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Miguel Martín

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Extremely non-complex $C(K)$ spaces ☆

Piotr Koszmider [a], Miguel Martín [b,*], Javier Merí [b]

[a] *Instytut Matematyki Politechniki Łódzkiej, ul. Wólczańska 215, 90-924 Łódź, Poland*
[b] *Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain*

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Dedicated to Isaac Namioka on his 80th birthday

Abstract

We show that there exist infinite-dimensional extremely non-complex Banach spaces, i.e. spaces X such that the norm equality $\|\mathrm{Id}+T^2\| = 1+\|T^2\|$ holds for every bounded linear operator $T : X \to X$. This answers in the positive Question 4.11 of [V. Kadets, M. Martín, J. Merí, Norm equalities for operators on Banach spaces, Indiana Univ. Math. J. 56 (2007) 2385–2411]. More concretely, we show that this is the case of some $C(K)$ spaces with few operators constructed in [P. Koszmider, Banach spaces of continuous functions with few operators, Math. Ann. 330 (2004) 151–183] and [G. Plebanek, A construction of a Banach space $C(K)$ with few operators, Topology Appl. 143 (2004) 217–239]. We also construct compact spaces K_1 and K_2 such that $C(K_1)$ and $C(K_2)$ are extremely non-complex, $C(K_1)$ contains a complemented copy of $C(2^\omega)$ and $C(K_2)$ contains a (1-complemented) isometric copy of ℓ_∞.

Keywords: Banach space; Few operators; Complex structure; Daugavet equation; Space of continuous functions

Our objective

Main Objective

We show that there exist (Hausdorff) compact topological spaces K such that

$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\| \qquad \big(\text{for every } T \in L(C(K))\big).$$

Actually, there are many different such a K's:

- Connected and with few operators.
- Totally disconnected, perfect and with few operators.
- Such that $C(K)$ contains a complemented copy of $C(\Delta)$.
- Such that $C(K)$ contains a $(1$-complemented$)$ isometric copy of ℓ_∞.

Outline

Motivation

Complex structures

Definition

X has complex structure if there is $T \in L(X)$ such that $T^2 = -\mathrm{Id}$.

Some remarks

- This gives a structure of vector space over $\mathbb{C}$:

$$(\alpha + i\beta)x = \alpha x + \beta T(x) \qquad \big(\alpha + i\beta \in \mathbb{C},\ x \in X\big)$$

- Defining

$$|||x||| = \max\Big\{\|e^{i\theta}x\| \ :\ \theta \in [0, 2\pi]\Big\} \qquad (x \in X)$$

  one gets that $(X, |||\cdot|||)$ is a complex Banach space.
- If T is an isometry, then actually the given norm of X is complex.
- Conversely, if X is a complex Banach space, then

$$T(x) = i\,x \qquad \big(x \in X\big)$$

  satisfies $T^2 = -\mathrm{Id}$ and T is an isometry.

Complex structures II

Some examples

1. If $\dim(X) < \infty$, X has complex structure iff $\dim(X)$ is even.
2. If $X \simeq Z \otimes Z$ (in particular, $X \simeq X^2$), then X has complex structure.
3. There are infinite-dimensional Banach spaces without complex structure:
   - **Dieudonné, 1952:** the James' space $\mathcal{J}$ (since $\mathcal{J}^{**} \equiv \mathcal{J} \oplus \mathbb{R}$).
   - **Szarek, 1986:** uniformly convex examples.
   - **Gowers-Maurey, 1993:** their H.I. space.
   - **Ferenczi-Medina Galego, 2007:** there are odd and even infinite-dimensional spaces X.
     - X is even if admits a complex structure but its hyperplanes does not.
     - X is odd if its hyperplanes are even (and so X does not admit a complex structure).

Definition

X is extremely non-complex if $\operatorname{dist}(T^2, -\mathrm{Id})$ is the maximum possible, i.e.

$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\| \qquad \big(T \in L(X)\big)$$

The Daugavet equation

What Daugavet did in 1963

The norm equality

$$\|\mathrm{Id} + T\| = 1 + \|T\|$$

holds for every compact T on $C[0,1]$.

The Daugavet equation

X Banach space, $T \in L(X)$, $\|\mathrm{Id} + T\| = 1 + \|T\|$ (DE).

Classical examples

1. **Daugavet, 1963:**
   Every compact operator on $C[0,1]$ satisfies (DE).
2. **Lozanoskii, 1966:**
   Every compact operator on $L_1[0,1]$ satisfies (DE).
3. **Abramovich, Holub, and more, 80's:**
   $X = C(K)$, K perfect compact space
   or $X = L_1(\mu)$, μ atomless measure
   $\Longrightarrow$ every weakly compact $T \in L(X)$ satisfies (DE).

The Daugavet property

The Daugavet property (Kadets–Shvidkoy–Sirotkin–Werner, 1997)

A Banach space X is said to have the Daugavet property iff every rank-one operator on X satisfies (DE).

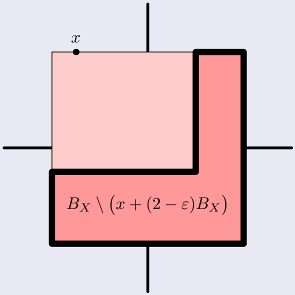
Some results

Let X be a Banach space with the Daugavet property. Then

- Every weakly compact operator on X satisfies (DE).
- X contains ℓ_1.
- X does not embed into a Banach space with unconditional basis.
- **Geometric characterization**: X has the Daugavet property iff for each $x \in S_X$

$$\overline{\mathrm{co}}\left(B_X \setminus \left(x + (2-\varepsilon)B_X\right)\right) = B_X\,.$$

(Kadets–Shvidkoy–Sirotkin–Werner, 1997 & 2000)



The Daugavet property II

For $C(K)$ spaces

K compact space, $C(K)$ has the Daugavet property if and only if K is perfect.

A related result

For every compact space K and every $T \in L\big(C(K)\big)$,

$$\|\mathrm{Id} + T\| = 1 + \|T\| \qquad \text{or} \qquad \|\mathrm{Id} - T\| = 1 + \|T\|.$$

More examples

The following spaces have the Daugavet property.

- **Wojtaszczyk, 1992:**
  The disk algebra $\mathbb{A}$ and H^∞.
- **Oikhberg, 2005:**
  Non-atomic C^*-algebras and preduals of non-atomic von Neumann algebras.
- **Ivankhno, Kadets, Werner, 2007:**
  Lip(K) when $K \subseteq \mathbb{R}^n$ is compact and convex.

Norm equalities for operators

V. Kadets, M. Martín, J. Merí,
Norm equalities for operators.
Indiana U. Math. J. (2007).

Motivating question

Are there other norm equalities which could define interesting properties of Banach spaces **?**

Concretely

We looked for non-trivial norm equalities of the forms

$$\|g(T)\| = f(\|T\|) \qquad \text{or} \qquad \|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$$

(g analytic, f arbitrary) satisfied by all rank-one operators on a Banach space.

Solution

We proved that there are few possibilities. . .

Norm equalities for operators: Occlusive results

Theorem

X real or complex with $\dim(X) \geqslant 2$. Suppose that the norm equality

$$\|g(T)\| = f(\|T\|)$$

holds for every rank-one operator $T \in L(X)$, where

- $g : \mathbb{K} \longrightarrow \mathbb{K}$ is analytic,
- $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ is arbitrary.

Then, there are $a, b \in \mathbb{K}$ such that

$$g(\zeta) = a + b\zeta \qquad (\zeta \in \mathbb{K}).$$

Corollary

Only three norm equalities of the form

$$\|g(T)\| = f(\|T\|)$$

are possible:

- $b = 0$: $\|a\,\mathrm{Id}\| = |a|$,
- $a = 0$: $\|bT\| = |b|\,\|T\|$,

  (trivial cases)
- $a \neq 0$, $b \neq 0$: $\|a\,\mathrm{Id} + bT\| = |a| + |b|\,\|T\|$,

  (Daugavet property)

Norm equalities for operators: Occlusive results II

Theorem

X complex with $\dim(X) \geqslant 2$. Suppose that the norm equality

$$\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$$

holds for every rank-one operator $T \in L(X)$, where

- $g : \mathbb{K} \longrightarrow \mathbb{K}$ is analytic, non constant and with $g(0) = 0$,
- $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ is *continuous*.

Then, X has the Daugavet property

Remarks

- We do not know if the result is true in the real case.
- It is true if g is onto.
- Even the simplest case, $g(t) = t^2$, is not solved. The only known thing is that, in this case, $f(t) = 1 + t$, leading to the equation

$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$$

The question

Godefroy, private communication

Is there any real Banach space X (with $\dim(X) > 1$) such that

$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$$

for every operator $T \in L(X)$ **?**

In other words, are there extremely non-complex Banach spaces other than $\mathbb{R}$ **?**

The examples

The first attempts

The first idea

We may try to check whether the known spaces without complex structure are actually extremely non-complex.

Some examples

1. If $\dim(X) < \infty$, X has complex structure iff $\dim(X)$ is even.
2. **Dieudonné, 1952:** the James' space $\mathcal{J}$ (since $\mathcal{J}^{**} \equiv \mathcal{J} \oplus \mathbb{R}$).
3. **Szarek, 1986:** uniformly convex examples.
4. **Gowers-Maurey, 1993:** their H.I. space.
5. **Ferenczi-Medina Galego, 2007:** there are odd and even infinite-dimensional spaces X.
   - X is even if admits a complex structure but its hyperplanes does not.
   - X is odd if its hyperplanes are even (and so X does not admit a complex structure).

(Un)fortunately...

This did not work and we moved to $C(K)$ spaces.

The first example: weak multiplications

Weak multiplication

Let K be a compact space. $T \in L\big(C(K)\big)$ is a weak multiplication if

$$T = g\,\mathrm{Id} + S$$

where $g \in C(K)$ and S is weakly compact.

Theorem

If K is perfect and all operators on $C(K)$ are weak multiplications, then $C(K)$ is extremely non-complex.

Example (Koszmider, 2004; Plebanek, 2004)

There are perfect compact spaces K such that all operators on $C(K)$ are weak multiplications.

Consequence

Therefore, there are extremely non-complex $C(K)$ spaces.

More examples: weak multipliers

Weak multiplier

Let K be a compact space. $T \in L\big(C(K)\big)$ is a weak multiplier if

$$T^* = g\,\mathrm{Id} + S$$

where g is a Borel function and S is weakly compact.

Theorem

If K is perfect and all operators on $C(K)$ are weak multipliers, then $C(K)$ is extremely non-complex.

Example (Koszmider, 2004)

There are infinitely many different perfect compact spaces K such that all operators on $C(K)$ are weak multipliers.

Corollary

There are infinitely many non-isomorphic extremely non-complex Banach spaces.

Further examples

Proposition

There is a compact infinite totally disconnected and perfect space K such that all operators on $C(K)$ are weak multipliers.

Consequence

There is a family $(K_i)_{i\in I}$ of pairwise disjoint perfect and totally disconnected compact spaces such that

- every operator on $C(K_i)$ is a weak multiplier,
- for $i \neq j$, every $T \in L(C(K_i), C(K_j))$ is weakly compact.

Theorem

There are some compactifications $\widetilde{K}$ of the above family $(K_i)_{i\in I}$ such that the corresponding $C(\widetilde{K})$'s are extremely non-complex.

Further examples II

Main consequence

There are perfect compact spaces K_1, K_2 such that:

- $C(K_1)$ and $C(K_2)$ are extremely non-complex,
- $C(K_1)$ contains a complemented copy of $C(\Delta)$.
- $C(K_2)$ contains a 1-complemented isometric copy of ℓ_∞.

Consequences

- $C(K_1)$ and $C(K_2)$ have operators which are not weak multipliers.
- There are perfect compact spaces K and L such that
  - $C(K) \simeq C(L)$,
  - $C(K)$ is extremely non-complex but $C(L)$ is not.

Consequences and open problems

Consequences

Consequences

Suppose that $C(K)$ is extremely non-complex. Then

- If $T \in L(C(K))$ is an onto isometry, then $T^2 = \mathrm{Id}$.
- So, if $\varphi : K \longrightarrow K$ is an homeomorphism, then $\varphi^2 = \mathrm{id}$.
- If $\psi : K \longrightarrow K$ is continuous, $U \subset K$ is open, and $\big(\psi|_U\big)^2 = \mathrm{id}|_U$, then $\phi|_U = \mathrm{id}|_U$.
- Therefore, the only homeomorphism of K is id.
- No finite-codimensional subspace of $C(K)$ admits a complex structure. So $C(K)$ is not odd nor even.

Open Questions

Question 1

Find topological characterization of the compact Hausdorff spaces K such that the spaces $C(K)$ are extremely non-complex.

Question 2

Find topological consequences on K when $C(K)$ is extremely non-complex.
For instance:
If $C(K)$ is extremely non-complex and $\psi : K \longrightarrow K$ is continuous, are there an open subset U of K such that $\psi|_U = \mathrm{id}$ and $\psi(K \setminus U)$ is finite **?**

Question 3

Find extremely non-complex Banach spaces which are not $C(K)$ spaces.