infinitely many different perfect compact spaces K such that all operators on $C(K)$ are weak multipliers.

Corollary

There are infinitely many non-isomorphic extremely non-complex Banach spaces.

Further examples

Proposition

There is a compact infinite totally disconnected and perfect space K such that all operators on $C(K)$ are weak multipliers.

Consequence

There is a family $(K_i)_{i\in I}$ of pairwise disjoint perfect and totally disconnected compact spaces such that

- every operator on $C(K_i)$ is a weak multiplier,
- for $i \neq j$, every $T \in L(C(K_i), C(K_j))$ is weakly compact.

Theorem

There are some compactifications $\widetilde{K}$ of the above family $(K_i)_{i\in I}$ such that the corresponding $C(\widetilde{K})$'s are extremely non-complex.

Further examples II

Main consequence

There are perfect compact spaces K_1, K_2 such that:

- $C(K_1)$ and $C(K_2)$ are extremely non-complex,
- $C(K_1)$ contains a complemented copy of $C(\Delta)$.
- $C(K_2)$ contains a 1-complemented isometric copy of ℓ_∞.

Observation

- $C(K_1)$ and $C(K_2)$ have operators which are not weak multipliers.
- They are not indecomposable spaces.

Related open questions

Question 1

Find topological characterization of the compact Hausdorff spaces K such that the spaces $C(K)$ are extremely non-complex.

Question 2

Find topological consequences on K when $C(K)$ is extremely non-complex.
For instance:
If $C(K)$ is extremely non-complex and $\psi : K \longrightarrow K$ is continuous, are there an open subset U of K such that $\psi|_U = \mathrm{id}$ and $\psi(K \setminus U)$ is finite **?**

- We will show latter than $\varphi : K \longrightarrow K$ homeomorphism $\implies$ $\varphi = \mathrm{id}$.

Extremely non-complex Banach spaces

Definition

X is extremely non-complex if $\operatorname{dist}(T^2, -\mathrm{Id})$ is the maximum possible, i.e.

$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\| \qquad (T \in L(X))$$

Examples

There are several extremely non-complex $C(K)$ spaces:

- If $T = g\mathrm{Id} + S$ for every $T \in L(C(K))$ (K Koszmider).
- If $T^* = g\mathrm{Id} + S$ for every $T \in L(C(K))$ (K weak Koszmider).
- One $C(K)$ containing a complemented copy of $C(\Delta)$.
- One $C(K)$ containing an isometric (1-complemented) copy of ℓ_∞.

Isometries on extremely non-complex spaces. I

Theorem

X extremely non-complex.

- $T \in \mathrm{Iso}(X) \implies T^2 = \mathrm{Id}$.
- $T_1, T_2 \in \mathrm{Iso}(X) \implies T_1 T_2 = T_2 T_1$.
- $T_1, T_2 \in \mathrm{Iso}(X) \implies \|T_1 - T_2\| \in \{0, 2\}$.
- $\Phi : \mathbb{R}_0^+ \longrightarrow \mathrm{Iso}(X)$ one-parameter semigroup $\implies \Phi(\mathbb{R}_0^+) = \{\mathrm{Id}\}$.

Consequences

- $\mathrm{Iso}(X)$ is a Boolean group for the composition operation.
- $\mathrm{Iso}(X)$ identifies with the set $\mathrm{Unc}(X)$ of unconditional projections on X:

$$\begin{aligned} P \in \mathrm{Unc}(X) &\iff P^2 = P,\ 2P - \mathrm{Id} \in \mathrm{Iso}(X) \\ &\iff P = \frac{1}{2}(\mathrm{Id} - T),\ T \in \mathrm{Iso}(X),\ T^2 = \mathrm{Id}. \end{aligned}$$

- $\mathrm{Iso}(X) \equiv \mathrm{Unc}(X)$ is a Boolean algebra
  $\iff P_1 P_2 \in \mathrm{Unc}(X)$ when $P_1, P_2 \in \mathrm{Unc}(X)$
  $\iff \left\| \frac{1}{2}\left(\mathrm{Id} + T_1 + T_2 - T_1 T_2\right) \right\| = 1$ for every $T_1, T_2 \in \mathrm{Iso}(X)$.

Extremely non-complex $C_E(K\|L)$ spaces.

Theorem

K perfect weak Koszmider, L closed nowhere dense, $E \subset C(L)$
$\Longrightarrow$ $C_E(K\|L)$ is extremely non-complex.

Proposition

K perfect $\Longrightarrow$ $\exists\ L \subset K$ closed nowhere dense with $C[0,1] \subset C(L)$.

Observation: exists a non $C(K)$ extremely non-complex space

$C_{\ell_2}(K\|L)$ is not isomorphic to a $C(K')$ space since $\ell_2 \overset{\text{comp}}{\hookrightarrow} C_{\ell_2}(K\|L)^*$.

Important consequence: Example

Take K perfect weak Koszmider, $L \subset K$ closed nowhere dense with
$E = \ell_2 \subset C[0,1] \subset C(L)$:

- $C_{\ell_2}(K\|L)$ has no non-trivial one-parameter semigroup of isometries.
- $C_{\ell_2}(K\|L)^* = \ell_2 \oplus_1 C_0(K\|L)^*$, so $\mathrm{Iso}\big(C_{\ell_2}(K\|L)^*\big) \supset \mathrm{Iso}(\ell_2)$.

But we are able to give a better result...

Isometries on extremely non-complex $C_E(K\|L)$ spaces

Theorem

$C_E(K\|L)$ extremely non-complex, $T \in \operatorname{Iso}(C_E(K\|L))$
$\Longrightarrow$ exists $\theta : K \setminus L \longrightarrow \{-1, 1\}$ continuous such that

$$[T(f)](x) = \theta(x) f(x) \qquad \big(x \in K \setminus L,\ f \in C_E(K\|L)\big)$$

Consequence: connected case

If K and $K \setminus L$ are connected, then

$$\operatorname{Iso}\big(C_E(K\|L)\big) = \{-\mathrm{Id}, +\mathrm{Id}\}$$

The main example

Koszmider, 2004

$\exists\ \mathcal{K}$ weak Koszmider space such that $\mathcal{K} \setminus F$ is connected if $|F| < \infty$.

Observation on the above construction

There is $\mathcal{L} \subset \mathcal{K}$ closed nowhere dense with

- $\mathcal{K} \setminus \mathcal{L}$ connected
- $C[0,1] \subseteq C(\mathcal{L})$

The best example

Consider $X = C_{\ell_2}(\mathcal{K}\|\mathcal{L})$. Then:

$$\operatorname{Iso}(X) = \{-\mathrm{Id}, +\mathrm{Id}\} \qquad \text{and} \qquad \operatorname{Iso}(X^*) \supset \operatorname{Iso}(\ell_2)$$

Proof.

- $\mathcal{K}$ weak Koszmider, $\mathcal{L}$ nowhere dense, $\ell_2 \subset C(\mathcal{L})$ $\Longrightarrow X$ well-defined and extremely non-complex.
- $\mathcal{K} \setminus \mathcal{L}$ connected $\Longrightarrow \operatorname{Iso}(X) = \{-\mathrm{Id}, +\mathrm{Id}\}$.
- $X^* = \ell_2 \oplus_1 C_0(\mathcal{K}\|\mathcal{L})^*$, so $\operatorname{Iso}(\ell_2) \subset \operatorname{Iso}(X^*)$. ✓

Open questions on extremely non-complex Banach spaces

Questions

X extremely non complex

- Does X have the Daugavet property **?**
- Stronger: Does Y have the Daugavet property if

$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\| \quad \text{for every rank-one } T \in L(Y) \text{ ?}$$

- Is it true that $n(X) = 1$ **?**
  - We actually know that $n(X) \geqslant C > 0$.
- Is $\mathrm{Iso}(X) \equiv \mathrm{Unc}(X)$ a Boolean algebra **?**
- If $Y \leqslant X$ is 1-codimensional, is Y extremely non complex **?**
- Is it possible that $X \simeq Z \oplus Z \oplus Z$ **?**