Numerical index theory

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Advanced Training School in Mathematics

Workshop on Geometry of Banach spaces and its Applications

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Schedule of the talk

Notation

Basic notation I

- $\mathbb{K}$ base field ($\mathbb{R}$ or $\mathbb{C}$):
  - $\mathbb{T}$ modulus-one scalars,
  - $\operatorname{Re} z$ real part of z ($\operatorname{Re} z = z$ if $\mathbb{K} = \mathbb{R}$).
- H Hilbert space: $(\cdot \mid \cdot)$ denotes the inner product.
- X Banach space:
  - S_X unit sphere, B_X unit ball,
  - X^* dual space,
  - $L(X)$ bounded linear operators,
  - $W(X)$ weakly compact linear operators,
  - $\operatorname{Iso}(X)$ surjective linear isometries,
- X Banach space, $T \in L(X)$:
  - $\operatorname{Sp}(T)$ spectrum of T.
  - $T^* \in L(X^*)$ adjoint operator of T.

Notation

Basic notation (II)

X Banach space, $B \subset X$, C convex subset of X:

- B is *rounded* if $\mathbb{T}B = B$,
- $\operatorname{co}(B)$ convex hull of B,
- $\overline{\operatorname{co}}(B)$ closed convex hull of B,
- $\operatorname{aconv}(B) = \operatorname{co}(\mathbb{T}\, B)$ absolutely convex hull of B,
- $\operatorname{ext}(C)$ extreme points of C,
- *slice* of C:

$$S(C, x^*, \alpha) = \{x \in C \ : \ \operatorname{Re} x^*(x) > \sup \operatorname{Re} x^*(C) - \alpha\}$$

where $x^* \in X^*$ and $0 < \alpha < \sup \operatorname{Re} x^*(C)$.

Numerical range of operators

2 Numerical range of operators
- Definitions and first properties
- The exponential function
- Numerical ranges and isometries

F. F. Bonsall and J. Duncan
Numerical Ranges. Vol I and II.
London Math. Soc. Lecture Note Series, 1971 & 1973.

Numerical range: Hilbert spaces

Hilbert space numerical range (Toeplitz, 1918)

- A $n \times n$ real or complex matrix

$$W(A) = \big\{(Ax \mid x) \ : \ x \in \mathbb{K}^n, \ (x \mid x) = 1\big\}.$$

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$$W(T) = \big\{(Tx \mid x) \ : \ x \in H, \ \|x\| = 1\big\}.$$

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Remark

★ Given $T \in L(H)$ we associate

- a sesquilinear form $\varphi_T(x,y) = (Tx \mid y) \qquad (x,y \in H)$,
- a quadratic form $\widehat{\varphi_T}(x) = \varphi_T(x,x) = (Tx \mid x) \qquad (x \in H)$.

★ Then, $W(T) = \widehat{\varphi_T}(S_H)$.

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★ Then, $W(T) = \widehat{\varphi_T}(S_H)$. Therefore:

- $\widehat{\varphi_T}(B_H) = [0,1]\, W(T)$,
- $\widehat{\varphi_T}(H) = \mathbb{R}^+\, W(T)$.
- But we cannot get $W(T)$ from $\widehat{\varphi_T}(B_H)$!

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- In the real case ($\dim(H) > 1$), there is $T \in L(H)$, $T \neq 0$ with $W(T) = \{0\}$.
- In the complex case,

$$\sup\{|(Tx \mid x)| \ : \ x \in S_H\} \geqslant \frac{1}{2}\,\|T\|.$$

  If T is actually self-adjoint, then

$$\sup\{|(Tx \mid x)| \ : \ x \in S_H\} = \|T\|.$$

Proving a result

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- For $x, y \in S_H$ fixed, use the polarization formula:

$$\begin{aligned}(Tx \mid y) = \frac{1}{4}\Big[&(T(x+y) \mid x+y) \ - \ (T(x-y) \mid x-y) \\ &+ \ i\,(T(x+iy) \mid x+iy) \ - i\,(T(x-iy) \mid x-iy)\Big].\end{aligned}$$

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- $|(Tx \mid y)| \leqslant \frac{1}{4}\,M\big[\|x+y\|^2 + \|x-y\|^2 + \|x+iy\|^2 + \|x-iy\|^2\big].$

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- By the parallelogram's law:

$$|(Tx \mid y)| \leqslant \frac{1}{4}\,M\big[2\|x\|^2 + 2\|y\|^2 + 2\|x\|^2 + 2\|iy\|^2\big] = 2M.$$

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- We just take supremum on $x, y \in S_H$ ✓

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Some reasons to study numerical ranges

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- It is useful to estimate spectral radii of small perturbations of matrices.

Example

Consider $A = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ \varepsilon & 0 \end{pmatrix}$.

- $\mathrm{Sp}(A) = \{0\}$, $\mathrm{Sp}(B) = \{0\}$.
- $\mathrm{Sp}(A+B) = \{\pm\sqrt{M\varepsilon}\} \subseteq W(A+B) \subseteq W(A) + W(B)$,
- so the spectral radius of $A+B$ is bounded above by $\frac{1}{2}(|M| + |\varepsilon|)$.

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- It is a comfortable way to study the spectrum.
- It is useful to estimate spectral radii of small perturbations of matrices.
- It is useful to work with some concepts like hermitian operator, skew-hermitian operator, dissipative operator. . .

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Banach spaces numerical range (Bauer 1962; Lumer, 1961)

X Banach space, $T \in L(X)$,

$$V(T) = \{x^*(Tx) \ : \ x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1\}$$

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- It allows to carry to the general case the concepts of hermitian operator, skew-hermitian operator, dissipative operators. . .
- It gives a description of the Lie algebra corresponding to the Lie group of all onto isometries on the space.
- It gives an easy and quantitative proof of the fact that Id is an strongly extreme point of $B_{L(X)}$ (MLUR point).

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$$\begin{aligned} v(T) &= \sup\left\{|\lambda| \ : \ \lambda \in V(T)\right\} \\ &= \sup\left\{|x^*(Tx)| \ : \ x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1\right\} \end{aligned}$$

Elementary properties

X Banach space, $T \in L(X)$

- $v(\cdot)$ is a seminorm, i.e.
  - $v(T+S) \leqslant v(T) + v(S)$ for every $T, S \in L(X)$.

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Numerical radius: definition and properties

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5. In particular, this is the case for $X = C(K)$.

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If $f_0(\xi_0) \sim 1$, then we were done. This our goal.

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If $X = L_1(\mu)$, then $X^* \equiv C(K_\mu)$. Therefore, $v(T) = v(T^*) = \|T^*\| = \|T\|$ ✓

Numerical radius: real and complex spaces

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X complex Banach space, define $T \in L(X_{\mathbb{R}})$ by

$$T(x) = i\,x \qquad (x \in X).$$

- $\|T\| = 1$ and $v(T) = 0$ if viewed in $X_{\mathbb{R}}$.
- $\|T\| = 1$ and $V(T) = \{i\}$, so $v(T) = 1$ if viewed in (complex) X.

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Theorem (Bohnenblust-Karlin; Glickfeld)

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$$v(T) \geqslant \frac{1}{\mathrm{e}}\|T\|.$$

The constant $\frac{1}{\mathrm{e}}$ is optimal:

$\exists\ X$ two-dimensional complex, $\exists\ T \in L(X)$ with $\|T\| = \mathrm{e}$ and $v(T) = 1$.

Numerical index: definition and properties

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- **We will improve this inequality in the sequel**

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Consequence

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- $\|\exp(\lambda T)\| \leqslant e^{|\lambda|\, v(T)}$ $(\lambda \in \mathbb{K})$.
- $v(T)$ is the best possible constant.

Semigroups of isometries: motivating example

A motivating example

A real or complex $n \times n$ matrix. TFAE:

- A is skew-adjoint (i.e. $A^* = -A$).
- $B = \exp(\rho A)$ is unitary for every $\rho \in \mathbb{R}$ (i.e. $B^* B = B B^* = \mathrm{Id}$).

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In term of Hilbert spaces

H (n-dimensional) Hilbert space, $T \in L(H)$. TFAE:

- $\mathrm{Re}\, W(T) = \{0\}$.
- $\exp(\rho T) \in \mathrm{Iso}(H)$ for every $\rho \in \mathbb{R}$.

Semigroups of isometries: motivating example

A motivating example

A real or complex $n \times n$ matrix. TFAE:

- A is skew-adjoint (i.e. $A^* = -A$).
- $\mathrm{Re}(Ax \mid x) = 0$ for every $x \in H$.
- $B = \exp(\rho A)$ is unitary for every $\rho \in \mathbb{R}$ (i.e. $B^*B = BB^* = \mathrm{Id}$).

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For general Banach spaces

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Semigroups of isometries: characterization

Theorem (Bonsall-Duncan, 1970's; Rosenthal, 1984)

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- $\operatorname{Re} V(T) = \{0\}$ (T is skew-hermitian).
- $\|\exp(\rho T)\| \leqslant 1$ for every $\rho \in \mathbb{R}$.
- $\{\exp(\rho T) \ : \ \rho \in \mathbb{R}_0^+\} \subset \operatorname{Iso}(X)$.
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This follows from the exponential formula

$$\sup \operatorname{Re} V(T) = \lim_{\beta \downarrow 0} \frac{\|\operatorname{Id} + \beta\, T\| - 1}{\beta} = \sup_{\alpha > 0} \frac{\log \|\exp(\alpha\, T)\|}{\alpha}.$$

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Remark

If X is complex, there always exists exponential one-parameter semigroups of surjective isometries:

$$t \longmapsto e^{it}\operatorname{Id} \qquad \text{generator: } i\operatorname{Id}.$$

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Main consequence

If X is a real Banach space such that

$$V(T) = \{0\} \quad \Longrightarrow \quad T = 0,$$

then $\operatorname{Iso}(X)$ is “small”:

- it does not contain any exponential one-parameter semigroup,
- the tangent space of $\operatorname{Iso}(X)$ at Id is zero.

Surjective isometries

3. Two results on surjective isometries
   - Isometries on finite-dimensional spaces
   - Isometries and duality

M. Martín
The group of isometries of a Banach space and duality.
J. Funct. Anal. (2008).

M. Martín, J. Merí, and A. Rodríguez-Palacios.
Finite-dimensional spaces with numerical index zero.
Indiana U. Math. J. (2004).

H. P. Rosenthal
The Lie algebra of a Banach space.
in: *Banach spaces* (Columbia, Mo., 1984), LNM, Springer, 1985.

Isometries in finite-dimensional spaces

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X finite-dimensional real space. TFAE:

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4. Moreover, if $X = X_0 \oplus X_1$ where X_1 is complex and

$$\left\| x_0 + e^{i\theta}\, x_1 \right\| = \| x_0 + x_1 \| \qquad \left(x_0 \in X_0,\ x_1 \in X_1,\ \theta \in \mathbb{R} \right).$$

(Note that the other 3 cases are included here)

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Question

Can every Banach space X with $n(X) = 0$ be decomposed as in 4 ?

Negative answer

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Infinite-dimensional case

There is an infinite-dimensional real Banach space X with $n(X) = 0$ but X is polyhedral. In particular, X does not contain $\mathbb{C}$ isometrically.

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An easy example is

$$X = \left[\bigoplus_{n \geqslant 2} X_n \right]_{c_0}$$

X_n is the two-dimensional space whose unit ball is the regular polygon of $2n$ vertices.

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Note

Such an example is not possible in the finite-dimensional case.

Quasi affirmative answer

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- $n(X) = 0$.
- $X = X_0 \oplus X_1 \oplus \cdots \oplus X_n$ such that
  - X_0 is a (possible null) real space,
  - $X_1, \dots, X_n$ are non-null complex spaces,

  there are $\rho_1, \dots, \rho_n$ **rational** numbers, such that

  $$\left\| x_0 + \mathrm{e}^{i\,\rho_1\,\theta}\, x_1 + \cdots + \mathrm{e}^{i\,\rho_n\,\theta}\, x_n \right\| = \left\| x_0 + x_1 + \cdots + x_n \right\|$$

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Remark

- The theorem is due to Rosenthal, but with real ρ's.
- The fact that the ρ's may be chosen as rational numbers is due to M.–Merí–Rodríguez-Palacios.

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- Use Kronecker's Approximation Theorem to change the eigenvalues of T^2 by rational numbers.✓

A simple case of getting rational numbers

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- Let $X = X_0 \oplus X_1 \oplus X_2$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ s.t.

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- But $\left\{ \dfrac{2\pi k}{\alpha - 1} \ : \ k \in \mathbb{Z} \right\}$ is dense in $\mathbb{T}$, so

$$\| x_0 + (x_1 + x_2) \| = \left\| x_0 + e^{i\rho} (x_1 + x_2) \right\| \quad \forall \rho \in \mathbb{R}$$

and $X = X_0 \oplus Z$ where $Z = X_1 \oplus X_2$ is a complex space

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X real space with $n(X) = 0$.

- If $\dim(X) = 2$, then $X \equiv \mathbb{C}$.
- If $\dim(X) = 3$, then $X \equiv \mathbb{R} \oplus \mathbb{C}$ (absolute sum).

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Example

$X = (\mathbb{R}^4, \|\cdot\|)$, $\|(a,b,c,d)\| = \dfrac{1}{4}\displaystyle\int_0^{2\pi} \left|\operatorname{Re}\left(e^{2it}(a+ib) + e^{it}(c+id)\right)\right| \, dt.$

Then $n(X) = 0$ but the unique possible decomposition is $X = \mathbb{C} \oplus \mathbb{C}$ with

$$\left\|e^{it}x_1 + e^{2it}x_2\right\| = \|x_1 + x_2\|.$$

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X real Banach space, $\mathcal{Z}(X) = \{T \in L(X) \ : \ v(T) = 0\}$.

- When X is finite-dimensional, $\mathrm{Iso}(X)$ is a Lie-group and $\mathcal{Z}(X)$ is the tangent space (i.e. its Lie-algebra).

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- $\dim(X) = n \quad \Longrightarrow \quad \dim(\mathcal{Z}(X)) \leqslant \dfrac{n(n-1)}{2}$.
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- If $\oplus \neq \oplus_2$, then isometries respect summands and $\dim(\mathcal{Z}(X)) = 1$. ✓

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The answer is yes. This is what we are going to present next.

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  - $\widetilde{\Phi} : C_E(K\|L)/C_0(K\|E) \longrightarrow E$ onto isometry:
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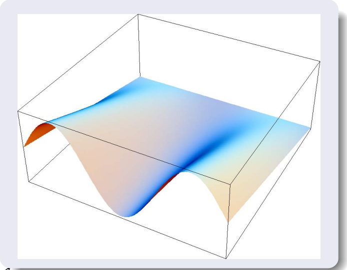

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- $|x^{**}(a^*)| = 1$ for every $x^{**} \in \operatorname{ext}(B_{X^{**}})$ and every $a^* \in \mathcal{A}$.
- This gives $n\big(C_E(K\|L)\big) = 1$:
  - $T \in L(X)$, $\varepsilon > 0$, take $a^* \in \mathcal{A}$ with $\|T^*(a^*)\| > \|T\| - \varepsilon$,
  - Take $x^{**} \in \operatorname{ext}(B_{X^{**}})$ with $|x^{**}(T^*(a^*))| > \|T\| - \varepsilon$,
  - Since $|x^{**}(a^*)| = 1$, we have

$$v(T) = v(T^*) \geqslant |x^{**}(T^*(a^*))| > \|T\| - \varepsilon. \checkmark$$

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Consequence: the example

Take $K = [0,1]$, $L = \Delta$ (Cantor set), $E = \ell_2 \subset C(\Delta)$.

- $\mathrm{Iso}\big(C_{\ell_2}([0,1]\|\Delta)\big)$ has no exponential one-parameter semigroups.
- $C_{\ell_2}([0,1]\|\Delta)^* \equiv \ell_2 \oplus_1 C_0([0,1]\|\Delta)^*$, so taken $S \in \mathrm{Iso}(\ell_2)$

$$\Longrightarrow \quad T = \begin{pmatrix} S & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \in \mathrm{Iso}\big(C_{\ell_2}([0,1]\|\Delta)^*\big)$$

Then, $\mathrm{Iso}\big(C_{\ell_2}([0,1]\|\Delta)^*\big)$ contains infinitely many exponential one-parameter semigroups.

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- In $C_{\ell_2}([0,1]\|\Delta)$ there is no $A \in L(X)$ such that the solution to the linear dynamical system

$$x' = A\,x \qquad \big(x : \mathbb{R}_0^+ \longrightarrow C_{\ell_2}([0,1]\|\Delta)\big)$$

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- Therefore, there is no semigroups in $\operatorname{Iso}(X)$, but there are infinitely many exponential one-parameter semigroups in $\operatorname{Iso}(X^*)$.

Numerical index of Banach spaces

4. Numerical index of Banach spaces
    - Basic definitions and examples
    - Stability properties
    - Duality
    - The isomorphic point of view
    - Banach spaces with numerical index one
        - Isomorphic properties
        - Isometric properties
        - Asymptotic behavior
    - How to deal with numerical index 1 property?

V. Kadets, M. Martín, and R. Payá.
Recent progress and open questions on the numerical index of Banach spaces.
RACSAM (2006)

Numerical index of Banach spaces: definitions

Numerical radius

X Banach space, $T \in L(X)$. The numerical radius of T is

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Numerical index (Lumer, 1968)

X Banach space, the numerical index of X is

$$\begin{aligned} n(X) &= \inf \{ v(T) \ : \ T \in L(X),\ \|T\| = 1\} \\ &= \max \{ k \geqslant 0 \ : \ k\,\|T\| \leqslant v(T) \ \ \forall\, T \in L(X)\} \\ &= \inf \left\{ M \geqslant 0 \ : \ \exists T \in L(X),\ \|T\| = 1,\ \|\exp(\rho T)\| \leqslant e^{\rho M}\ \forall \rho \in \mathbb{R} \right\} \end{aligned}$$

Numerical index of Banach spaces: basic properties

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- Actually,

$$\{n(X) \ : \ X \text{ complex}, \ \dim(X) = 2\} = [\mathrm{e}^{-1}, 1]$$

$$\{n(X) \ : \ X \text{ real}, \ \dim(X) = 2\} = [0, 1]$$

(Duncan–McGregor–Pryce–White, 1970)

Numerical index of Banach spaces: examples (I)

Some examples

1. H Hilbert space, $\dim(H) > 1$,

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Numerical index of Banach spaces: some examples (II)

More examples

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6. Every finite-codimensional subspace of $C[0,1]$ has numerical index 1

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Numerical index of Banach spaces: some examples (III)

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and $M_p = v\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \max\limits_{t\in[0,1]} \dfrac{|t^{p-1} - t|}{1 + t^p}$

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   - In the real case, $n\big(L_p(\mu)\big) \geqslant \dfrac{M_p}{8e}$.
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Numerical index: open problems on computing

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4. Compute the numerical index of real C^*-algebras.
5. Compute the numerical index of more classical Banach spaces: $C^m[0,1]$, Lip(K), Lorentz spaces, Orlicz spaces. . .

Stability properties

Direct sums of Banach spaces (M.–Payá, 2000)

$$n\Big([\oplus_{\lambda\in\Lambda}X_\lambda]_{c_0}\Big) = n\Big([\oplus_{\lambda\in\Lambda}X_\lambda]_{\ell_1}\Big) = n\Big([\oplus_{\lambda\in\Lambda}X_\lambda]_{\ell_\infty}\Big) = \inf_\lambda n(X_\lambda)$$

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  (i.e. $v(\cdot)$ is a norm on $L(X)$ which is not equivalent to the operator norm).
- For every $t \in [0,1]$, there exist a real X_t isomorphic to c_0 (or ℓ_1 or ℓ_∞) with $n(X_t) = t$.
- For every $t \in [e^{-1},1]$, there exist a complex Y_t isomorphic to c_0 (or ℓ_1 or ℓ_∞) with $n(Y_t) = t$.

Stability properties (II)

Vector-valued function spaces (López-M.-Merí-Payá-Villena, 2000's)

E Banach space, μ positive σ-finite measure, K compact space. Then

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Tensor products (Lima, 1980)

There is no general formula for $n\big(X\widetilde{\otimes}_\varepsilon Y\big)$ nor for $n\big(X\widetilde{\otimes}_\pi Y\big)$:

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L_p-spaces (Askoy–Ed-Dari–Khamsi, 2007)

$$n\big(L_p([0,1],E)\big) = n\big(\ell_p(E)\big) = \lim_{m\to\infty} n\big(E\oplus_p \overset{m}{\cdots} \oplus_p E\big).$$

Numerical index and duality

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Proposition

X Banach space, $T \in L(X)$. Then

- $\sup \operatorname{Re} V(T) = \lim_{\alpha \to 0^+} \dfrac{\|\mathrm{Id} + \alpha\, T\| - 1}{\alpha}$.

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Negative answer (Boyko–Kadets–M.–Werner, 2007)

Consider the space

$$X = \{(x, y, z) \in c \oplus_\infty c \oplus_\infty c \ : \ \lim x + \lim y + \lim z = 0\}.$$

Then, $n(X) = 1$ but $n(X^*) < 1$.

Numerical index and duality. Proof of main example

$X = \big\{(x,y,z) \in c \oplus_\infty c \oplus_\infty c \ : \ \lim x + \lim y + \lim z = 0\big\}$:

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  $$|x^{**}(T^*(a))| = \|T^*(a)\| > \|T^*\| - \varepsilon.$$
- Since $|x^{**}(a)| = 1$, this gives that $v(T^*) > \|T^*\| - \varepsilon$, so $v(T) = \|T\|$ and $n(X) = 1$. ✓

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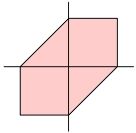

Figure: B_Z

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Example

$X = C_{K(\ell_2)}([0,1]\|\Delta)$. Then $n(X) = 1$ and

$$X^* \equiv K(\ell_2)^* \oplus_1 C_0(K\|\Delta)^* \qquad \text{and} \qquad X^{**} \equiv L(\ell_2) \oplus_\infty C_0(K\|\Delta)^{**}.$$

Therefore, X^{**} is a C^*-algebra, but $n(X^*) = 1/2 < n(X) = 1$.

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Question 4

If X has the RNP, does $n(X) = n(X^*)$ **?**

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★ What about the value $\mathbf{1}$ **?**

Banach spaces with numerical index one

Numerical index 1

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Observation

For Hilbert spaces, the above formula is equivalent to

$$\|T\| = \sup\{|\langle Tx, x\rangle| \ : \ x \in S_X\}$$

which is known to be valid for every self-adjoint operator T.

Banach spaces with numerical index one

Numerical index 1

Recall that X has numerical index one ($n(X) = 1$) iff

$$\|T\| = \sup\{|x^*(Tx)| \ : \ x \in S_X, \ x^* \in S_{X^*}, \ x^*(x) = 1\}$$

(i.e. $v(T) = \|T\|$) for every $T \in L(X)$.

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Examples

$C(K)$, $L_1(\mu)$, $A(\mathbb{D})$, H^∞, finite-codimensional subspaces of $C[0,1]$...

Isomorphic properties (prohibitive results)

Question

Does every Banach space admit an equivalent norm with numerical index 1 **?**

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Not every real Banach space can be renormed to have numerical index 1.
Concretely:

- If X is real, reflexive, and $\dim(X) = \infty$, then $n(X) < 1$.

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- Moreover, if X is real, RNP, $\dim(X) = \infty$, and $n(X) = 1$, then $X \supset \ell_1$.

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A very recent result (Avilés–Kadets–M.–Merí–Shepelska)

If X is real, $\dim(X) = \infty$ and $n(X) = 1$, then $X^* \supset \ell_1$.

More details on this later on.

Proving the 1999 results (I)

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Lemma

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- (Choquet's lemma): $x_0^* \in \operatorname{ext}(B_{X^*})$, $\exists y \in S_X$ and $\beta > 0$ such that
$$|z^*(x_0) - x_0^*(x_0)| < \varepsilon \quad \text{whenever } z^* \in B_{X^*} \text{ satisfies } \operatorname{Re} z^*(y) > 1 - \beta.$$

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- It follows that $\|sx - x_0\| < \varepsilon$ and $|tx^*(x_0) - x_0^*(x_0)| < \varepsilon$, and so
$$\begin{aligned} 1 - |x_0^*(x_0)| &\leqslant |tx^*(sx) - x_0^*(x_0)| \leqslant \\ &\leqslant |tx^*(sx) - tx^*(x_0)| + |tx^*(x_0) - x_0^*(x_0)| < 2\varepsilon. \checkmark \end{aligned}$$

Proving the 1999 results (II)

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Proposition

X real, $A \subset S_X$ infinite with $|x^*(a)| = 1$ $\forall x^* \in \operatorname{ext}(B_{X^*})$, $\forall a \in A$.
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- $\{a_n\}$ separates points of Y^* $\implies$ E_k finite, so $\operatorname{ext}(B_{Y^*})$ countable.
- (Fonf): $Y \supseteq c_0$. So, $X \supseteq c_0$. ✓

Proving the 1999 results (III)

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- X RNP, $\dim(X) = \infty \implies \exists$ infinitely many denting points of B_X.
- Therefore, $X \supseteq c_0$ or $X \supseteq \ell_1$.
- If X RNP, then $X \not\supseteq c_0$. ✓

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Main consequence

X real, RNP, $\dim(X) = \infty$, and $n(X) = 1 \Longrightarrow X \supseteq \ell_1$.

Corollary

X real, $\dim(X) = \infty$, $n(X) = 1$.

- X is not reflexive.
- X^{**}/X is non-separable.

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Main consequence

X real, RNP, $\dim(X) = \infty$, and $n(X) = 1 \Longrightarrow X \supseteq \ell_1$.

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Negative result (Bourgain–Delbaen, 1980)

There is X such that $X^* \simeq \ell_1$ and X has the RNP. Then, X can not be renormed with numerical index 1 (in such a case, $X \supset \ell_1$ **!**)

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What is the situation in the infinite-dimensional case **?**

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- But, since $T_n \longrightarrow T$ and $T^2 = 0$, then $\left[T_n^{**}\right]^2 \longrightarrow 0$ **!!** ✓

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$X = C(\mathbb{T})/A(\mathbb{D})$. $X^* = H^1$ is smooth $\Longrightarrow$ $n(X)<1$ & $n(H^1)<1$.

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- There is X (non-complete) strictly convex with $X^* \equiv L_1(\mu)$, so $n(X) = 1$.
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Open question

Is there X with $n(X) = 1$ which is smooth or strictly convex **?**

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There is a universal constant c such that

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  for every $m \in \mathbb{N}$ and every m-dimensional X's with $n(X) = 1$ **?**
- What is the diameter of the set of all m-dimensional X's with $n(X) = 1$ **?**

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- The weakest property is called lushness.

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Relationship between the properties

- One of the key ideas to get interesting results for Banach spaces with numerical index 1 is to study when the three properties below are equivalent.

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The alternative Daugavet property

M. Martín and T. Oikberg
An alternative Daugavet property
J. Math. Anal. Appl. (2004)

M. Martín
The alternative Daugavet property of C^*-algebras and JB^*-triples
Math. Nachr. (2008)

The Daugavet property: motivation

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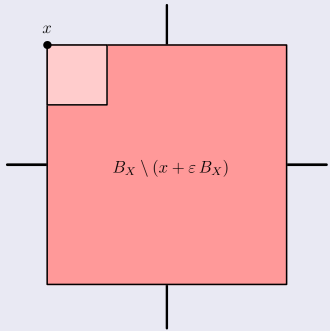

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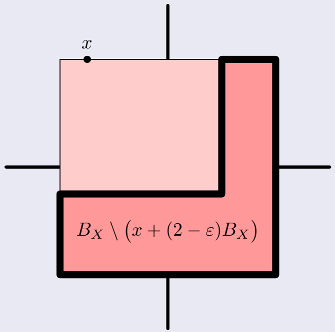

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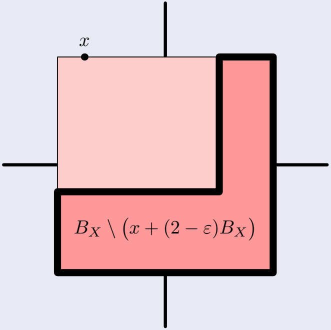
- This geometric property is equivalent to a property of operators on the space.



The Daugavet property: definition

The Daugavet equation

X Banach space, $T \in L(X)$

$$\|\mathrm{Id} + T\| = 1 + \|T\| \qquad \text{(DE)}$$

The Daugavet property: definition

The Daugavet equation

X Banach space, $T \in L(X)$

$$\|\mathrm{Id} + T\| = 1 + \|T\| \qquad \text{(DE)}$$

Classical examples

1. **Daugavet, 1963:**
   Every compact operator on $C[0,1]$ satisfies (DE).
2. **Lozanoskii, 1966:**
   Every compact operator on $L_1[0,1]$ satisfies (DE).
3. **Abramovich, Holub, and more, 80's:**
   $X = C(K)$, K perfect compact space
   or $X = L_1(\mu)$, μ atomless measure
   $\Longrightarrow$ every weakly compact $T \in L(X)$ satisfies (DE).

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The Daugavet property

A Banach space X is said to have the Daugavet property iff every rank-one operator on X satisfies (DE).

★ Then, every weakly compact operator on X satisfies (DE).

(Kadets–Shvidkoy–Sirotkin–Werner, 1997 & 2000)

The Daugavet property: geometric characterizations

Theorem [KSSW]

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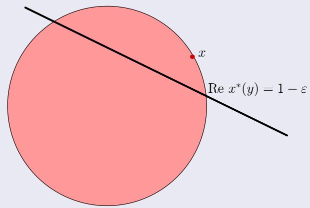

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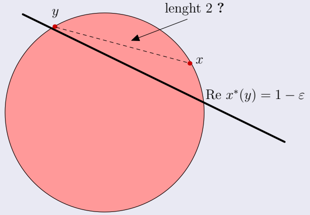

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- For every $x \in S_X$ and every $\varepsilon > 0$, we have
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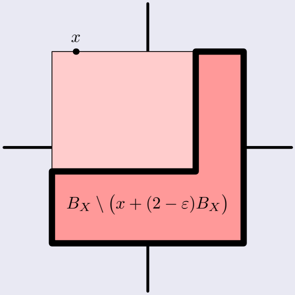

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The DPr, the ADP and numerical index 1

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The alternative Daugavet property (M.–Oikhberg, 2004)

alternative Daugavet property (ADP): every rank-one $T \in L(X)$ satisfies (aDE).

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Relations between the properties

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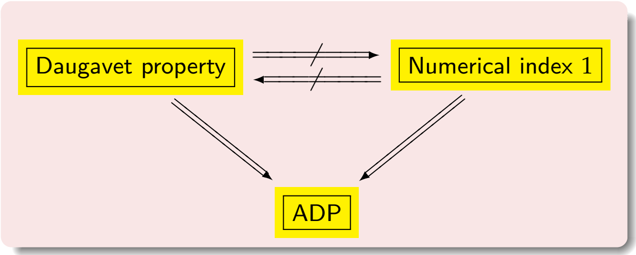

Examples

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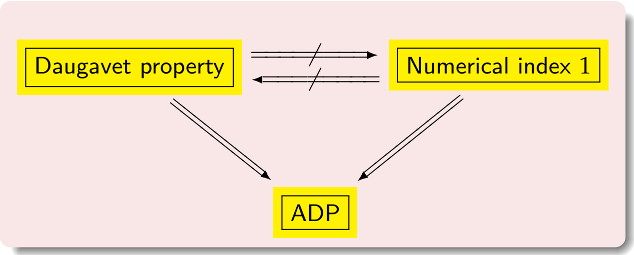

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Remarks

- For RNP or Asplund spaces, $\boxed{\text{ADP}} \Longrightarrow \boxed{\text{numerical index 1}}$.
- Every Banach space with the ADP can be renormed still having the ADP but failing the Daugavet property.

Geometric characterizations of the ADP

Theorem

X Banach space. TFAE:

- X has the ADP.

Every rank-one operator $T \in L(X)$ (equivalently, every weakly compact operator) satisfies

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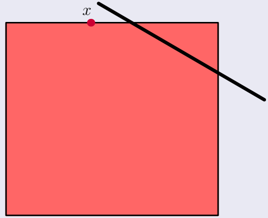

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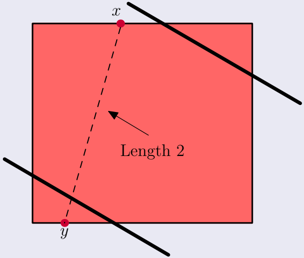

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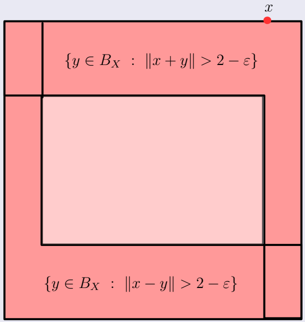

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- $|v(v_*)| = 1$ for $v \in \operatorname{ext}(B_V)$ and $v_* \in \operatorname{ext}(B_{V_*})$, or
- $V = C \oplus_\infty N$, where C is commutative and N has no atomic projections.

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- $\exists$ a commutative ideal Y such that X/Y has the Daugavet property.

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A renorming result (Boyko–Kadets–M.–Merí, 2009)

If X is separable, $X \supset c_0$, then X can be renormed with the ADP.

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Open question

Is there X with the ADP which is smooth or strictly convex **?**

Lush spaces

6 Lush spaces

- Definition and examples
- Lush renorming
- Reformulations of lushness and applications
- Lushness is not equivalent to numerical index one

K. Boyko, V. Kadets, M. Martín, and J. Merí.
Properties of lush spaces and applications to Banach spaces with numerical index 1.
Studia Math. (2009).

K. Boyko, V. Kadets, M. Martín, and D. Werner.
Numerical index of Banach spaces and duality.
Math. Proc. Cambridge Philos. Soc. (2007).

V. Kadets, M. Martín, J. Merí, and R. Payá.
Convexity and smoothnes of Banach spaces with numerical index one.
Illinois J. Math. (to appear).

V. Kadets, M. Martín, J. Merí, and V. Shepelska.
Lushness, numerical index one and duality.
J. Math. Anal. Appl. (2009).

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Showing that (c) $\Longrightarrow$ $n(X) = 1$, one realizes that (c) is too much.

Motivation

Some sufficient conditions

Let X be a Banach space. Consider:

(a) **Lindenstrauss, 1964:** X has the 3.2.I.P. if the intersection of every family of three mutually intersecting balls is not empty.

(b) **Fullerton, 1961:** X is a CL-space if B_X is the absolutely convex hull of every maximal face of S_X.

(c) **Lima, 1978:** X is an almost-CL-space if B_X is the closed absolutely convex hull of every maximal face of S_X.

$$\boxed{(a)} \underset{\not\Longleftarrow}{\Longrightarrow} \boxed{(b)} \underset{\not\Longleftarrow}{\Longrightarrow} \boxed{(c)} \underset{\not\Longleftarrow}{\Longrightarrow} \boxed{n(X)=1}$$

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Lushness (Boyko–Kadets–M.–Werner, 2007)

X is lush if given $x,y \in S_X$, $\varepsilon > 0$, there is $x^* \in S_{X^*}$ such that

$$x \in S(B_X, x^*, \varepsilon) \quad \text{and} \quad \operatorname{dist}\big(y, \operatorname{aconv}\big(S(B_X, x^*, \varepsilon)\big)\big) < \varepsilon.$$

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- Use lushness for $x_0 = Ty_0/\|Ty_0\|$ and y_0 to get $x^* \in S_{X^*}$ and

$$v = \sum_{i=1}^{n} \lambda_i \theta_i x_i \quad \text{where} \quad x_i \in S(B_X, x^*, \varepsilon),\ \lambda_i \in [0,1],\ \sum \lambda_i = 1,\ \theta_i \in \mathbb{T},$$

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- By a convexity argument, $\exists\, i$ such that $|x^*(Tx_i)| \sim \|T\|$ and $\operatorname{Re} x^*(x_i) \sim 1$.
- Then $\max_{\omega \in \mathbb{T}} \|\mathrm{Id} + \omega T\| \sim 1 + \|T\| \implies v(T) \sim \|T\|$. ✓

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K compact, X subspace of $C(K)$ is C-rich iff $\forall U$ open nonempty and $\forall \varepsilon > 0$ exists $h : K \longrightarrow [0,1]$ continuous, $\text{supp}(h) \subseteq U$ such that $\text{dist}\big(h, X\big) < \varepsilon$.

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X separable, $X \supseteq c_0 \implies$ exists $|\!|\!| \cdot |\!|\!| \simeq \| \cdot \|$ and $T : (X, |\!|\!| \cdot |\!|\!|) \longrightarrow \ell_\infty$ with T isometric embedding & $c_0 \subseteq T(X)$ (canonical copy).

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X Banach space. Consider the following assertions.

(a) Exists $A \subset B_{X^*}$ norming, $|x^{**}(a^*)| = 1$ $\forall a^* \in A$ and $\forall x^{**} \in \operatorname{ext}(B_{X^{**}})$.

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Definition (Werner, 1997)

X is nicely embedded in $C_b(\Omega)$ if exists $J : X \longrightarrow C_b(\Omega)$ linear isometry with

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8. Nicely embedded Banach spaces (they fulfil (a)).
9. In particular, function algebras (as $A(\mathbb{D})$ and H^∞).

Some reformulations of lushness

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We almost returned to the almost-CL-space definition **!!**

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Consequence (real case)

$X \subseteq C[0,1]$ strictly convex or smooth $\implies$ $C[0,1]/X$ contains $C[0,1]$.

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Then, $X \supseteq c_0$ or $X \supseteq \ell_1$.

Main consequence

X real lush, $\dim(X) = \infty \implies X^* \supseteq \ell_1$.

Proof.

- There is $E \subseteq X$ separable and lush.

An important consequence

Remark

X lush separable, $\dim(X) = \infty \implies$ there is $G \in S_{X^*}$ infinite such that

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- By "lifting" property of $\ell_1 \implies X^* \supseteq \ell_1$. ✓

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What happens if just $n(X) = 1$ **?**

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What happens if just $n(X) = 1$ **?** The same, we will prove later.

Lushness is not equivalent to numerical index one

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$$\boxed{X \text{ lush}} \;\; \underset{\not\Longleftarrow}{\not\Longrightarrow} \;\; \boxed{X^* \text{ lush}}$$

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Slicely countably determined spaces

7. Slicely countably determined spaces
   - Slicely Countably Determined sets and spaces
   - Applications to numerical index 1 spaces
   - SCD operators
   - Open questions

A. Avilés, V. Kadets, M. Martín, J. Merí, and V. Shepelska
Slicely Countably Determined Banach spaces
Trans. Amer. Math. Soc. (to appear)

SCD sets: Definitions and preliminary remarks

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X Banach space, $A \subset X$ bounded and convex.

SCD sets

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SCD sets: Definitions and preliminary remarks

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Remarks

- A is SCD iff $\overline{A}$ is SCD.
- If A is SCD, then it is separable.

SCD sets: Elementary examples I

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Example

A separable and $A = \overline{\text{conv}}(\text{dent}(A)) \Longrightarrow A$ is SCD.

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- For every $n, m \in \mathbb{N}$, take a slice $S_{n,m}$ containing a_n and of diameter $1/m$.
- If $B \cap S_{n,m} \neq \emptyset \ \forall n, m \in \mathbb{N} \implies a_n \in \overline{B} \ \forall n \in \mathbb{N}$.
- Therefore, $A = \overline{\text{conv}}\big(\{a_n : n \in \mathbb{N}\}\big) \subseteq \overline{\text{conv}}(\overline{B}) = \overline{\text{conv}}(B)$. ✓

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In particular, A RNP separable $\Longrightarrow A$ SCD.

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Example

In particular, A RNP separable $\Longrightarrow A$ SCD.

Corollary

- If X is separable LUR $\Longrightarrow B_X$ is SCD.
- So, every separable space can be renormed such that $B_{(X,|\cdot|)}$ is SCD.

SCD sets: Elementary examples II

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Negative example

If X has the Daugavet property $\Longrightarrow B_X$ is not SCD.
Therefore, $B_{C[0,1]}$, $B_{L_1[0,1]}$ are not SCD.

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- By [KSSW] there is a sequence $(x_n) \subset B_X$ such that
  - $x_n \in S_n$ for every $n \in \mathbb{N}$,
  - $(x_n)_{n \geqslant 0}$ is equivalent to the basis of ℓ_1,
  - so $x_0 \notin \overline{\text{lin}}\{x_n : n \in \mathbb{N}\}$. ✓

SCD sets: Further examples I

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Convex combination of slices

$W = \sum_{k=1}^{m} \lambda_k S_k \subset A$ where $\lambda_k \geqslant 0$, $\sum \lambda_k = 1$, S_k slices.

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Proposition

In the definition of SCD we can use a sequence $\{S_n : n \in \mathbb{N}\}$ of convex combination of slices.

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Small combinations of slices

A has small combinations of slices iff every slice of A contains convex combinations of slices of A with arbitrary small diameter.

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Particular case

A strongly regular + separable $\Longrightarrow$ A is SCD.

SCD sets: Further examples II

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Bourgain's lemma

Every relative weak open subset of A contains a convex combination of slices.

SCD sets: Further examples II

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Corollary

In the definition of SCD we can use a sequence $\{S_n : n \in \mathbb{N}\}$ of relative weak open subsets.

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A π-base of the weak topology of A is a family $\{V_i : i \in I\}$ of weak open sets of A such that every weak open subset of A contains one of the V_i's.

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Proposition

If $(A, \sigma(X, X^*))$ has a countable π-base $\Longrightarrow$ A is SCD.

SCD sets: Further examples III

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Theorem

A separable without ℓ_1-sequences $\Longrightarrow$ $(A, \sigma(X, X^*))$ has a countable π-base.

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SCD sets: Further examples III

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- We see $(A, \sigma(X, X^*)) \subset C(T)$ where $T = (B_{X^*}, \sigma(X^*, X))$.

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SCD spaces: definition and examples

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Examples of SCD spaces

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Examples of SCD spaces

1. X separable strongly regular. In particular, RNP, CPCP spaces.

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1. X having the Daugavet property.
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SCD spaces: definition and examples

SCD space

X is Slicely Countably Determined (SCD) if so are its convex bounded subsets.

Examples of SCD spaces

1. X separable strongly regular. In particular, RNP, CPCP spaces.
2. X separable $X \not\supseteq \ell_1$. In particular, if X^* is separable.

Examples of NOT SCD spaces

1. X having the Daugavet property.
2. In particular, $C[0,1]$, $L_1[0,1]$
3. There is X with the Schur property which is not SCD.

Remark

- Every subspace of a SCD space is SCD.
- This is false for quotients.

SCD spaces: stability properties

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Theorem

$Z \subset X$. If Z and X/Z are SCD $\Longrightarrow$ X is SCD.

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Corollary

$X_1, \ldots, X_m$ SCD $\Longrightarrow X_1 \oplus \cdots \oplus X_m$ SCD.

SCD spaces: stability properties II

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Theorem

$X_1, X_2, \ldots$ SCD, E with unconditional basis.

- $E \not\supseteq c_0 \Longrightarrow \left[\bigoplus_{n\in\mathbb{N}} X_n\right]_E$ SCD.
- $E \not\supseteq \ell_1 \Longrightarrow \left[\bigoplus_{n\in\mathbb{N}} X_n\right]_E$ SCD.

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Examples

1. $c_0(\ell_1)$ and $\ell_1(c_0)$ are SCD.
2. $c_0 \otimes_\varepsilon c_0$, $c_0 \otimes_\pi c_0$, $c_0 \otimes_\varepsilon \ell_1$, $c_0 \otimes_\pi \ell_1$, $\ell_1 \otimes_\varepsilon \ell_1$, and $\ell_1 \otimes_\pi \ell_1$ are SCD.
3. $K(c_0)$ and $K(c_0, \ell_1)$ are SCD.
4. $\ell_2 \otimes_\varepsilon \ell_2 \equiv K(\ell_2)$ and $\ell_2 \oplus_\pi \ell_2 \equiv \mathcal{L}_1(\ell_2)$ are SCD

The DPr, the ADP and numerical index 1

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Recalling the properties

1. **Kadets-Shvidkoy-Sirotkin-Werner, 1997:**
   X has the Daugavet property (DPr) if

$$\|\mathrm{Id} + T\| = 1 + \|T\| \qquad \text{(DE)}$$

   for every rank-one $T \in L(X)$.
   ★Then every weakly compact T also satisfies (DE).

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2. **Lumer, 1968:** X has numerical index 1 if EVERY operator on X satisfies

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3. **M.-Oikhberg, 2004:** X has the alternative Daugavet property (ADP) if every rank-one $T \in L(X)$ satisfies (aDE).
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Relations between these properties

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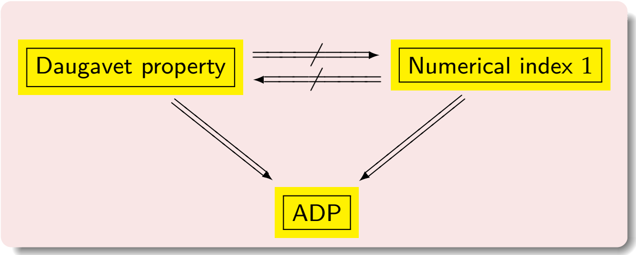

Examples

- $C\big([0,1],K(\ell_2)\big)$ has DPr, but has not numerical index 1
- c_0 has numerical index 1, but has not DPr
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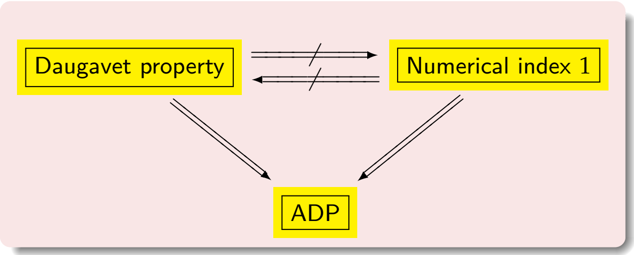

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Remarks

- For RNP or Asplund spaces, $\boxed{\text{ADP}} \Longrightarrow \boxed{\text{numerical index 1}}$.
- Every Banach space with the ADP can be renormed still having the ADP but failing the Daugavet property.

ADP + SCD $\Longrightarrow$ numerical index 1

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X Banach space. TFAE:

- X has ADP (i.e. $\max_{\theta\in\mathbb{T}} \|\mathrm{Id} + \theta\, T\| = 1 + \|T\|$ for all T rank-one).

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Theorem

X ADP + B_X SCD $\Longrightarrow$ given $x \in S_X$ and $\varepsilon > 0$, there is $y^* \in S_{X^*}$ such that
$$x \in S(B_X, y^*, \varepsilon) \quad \text{and} \quad B_X = \overline{\mathrm{conv}}\big(\mathbb{T}\, S(B_X, y^*, \varepsilon)\big).$$

★ This implies lushness and so, numerical index 1.

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- Lush + $\dim(X) = \infty$ $\implies$ $X^* \supseteq \ell_1$ ✓

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Open question

X real, $\dim(X) = \infty$, $n(X) = 1$ $\Longrightarrow$ $X \supset c_0$ or $X \supset \ell_1$ **?**

SCD operators

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Examples

T is an SCD-operator when $T(B_X)$ is separable and

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- X DPr + T

Remark

Separability is not needed !

Main corollary

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- Find more sufficient conditions for a set to be SCD.
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On SCD-operators

- T_1, T_2 SCD-operators, is $T_1 + T_2$ an SCD-operator **?**
- $T : X \longrightarrow Y$ hereditary SCD, is there Z SCD-space such that T factor through Z **?**

Remarks on two recent results

8 Remarks on two recent results
- Containment of c_0 or ℓ_1
- On the numerical index of $L_p(\mu)$

A. Avilés, V. Kadets, M. Martín, J. Merí, and V. Shepelska.
Slicely countably determined Banach spaces.
Trans. Amer. Math. Soc. (to appear).

V. Kadets, M. Martín, J. Merí, and R. Payá.
Smoothness and convexity for Banach spaces with numerical index 1.
Illinois J. Math. (to appear).

M. Martín, J. Merí, and M. Popov.
On the numerical index of real $L_p(\mu)$-spaces.
Preprint.

Containment of c_0 or ℓ_1

Open question (Godefroy, private communication)

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- (LMP 1999): This gives $X^* \supseteq c_0$ or $X^* \supseteq \ell_1 \implies X^* \supseteq \ell_1$ ✓

Containment of c_0 or ℓ_1

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X real, $\dim(X) = \infty$, $n(X) = 1 \implies X \supset c_0$ or $X \supset \ell_1$ **?**

★ Old approaches to this problem:

- López–M.–Payá, 1999:
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★ Equivalent reformulation of the problem:

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★ Equivalent reformulation of the problem:

Equivalent open problem

X real separable, $X \not\supset \ell_1$, exists $G \subseteq S_{X^*}$ norming with

$$B_X = \overline{\operatorname{aconv}}\left(\{x \in B_X \ : \ x^*(x) = 1\}\right) \qquad (x^* \in G).$$

Does $X \supseteq c_0$ **?**

On the numerical index of $L_p(\mu)$. I

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The numerical radius for $L_p(\mu)$

For $T \in L\big(L_p(\mu)\big)$, $1 < p < \infty$, one has

$$v(T) = \sup\left\{\left|\int_\Omega x^\# Tx\, d\mu\right| \ : \ x \in L_p(\mu),\ \|x\|_p = 1\right\}.$$

where for $x \in L_p(\mu)$, $x^\# = |x|^{p-1}\operatorname{sign}(x) \in L_q(\mu)$ satisfies (unique)

$$\|x\|_p^p = \|x^\#\|_q^q \qquad \text{and} \qquad \int_\Omega x\, x^\#\, d\mu = \|x\|_p\, \|x^\#\|_q = \|x\|_p^p.$$

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The absolute numerical radius

For $T \in L\big(L_p(\mu)\big)$ we write

$$\begin{aligned}|v|(T) &:= \sup\left\{\int_\Omega |x^\# Tx|\,d\mu \ :\ x \in L_p(\mu),\ \|x\|_p = 1\right\}\\ &= \sup\left\{\int_\Omega |x|^{p-1}|Tx|\,d\mu \ :\ x \in L_p(\mu),\ \|x\|_p = 1\right\}\end{aligned}$$

On the numerical index of $L_p(\mu)$ (II)

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For $T \in L\big(L_p(\mu)\big)$, $1 < p < \infty$, one has

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For $T \in L\big(L_p(\mu)\big)$, $1 < p < \infty$, one has

$$2\,|v|(T) \geqslant v(T_{\mathbb{C}}) \geqslant n\big(L_p^{\mathbb{C}}(\mu)\big)\,\|T\|,$$

- $T_{\mathbb{C}}$ complexification of T, $n\big(L_p^{\mathbb{C}}(\mu)\big)$ numerical index *complex case*.

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Consequence

For $1 < p < \infty$, $n\big(L_p(\mu)\big) \geqslant \dfrac{M_p}{8e}$.

- If $p \neq 2$, then $n\big(L_p(\mu)\big) > 0$, so v and $\|\cdot\|$ are equivalent in $L\big(L_p(\mu)\big)$.

Extremely non-complex Banach spaces

9. Extremely non-complex Banach spaces
   - Motivation
   - Extremely non-complex Banach spaces
   - Surjective isometries

V. Kadets, M. Martín, and J. Merí.
Norm equalities for operators on Banach spaces.
Indiana U. Math. J. (2007).

P. Koszmider, M. Martín, and J. Merí.
Extremely non-complex $C(K)$ spaces.
J. Math. Anal. Appl. (2009).

P. Koszmider, M. Martín, and J. Merí.
Isometries on extremely non-complex Banach spaces.
Preprint (2008).

Isometries and duality. Reminder

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Example (produced with numerical ranges)

There is a Banach space X such that

- $\mathrm{Iso}(X)$ has no exponential one-parameter semigroups.
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★ In terms of linear dynamical systems:

- There is no $A \in L(X)$ such that

$$x' = A\,x \qquad \left(x : \mathbb{R}_0^+ \longrightarrow X\right)$$

  is given by a semigroup of isometries.
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We would like to find $\mathcal{X}$ such that

- $\mathrm{Iso}(\mathcal{X})$ has no C_0 semigroup of isometries.
- $\mathrm{Iso}(\mathcal{X}^*)$ has exponential semigroup of isometries

Numerical range of unbounded operators

Numerical range of unbounded operators (1960's)

X Banach space, $T : D(T) \longrightarrow X$ linear,

$$V(T) = \{x^*(Tx) \ : \ x^* \in X^*,\ x \in D(T),\ x^*(x) = \|x^*\| = \|x\| = 1\}.$$

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Teorema (Stone, 1932)

H Hilbert space, A densely defined operator. TFAE:

- A generates an strongly continuous one-parameter semigroup of unitary operators (onto isometries).
- $A^* = -A$.
- $\mathrm{Re}(Ax \mid x) = 0$ for every $x \in D(A)$.

Numerical range of unbounded operators. II

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Difficulty

Which Banach spaces have unbounded operators with numerical range zero?

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Examples

- In $C_0(\mathbb{R})$, $\Phi(t)(f)(s) = f(t+s)$ is an strongly continuous one-parameter semigroup of isometries (generated by the derivative).
- In $C_E([0,1]\|\Delta)$ there are also strongly continuous one-parameter semigroup of isometries.

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Consequence

We have to completely change our approach to the problem.

Complex structures

Definition

X has complex structure if there is $T \in L(X)$ such that $T^2 = -\mathrm{Id}$.

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Some remarks

- This gives a structure of vector space over $\mathbb{C}$:

$$(\alpha + i\,\beta)\,x = \alpha\,x + \beta\,T(x) \qquad \big(\alpha + i\,\beta \in \mathbb{C},\ x \in X\big)$$

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- Defining

$$|||x||| = \max\{\|e^{i\theta}x\| \ :\ \theta \in [0,2\pi]\} \qquad (x \in X)$$

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  one gets that $(X, |||\cdot|||)$ is a complex Banach space.
- If T is an isometry, then actually the given norm of X is complex.
- Conversely, if X is a complex Banach space, then

$$T(x) = i\,x \qquad \big(x \in X\big)$$

  satisfies $T^2 = -\mathrm{Id}$ and T is an isometry.

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2. If $X \simeq Z \oplus Z$ (in particular, $X \simeq X^2$), then X has complex structure.
3. There are infinite-dimensional Banach spaces without complex structure:
   - **Dieudonné, 1952:** the James' space $\mathcal{J}$ (since $\mathcal{J}^{**} \equiv \mathcal{J} \oplus \mathbb{R}$).
   - **Szarek, 1986:** uniformly convex examples.
   - **Gowers-Maurey, 1993:** their H.I. space.
   - **Ferenczi-Medina Galego, 2007:** there are odd and even infinite-dimensional spaces X.
     - X is even if admits a complex structure but its hyperplanes does not.
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Definition

X is extremely non-complex if $\operatorname{dist}(T^2, -\mathrm{Id})$ is the maximum possible, i.e.

$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\| \qquad (T \in L(X))$$

The Daugavet equation

What Daugavet did in 1963

The norm equality

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Classical examples

1. **Daugavet, 1963:**
   Every compact operator on $C[0,1]$ satisfies (DE).
2. **Lozanoskii, 1966:**
   Every compact operator on $L_1[0,1]$ satisfies (DE).
3. **Abramovich, Holub, and more, 80's:**
   $X = C(K)$, K perfect compact space
   or $X = L_1(\mu)$, μ atomless measure
   $\Longrightarrow$ every weakly compact $T \in L(X)$ satisfies (DE).

The Daugavet property

The Daugavet property (Kadets–Shvidkoy–Sirotkin–Werner, 1997)

A Banach space X is said to have the Daugavet property iff every rank-one operator on X satisfies (DE).

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Let X be a Banach space with the Daugavet property. Then

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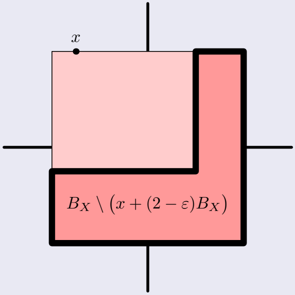
- X contains ℓ_1.
- X does not embed into a Banach space with unconditional basis.
- **Geometric characterization**: X has the Daugavet property iff for each $x \in S_X$

$$\overline{\text{co}}\left(B_X \setminus \big(x + (2-\varepsilon)B_X\big)\right) = B_X\,.$$

(Kadets–Shvidkoy–Sirotkin–Werner, 1997 & 2000)



The Daugavet property II

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More examples

The following spaces have the Daugavet property.

- **Wojtaszczyk, 1992:**
  The disk algebra and H^∞.
- **Werner, 1997:**
  "Nonatomic" function algebras.
- **Oikhberg, 2005:**
  Non-atomic C^*-algebras and preduals of non-atomic von Neumann algebras.
- **Becerra–M., 2005**:
  Non-atomic JB^*-triples and their preduals.
- **Becerra–M., 2006:**
  Preduals of $L_1(\mu)$ without Fréchet-smooth points.
- **Ivankhno, Kadets, Werner, 2007:**
  Lip(K) when $K \subseteq \mathbb{R}^n$ is compact and convex.

Daugavet–type inequalities

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- **Benyamini–Lin, 1985:**
  For every $1 < p < \infty$, $p \neq 2$, there exists $\psi_p : (0,\infty) \longrightarrow (0,\infty)$ such that
  $$\|\mathrm{Id} + T\| \geqslant 1 + \psi_p(\|T\|)$$
  for every compact operator T on $L_p[0,1]$.

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  If ψ_p is the best possible function above, then
  $$\lim_{p\to 1^+} \psi_p(t) = t \qquad (t > 0).$$

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- **Oikhberg, 2005:**
  If $K(\ell_2) \subseteq X \subseteq L(\ell_2)$, then
  $$\|\mathrm{Id} + T\| \geqslant 1 + \frac{1}{8\sqrt{2}}\|T\|$$
  for every compact T on X.

Norm equalities for operators

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Motivating question

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Concretely

We looked for non-trivial norm equalities of the forms

$$\|\mathrm{Id} + T\| = f(\|T\|) \quad \text{or} \quad \|g(T)\| = f(\|T\|) \quad \text{or} \quad \|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$$

(g analytic, f arbitrary) satisfied by all rank-one operators on a Banach space.

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Solution

We proved that there are few possibilities. . .

Equalities of the form $\|\mathrm{Id} + T\| = f(\|T\|)$

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Proposition

X real or complex, $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ arbitrary, $a, b \in \mathbb{K}$. If the norm equality

$$\|a\,\mathrm{Id} + b\,T\| = f(\|T\|)$$

holds for every rank-one operator $T \in L(X)$, then

$$f(t) = |a| + |b|\,t \qquad \big(t \in \mathbb{R}_0^+\big).$$

If $a \neq 0$, $b \neq 0$, then X has the Daugavet property.

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Then, we have to look for Daugavet-type equalities in which $\mathrm{Id} + T$ is replaced by something different.

Proof

We have. . .

$$\|a\,\mathrm{Id} + b\,T\| = f(\|T\|)\ \forall T \in L(X)\ \text{rank-one}$$

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$\overset{?}{\Longrightarrow}$

We want...

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- Trivial if $a \cdot b = 0$. Suppose $a \neq 0$ and $b \neq 0$ and write $\omega_0 = \frac{\overline{b}}{|b|}\,\frac{a}{|a|} \in \mathbb{T}$.

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$$= \|a\,x_0 + b\,\omega_0 t\,x_0\| = |a + b\,\omega_0 t|\,\|x_0\| = \left|a + b\frac{\overline{b}}{|b|}\frac{a}{|a|}\,t\right| = |a| + |$$

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- Finally, for rank-one $T \in L(X)$, write $S = \frac{a}{b}\,T$ and observe

$$|a|\big(1 + \|T\|\big) = |a| + |b|\,\|S\| = \|a\mathrm{Id} + b\,S\| = |a|\,\|\mathrm{Id} + T\|. \checkmark$$

Equalities of the form $\|g(T)\| = f(\|T\|)$

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Theorem

X real or complex with $\dim(X) \geqslant 2$.
Suppose that the norm equality

$$\|g(T)\| = f(\|T\|)$$

holds for every rank-one operator $T \in L(X)$, where

- $g : \mathbb{K} \longrightarrow \mathbb{K}$ is analytic,
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Corollary

Only three norm equalities of the form

$$\|g(T)\| = f(\|T\|)$$

are possible:

- $b = 0:\quad \|a\,\mathrm{Id}\| = |a|$,
- $a = 0:\quad \|b\,T\| = |b|\,\|T\|$,

  (trivial cases)
- $a \neq 0,\ b \neq 0:$
  $\|a\,\mathrm{Id} + b\,T\| = |a| + |b|\,\|T\|$,

  (Daugavet property)

Proof (complex case)

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$$|\widetilde{g}(\lambda)| \leqslant 2|a_0| + |a_1||\lambda| \qquad (\lambda \in \mathbb{C}),$$
- and so $\widetilde{g}$ is a degree-one polynomial by Cauchy inequalities. ✓

Equalities of the form $\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$

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Remark

If X has the Daugavet property and g is analytic, then

$$\|\mathrm{Id} + g(T)\| = |1 + g(0)| - |g(0)| + \|g(T)\|$$

for every rank-one $T \in L(X)$.

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- Our aim here is not to show that g has a suitable form,
- but it is to see that for every g another simpler equation can be found.
- From now on, we have to separate the complex and the real case.

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Proposition

X complex, $\dim(X) \geqslant 2$. Suppose that

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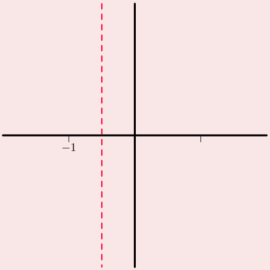
We obtain two different cases:

- $|1 + g(0)| - |g(0)| \neq 0$ or
- $|1 + g(0)| - |g(0)| = 0$.



Equalities of the form $\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$. Complex case

Theorem

If $\operatorname{Re} g(0) \neq -1/2$ and

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Example

If $X = C[0,1] \oplus_2 C[0,1]$, then

- $\left\|\mathrm{Id} + \mathrm{e}^{i\theta}\, T\right\| = \|\mathrm{Id} + T\|$
  for every $\theta \in \mathbb{R}$, rank-one $T \in L(X)$.
- X does not have the Daugavet property.

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Example

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- X does not have the Daugavet property.

The question

Godefroy, private communication

Is there any real Banach space X (with $\dim(X) > 1$) such that

$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$$

for every operator $T \in L(X)$ **?**

In other words, are there extremely non-complex spaces other than $\mathbb{R}$ **?**

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We may try to check whether the known spaces without complex structure are actually extremely non-complex.

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Some examples

1. If $\dim(X) < \infty$, X has complex structure iff $\dim(X)$ is even.
2. **Dieudonné, 1952:** the James' space $\mathcal{J}$ (since $\mathcal{J}^{**} \equiv \mathcal{J} \oplus \mathbb{R}$).
3. **Szarek, 1986:** uniformly convex examples.
4. **Gowers-Maurey, 1993:** their H.I. space.
5. **Ferenczi-Medina Galego, 2007:** there are odd and even infinite-dimensional spaces X.
    - X is even if admits a complex structure but its hyperplanes does not.
    - X is odd if its hyperplanes are even (and so X does not admit a complex structure).

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(Un)fortunately...

This did not work and we moved to $C(K)$ spaces.

The first example: weak multiplications

Weak multiplication

Let K be a compact space. $T \in L\big(C(K)\big)$ is a weak multiplication if

$$T = g\,\mathrm{Id} + S$$

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Theorem

K perfect, $T = g\,\mathrm{Id} + S \in L\big(C(K)\big)$ weak multiplication
$\Longrightarrow \ \|\mathrm{Id} + T^2\| = 1 + \|T^2\|$

Proof of the theorem

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We have $X = C(K)$, K perfect, $T = g\mathrm{Id} + S$

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- $\|\mathrm{Id} - (g^2\,\mathrm{Id} + S)\| \leqslant \|(1 - g^2)\mathrm{Id}\| + \|S\| = 1 - \min g^2(K) + \|S\|.$

Proof of the theorem

We have $X = C(K)$, K perfect, $T = g\mathrm{Id} + S$

- $\max \|\mathrm{Id} \pm T\| = 1 + \|T\|$ (true for every K and every T)
- $\|\mathrm{Id} + S\| = 1 + \|S\|$ (if $S \in W(X)$, K perfect)

We need

$\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$

- If $T = g\mathrm{Id} + S$, then $T^2 = g^2\,\mathrm{Id} + S'$ with S' weakly compact.
- We will prove that $\|\mathrm{Id} + g^2\,\mathrm{Id} + S\| = 1 + \|g^2\,\mathrm{Id} + S\|$ for $g \in C(K)$ and S weakly compact.
- **Step 1:** We assume $\|g^2\| \leqslant 1$ and $\min g^2(K) > 0$.

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- $\|\mathrm{Id} - (g^2\,\mathrm{Id} + S)\| \leqslant \|(1 - g^2)\mathrm{Id}\| + \|S\| = 1 - \min g^2(K) + \|S\|.$
- $$\begin{aligned}\|g^2\,\mathrm{Id} + S\| &= \|\mathrm{Id} + S + (g^2\,\mathrm{Id} - \mathrm{Id})\| \geqslant \|\mathrm{Id} + S\| - \|g^2\,\mathrm{Id} - \mathrm{Id}\| \\ &= 1 + \|S\| - \big(1 - \min g^2(K)\big) = \|S\| + \min g^2(K).\end{aligned}$$

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Proof

Just think that the set of operators satisfying (DE) is closed.

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which gives us the result. ✓

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Proof

If $\|u + v\| = \|u\| + \|v\| \Longrightarrow \|\alpha\, u + \beta\, v\| = \alpha\|u\| + \beta\|v\|$ for $\alpha, \beta \in \mathbb{R}_0^+$.

The first example: weak multiplications. II

Weak multiplication

Let K be a compact space. $T \in L\big(C(K)\big)$ is a weak multiplication if

$$T = g\,\mathrm{Id} + S$$

where $g \in C(K)$ and S is weakly compact.

Theorem

K perfect, $T = g\,\mathrm{Id} + S \in L\big(C(K)\big)$ weak multiplication
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Example (Koszmider, 2004; Plebanek, 2004)

There are perfect compact spaces K such that all operators on $C(K)$ are weak multiplications.

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Example (Koszmider, 2004; Plebanek, 2004)

There are perfect compact spaces K such that all operators on $C(K)$ are weak multiplications.

Consequence

Therefore, there are extremely non-complex $C(K)$ spaces.

More examples: weak multipliers

Weak multiplier

Let K be a compact space. $T \in L\big(C(K)\big)$ is a weak multiplier if

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Example (Koszmider, 2004)

There are infinitely many different perfect compact spaces K such that all operators on $C(K)$ are weak multipliers.

Corollary

There are infinitely many non-isomorphic extremely non-complex Banach spaces.

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There is a family $(K_i)_{i\in I}$ of pairwise disjoint perfect and totally disconnected compact spaces such that

- every operator on $C(K_i)$ is a weak multiplier,
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Theorem

There are some compactifications $\widetilde{K}$ of the above family $(K_i)_{i\in I}$ such that the corresponding $C(\widetilde{K})$'s are extremely non-complex.

Further examples II

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Main consequence

There are perfect compact spaces K_1, K_2 such that:

- $C(K_1)$ and $C(K_2)$ are extremely non-complex,
- $C(K_1)$ contains a complemented copy of $C(\Delta)$.
- $C(K_2)$ contains a 1-complemented isometric copy of ℓ_∞.

Further examples II

Main consequence

There are perfect compact spaces K_1, K_2 such that:

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Observation

- $C(K_1)$ and $C(K_2)$ have operators which are not weak multipliers.
- They are not indecomposable spaces.

Related open questions

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Find topological characterization of the compact Hausdorff spaces K such that the spaces $C(K)$ are extremely non-complex.

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Find topological characterization of the compact Hausdorff spaces K such that the spaces $C(K)$ are extremely non-complex.

Question 2

Find topological consequences on K when $C(K)$ is extremely non-complex.
For instance:
If $C(K)$ is extremely non-complex and $\psi : K \longrightarrow K$ is continuous, are there an open subset U of K such that $\psi|_U = \mathrm{id}$ and $\psi(K \setminus U)$ is finite **?**

Related open questions

Question 1

Find topological characterization of the compact Hausdorff spaces K such that the spaces $C(K)$ are extremely non-complex.

Question 2

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For instance:
If $C(K)$ is extremely non-complex and $\psi : K \longrightarrow K$ is continuous, are there an open subset U of K such that $\psi|_U = \mathrm{id}$ and $\psi(K \setminus U)$ is finite **?**

- We will show latter than $\varphi : K \longrightarrow K$ homeomorphism $\implies \varphi = \mathrm{id}$.

Extremely non-complex Banach spaces

Definition

X is extremely non-complex if $\operatorname{dist}(T^2, -\mathrm{Id})$ is the maximum possible, i.e.

$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\| \qquad \big(T \in L(X)\big)$$

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Examples

There are several extremely non-complex $C(K)$ spaces:

- If $T = g\mathrm{Id} + S$ for every $T \in L(C(K))$ (K Koszmider).
- If $T^* = g\mathrm{Id} + S$ for every $T \in L(C(K))$ (K weak Koszmider).
- One $C(K)$ containing a complemented copy of $C(\Delta)$.
- One $C(K)$ containing an isometric (1-complemented) copy of ℓ_∞.

Isometries on extremely non-complex spaces. I

Theorem

X extremely non-complex.

- $T \in \mathrm{Iso}(X) \implies T^2 = \mathrm{Id}.$
- $T_1, T_2 \in \mathrm{Iso}(X) \implies T_1T_2 = T_2T_1.$
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- Take $S = \frac{1}{\sqrt{2}}\left(T - T^{-1}\right) \implies S^2 = \frac{1}{2}T^2 - \mathrm{Id} + \frac{1}{2}T^{-2}.$

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- Then $\mathrm{Id} = \frac{1}{2}T^2 + \frac{1}{2}T^{-2}.$

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- Then $\mathrm{Id} = \frac{1}{2}T^2 + \frac{1}{2}T^{-2}.$
- Since Id is an extreme point of $B_{L(X)} \implies T^2 = T^{-2} = \mathrm{Id}.$ ✓

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$$\implies T_1 T_2 = T_1 (T_1 T_2 T_1 T_2) T_2 = (T_1 T_1) T_2 T_1 (T_2 T_2) = T_2 T_1. \ \checkmark$$

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- $(\mathrm{Id} - T)^2 = 2(\mathrm{Id} - T) \implies 2\|\mathrm{Id} - T\| = \|(\mathrm{Id} - T)^2\| \leqslant \|\mathrm{Id} - T\|^2.$

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- $\|T_1 - T_2\| = \|T_1(\mathrm{Id} - T_1T_2)\| = \|\mathrm{Id} - T_1T_2\| \in \{0,2\}$. ✓

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Proof.

$\Phi(t) = \Phi(t/2 + t/2) = \Phi(t/2)^2 = \mathrm{Id}$. ✓

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Consequences

- $\mathrm{Iso}(X)$ is a Boolean group for the composition operation.
- $\mathrm{Iso}(X)$ identifies with the set $\mathrm{Unc}(X)$ of unconditional projections on X:

$$P \in \mathrm{Unc}(X) \iff P^2 = P,\ 2P - \mathrm{Id} \in \mathrm{Iso}(X)$$
$$\iff P = \frac{1}{2}(\mathrm{Id} - T),\ T \in \mathrm{Iso}(X),\ T^2 = \mathrm{Id}.$$

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- $\mathrm{Iso}(X) \equiv \mathrm{Unc}(X)$ is a Boolean algebra
  $\iff P_1 P_2 \in \mathrm{Unc}(X)$ when $P_1, P_2 \in \mathrm{Unc}(X)$
  $\iff \left\| \frac{1}{2}\left(\mathrm{Id} + T_1 + T_2 - T_1 T_2\right) \right\| = 1$ for every $T_1, T_2 \in \mathrm{Iso}(X)$.

Extremely non-complex $C_E(K\|L)$ spaces.

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Theorem

K perfect weak Koszmider, L closed nowhere dense, $E \subset C(L)$
$\Longrightarrow$ $C_E(K\|L)$ is extremely non-complex.

Extremely non-complex $C_E(K\|L)$ spaces.

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K perfect weak Koszmider, L closed nowhere dense, $E \subset C(L)$
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Important consequence: Example

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- $C_{\ell_2}(K\|L)^* = \ell_2 \oplus_1 C_0(K\|L)^*$, so $\operatorname{Iso}\big(C_{\ell_2}(K\|L)^*\big) \supset \operatorname{Iso}(\ell_2)$.

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But we are able to give a better result...

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- θ is continuous. ✓

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Consequences: cases $E = C(L)$ and $E = 0$

- $C(K)$ extremely non-complex, $\varphi : K \longrightarrow K$ homeomorphism $\Longrightarrow$ $\varphi = \mathrm{id}$
- $C_0(K \setminus L) \equiv C_0(K\|L)$ extremely non-complex, $\varphi : K \setminus L \longrightarrow K \setminus L$ homeomorphism $\Longrightarrow$ $\varphi = \mathrm{id}$
- In both cases, the group of surjective isometries identifies with a Boolean algebra of clopen sets.

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Consequences: general case

- If for every $x \in L$, there is $f \in E$ with $f(x) \neq 0$
  $\Longrightarrow$ θ extends to the whole K and

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  for every surjective isometry T.

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- But for $E = \ell_2$, $0 \in \overline{\operatorname{ext}(B_{E^*})}^{w^*}$.

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Consequence: connected case

If K and $K \setminus L$ are connected, then

$$\mathrm{Iso}\big(C_E(K\|L)\big) = \{-\mathrm{Id}, +\mathrm{Id}\}$$

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$\exists\ \mathcal{K}$ weak Koszmider space such that $\mathcal{K} \setminus F$ is connected if $|F| < \infty$.

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Open questions on extremely non-complex Banach spaces

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- Does X have the Daugavet property **?**

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- Is it possible that $X \simeq Z \oplus Z \oplus Z$ **?**

Schedule of the talk

1. Basic notation
2. Numerical range of operators
3. Two results on surjective isometries
4. Numerical index of Banach spaces
5. The alternative Daugavet property
6. Lush spaces
7. Slicely countably determined spaces
8. Remarks on two recent results
9. Extremely non-complex Banach spaces