The numerical index of Banach spaces

Miguel Martín

http://www.ugr.es/local/mmartins

May 23rd, 2008 – Alcoy, Alicante

Schedule of the talk

Notation

Basic notation

X Banach space.

- $\mathbb{K}$ base field (it may be $\mathbb{R}$ or $\mathbb{C}$),
- $\mathbb{T}$ modulus-one scalars,
- S_X unit sphere, B_X unit ball,
- X^* dual space,
- $L(X)$ bounded linear operators,
- $\mathrm{Iso}(X)$ surjective linear isometries,
- $T^* \in L(X^*)$ adjoint operator of $T \in L(X)$.

Numerical range of operators

Numerical range: Hilbert spaces

Hilbert space numerical range (Toeplitz, 1918)

- A $n \times n$ real or complex matrix

$$W(A) = \big\{ (Ax \mid x) \ : \ x \in \mathbb{K}^n, \ (x \mid x) = 1 \big\}.$$

- H real or complex Hilbert space, $T \in L(H)$,

$$W(T) = \big\{ (Tx \mid x) \ : \ x \in H, \ \|x\| = 1 \big\}.$$

Some properties

H Hilbert space, $T \in L(H)$:

- $W(T)$ is convex.
- In the complex case, $\overline{W(T)}$ contains the spectrum of T.
- If T is normal, then $\overline{W(T)} = \overline{\mathrm{co}}\,\mathrm{Sp}(T)$.

Numerical range: Banach spaces

Banach spaces numerical range (Bauer 1962; Lumer, 1961)

X Banach space, $T \in L(X)$,

$$V(T) = \left\{ x^*(Tx) \ : \ x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1 \right\}$$

Some properties

X Banach space, $T \in L(X)$:

- $V(T)$ is connected (not necessarily convex).
- In the complex case, $\overline{W(T)}$ contains the spectrum of T.
- In fact,

$$\overline{\mathrm{co}}\,\mathsf{Sp}(T) = \bigcap \overline{\mathrm{co}}\, V(T),$$

  the intersection taken over all numerical ranges $V(T)$ corresponding to equivalent norms on X.

Some motivations for the numerical range

For Hilbert spaces

- It is a comfortable way to study the spectrum.
- It is useful to work with some concept like hermitian operator, skew-hermitian operator, dissipative operator...

For Banach spaces

- It allows to carry to the general case the concepts of hermitian operator, skew-hermitian operator, dissipative operators...
- It gives a description of the Lie algebra corresponding to the Lie group of all onto isometries on the space.
- It gives an easy and quantitative proof of the fact that Id is an strongly extreme point of $B_{L(X)}$ (MLUR point).

Relationship with semigroups of operators

Theorem (Bonsall-Duncan, 1970's; Rosenthal, 1984)

X real or complex Banach space, $T \in L(X)$. TFAE:

- $\operatorname{Re} V(T) = \{0\}$ (T is skew-hermitian).
- $\|\exp(\rho T)\| \leqslant 1$ for every $\rho \in \mathbb{R}$.
- $\left\{\exp(\rho T) \ : \ \rho \in \mathbb{R}_0^+\right\} \subset \operatorname{Iso}(X)$.
- T belongs to the tangent space to $\operatorname{Iso}(X)$ at Id.
- $\displaystyle\lim_{\rho \to 0} \frac{\|\operatorname{Id} + \rho T\| - 1}{\rho} = 0$.

Main consequence

If X is a real Banach space such that

$$V(T) = \{0\} \quad \Longrightarrow \quad T = 0,$$

then $\operatorname{Iso}(X)$ is "small":

- it does not contain any uniformly continuous one-parameter semigroups,
- the tangent space of $\operatorname{Iso}(X)$ at Id is zero.

Isometries on finite-dimensional spaces

Theorem (Rosenthal, 1984)

X real finite-dimensional Banach space. TFAE:

- $\operatorname{Iso}(X)$ is infinite.
- There is $T \in L(X)$, $T \neq 0$, with $V(T) = \{0\}$.

Theorem (Rosenthal, 1984; M.–Merí–Rodríguez-Palacios, 2004)

X finite-dimensional real space. TFAE:

- $\operatorname{Iso}(X)$ is infinite.
- $X = X_0 \oplus X_1 \oplus \cdots \oplus X_n$ such that
  - X_0 is a (possible null) real space,
  - $X_1, \ldots, X_n$ are non-null complex spaces,

  there are $\rho_1, \ldots, \rho_n$ **rational** numbers, such that

  $$\left\| x_0 + e^{i\rho_1\theta} x_1 + \cdots + e^{i\rho_n\theta} x_n \right\| = \left\| x_0 + x_1 + \cdots + x_n \right\|$$

  for every $x_i \in X_i$ and every $\theta \in \mathbb{R}$.

Isometries on finite-dimensional spaces II

Remark

- The theorem is due to Rosenthal, but with real ρ's.
- The fact that the ρ's may be chosen as rational numbers is due to M.–Merí–Rodríguez-Palacios.

Corollary

X real space with infinitely many isometries.

- If $\dim(X)=2$, then $X\equiv\mathbb{C}$.
- If $\dim(X)=3$, then $X\equiv\mathbb{R}\oplus\mathbb{C}$ (absolute sum).

Example

$$X=(\mathbb{R}^4,\|\cdot\|),\ \|(a,b,c,d)\|=\frac{1}{4}\int_0^{2\pi}\left|\mathrm{Re}\left(\mathrm{e}^{2it}(a+ib)+\mathrm{e}^{it}(c+id)\right)\right|\,dt.$$

Then, $\mathrm{Iso}(X)$ is infinite but the unique possible decomposition is $X=\mathbb{C}\oplus\mathbb{C}$ with

$$\left\|\mathrm{e}^{it}x_1+\mathrm{e}^{2it}x_2\right\|=\|x_1+x_2\|.$$

Semigroups of surjective isometries and duality

The construction (M., 20??)

$E \subset C[0,1]$ separable Banach space. We consider the Banach space

$$X(E) = \Big\{ f \in C\big([0,1]\times[0,1]\big) \ : \ f(\cdot,0) \in E \Big\}.$$

Then, every $T \in L\big(X(E)\big)$ satisfies $\sup |V(T)| = \|T\|$ and

$$X(E)^* \equiv E^* \oplus_1 L_1(\mu).$$

The main consequence

Take $E = \ell_2$ (real). Then

- $\operatorname{Iso}\big(X(\ell_2)\big)$ is "small" (there is no uniformly continuous semigroups).
- Since $X(\ell_2)^* \equiv \ell_2 \oplus_1 L_1(\mu)$, given $S \in \operatorname{Iso}(\ell_2)$, the operator

$$T = \begin{pmatrix} S & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$$

  is a surjective isometry of $X(\ell_2)^*$.
- Therefore, $\operatorname{Iso}\big(X(\ell_2)^*\big)$ contains infinitely many semigroups of isometries.

Numerical index of Banach spaces

Numerical index of Banach spaces: definitions

Numerical radius

X Banach space, $T \in L(X)$. The numerical radius of T is

$$v(T) = \sup\left\{|x^*(Tx)| \; : \; x^* \in S_{X^*},\; x \in S_X,\; x^*(x) = 1\right\}$$

Remark

The numerical radius is a continuous seminorm in $L(X)$. Actually, $v(\cdot) \leqslant \|\cdot\|$

Numerical index (Lumer, 1968)

X Banach space, the numerical index of X is

$$\begin{aligned} n(X) &= \inf\left\{v(T) \; : \; T \in L(X),\; \|T\| = 1\right\} \\ &= \max\left\{k \geqslant 0 \; : \; k\|T\| \leqslant v(T) \quad \forall\, T \in L(X)\right\} \end{aligned}$$

Using exponentials

$$n(X) = \inf\left\{M \geqslant 0 \; : \; \exists T \in L(X), \|T\| = 1,\; \|\exp(\rho T)\| \leqslant e^{\rho M} \;\; \forall \rho \in \mathbb{R}\right\}$$

Numerical index of Banach spaces: basic properties

Some basic properties

- $n(X) = 1$ iff v and $\|\cdot\|$ coincide.
- $n(X) = 0$ iff v is not an equivalent norm in $L(X)$

- X complex $\Rightarrow$ $n(X) \geqslant 1/\mathrm{e}$.

  (Bohnenblust–Karlin, 1955; Glickfeld, 1970)

- Actually,

$$\{n(X) \ : \ X \text{ complex}, \ \dim(X) = 2\} = [\mathrm{e}^{-1}, 1]$$
$$\{n(X) \ : \ X \text{ real}, \ \dim(X) = 2\} = [0, 1]$$

(Duncan–McGregor–Pryce–White, 1970)

Numerical index of Banach spaces: some examples

Examples

1. H Hilbert space, $\dim(H) > 1$,

$$\begin{array}{ll} n(H) = 0 & \text{if } H \text{ is real} \\ n(H) = 1/2 & \text{if } H \text{ is complex} \end{array}$$

2. $n\big(L_1(\mu)\big) = 1$ $\quad \mu$ positive measure
   $n\big(C(K)\big) = 1$ $\quad K$ compact Hausdorff space

   (Duncan et al., 1970)

3. If A is a C^*-algebra $\Rightarrow \begin{cases} n(A) = 1 & A \text{ commutative} \\ n(A) = 1/2 & A \text{ not commutative} \end{cases}$

   (Huruya, 1977; Kaidi–Morales–Rodríguez, 2000)

4. If A is a function algebra $\Rightarrow$ $n(A) = 1$

   (Werner, 1997)

Numerical index of Banach spaces: some examples II

More examples

5. For $n \geqslant 2$, the unit ball of X_n is a $2n$ regular polygon:

$$n(X_n) = \begin{cases} \tan\left(\frac{\pi}{2n}\right) & \text{if } n \text{ is even,} \\ \\ \sin\left(\frac{\pi}{2n}\right) & \text{if } n \text{ is odd.} \end{cases}$$

(M.–Merí, 2007)

6. Every finite-codimensional subspace of $C[0,1]$ has numerical index 1

(Boyko–Kadets–M.–Werner, 2007)

Numerical index of Banach spaces: some examples III

Even more examples

7. Numerical index of L_p-spaces, $1 < p < \infty$:
   - $n\big(L_p[0,1]\big) = n(\ell_p) = \lim_{m\to\infty} n\big(\ell_p^{(m)}\big)$.

     (Ed-Dari, 2005 & Ed-Dari-Khamsi, 2006)
   - $n\big(\ell_p^{(2)}\big)$ **?**
   - In the real case,

$$\max\left\{\frac{1}{2^{1/p}}, \frac{1}{2^{1/q}}\right\} v\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \leqslant n\big(\ell_p^{(2)}\big) \leqslant v\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $v\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \max_{t\in[0,1]} \dfrac{|t^{p-1}-t|}{1+t^p} = \max \tan\big(\measuredangle OPN\big)$

(M.–Merí, 200?)

Open problem

Compute $n\big(L_p[0,1]\big)$ for $1 < p < \infty$, $p \neq 2$. Even more, compute $n\big(\ell_p^{(2)}\big)$.

Stability properties

Direct sums of Banach spaces (M.–Payá, 2000)

$$n\Big([\oplus_{\lambda\in\Lambda} X_\lambda]_{c_0}\Big) = n\Big([\oplus_{\lambda\in\Lambda} X_\lambda]_{\ell_1}\Big) = n\Big([\oplus_{\lambda\in\Lambda} X_\lambda]_{\ell_\infty}\Big) = \inf_\lambda n(X_\lambda)$$

Consequences

- There is a real Banach space X such that

  $$v(T) > 0 \quad \text{when } T \neq 0,$$

  but $n(X) = 0$
  (i.e. $v(\cdot)$ is a norm on $L(X)$ which is not equivalent to the operator norm).
- For every $t \in [0,1]$, there exist a real X_t isomorphic to c_0 (or ℓ_1 or ℓ_∞) with $n(X_t) = t$.
- For every $t \in [\mathrm{e}^{-1}, 1]$, there exist a complex Y_t isomorphic to c_0 (or ℓ_1 or ℓ_∞) with $n(Y_t) = t$.

Stability properties II

Vector-valued function spaces (López–M.–Merí–Payá–Villena, 200's)

E Banach space, μ positive measure, K compact space. Then

$$n\big(C(K,E)\big) = n\big(C_w(K,E)\big) = n\big(L_1(\mu,E)\big) = n\big(L_\infty(\mu,E)\big) = n(E),$$

and $n\big(C_{w^*}(K,E^*)\big) \leqslant n(E)$

Tensor products (Lima, 1980)

There is no general formula neither for $n\big(X\widetilde{\otimes}_\varepsilon Y\big)$ nor for $n\big(X\widetilde{\otimes}_\pi Y\big)$:

- $n\big(\ell_1^{(4)}\widetilde{\otimes}_\pi\ell_1^{(4)}\big) = n\big(\ell_\infty^{(4)}\widetilde{\otimes}_\varepsilon\ell_\infty^{(4)}\big) = 1.$
- $n\big(\ell_1^{(4)}\widetilde{\otimes}_\varepsilon\ell_1^{(4)}\big) = n\big(\ell_\infty^{(4)}\widetilde{\otimes}_\pi\ell_\infty^{(4)}\big) < 1.$

L_p-spaces (Askoy–Ed-Dari–Khamsi, 2007)

$$n\big(L_p([0,1],E)\big) = n\big(\ell_p(E)\big) = \lim_{m\to\infty} n\big(E\oplus_p \overset{m}{\cdots} \oplus_p E\big).$$

Numerical index and duality

Proposition

X Banach space, $T \in L(X)$. Then

- $\sup \operatorname{Re} V(T) = \lim\limits_{\alpha \to 0^+} \dfrac{\|\mathrm{Id} + \alpha T\| - 1}{\alpha}$.
- $v(T^*) = v(T)$ for every $T \in L(X)$.
- Therefore, $n(X^*) \leqslant n(X)$.

(Duncan–McGregor–Pryce–White, 1970)

Question (From the 1970's)

Is $n(X) = n(X^*)$ **?**

Negative answer (Boyko–Kadets–M.–Werner, 2007)

Consider the space

$$X = \Big\{ (x, y, z) \in c \oplus_\infty c \oplus_\infty c \ : \ \lim x + \lim y + \lim z = 0 \Big\}.$$

Then, $n(X) = 1$ but $n(X^*) < 1$.

Numerical index and duality II

Some positive partial answers

One has $n(X) = n(X^*)$ when

- X is reflexive (evident).
- X is a C^*-algebra or a von Neumann predual (1970's – 2000's).
- X is L-embedded in X^{**} (M., 20??).
- If X has RNP and $n(X) = 1$, then $n(X^*) = 1$ (M., 2002).

Open question

Find isometric or isomorphic properties assuring that $n(X) = n(X^*)$.

More examples (M. 20??)

- There is X with $n(X) > n(X^*)$ such that X^{**} is a von Neumann algebra.
- If X is separable and $X \supset c_0$, then X can be renormed to fail the equality.

The isomorphic point of view

Renorming and numerical index (Finet–M.–Payá, 2003)

$(X, \|\cdot\|)$ (separable or reflexive) Banach space. Then

- Real case:
$$[0,1[\subseteq \{n(X,|\cdot|)\ :\ |\cdot| \simeq \|\cdot\|\}$$
- Complex case:
$$[e^{-1},1[\subseteq \{n(X,|\cdot|)\ :\ |\cdot| \simeq \|\cdot\|\}$$

Open question

The result is known to be true when X has a long biorthogonal system.
Is it true in general **?**

Banach spaces with numerical index one

Banach spaces with numerical index 1

Definition

Numerical index 1 Recall that X has numerical index one ($n(X) = 1$) iff

$$\|T\| = \sup\left\{|x^*(Tx)| \;:\; x \in S_X,\; x^* \in S_{X^*},\; x^*(x) = 1\right\}$$

(i.e. $v(T) = \|T\|$) for every $T \in L(X)$.

Observation

For Hilbert spaces, the above formula is equivalent to the classical formula

$$\|T\| = \sup\{|\langle Tx, x\rangle| \;:\; x \in S_X\}$$

for the norm of a self-adjoint operator T.

Examples

$C(K)$, $L_1(\mu)$, $A(\mathbb{D})$, H^∞, finite-codimensional subspaces of $C[0,1]$...

Isomorphic properties (occlusive results)

Question

Does every Banach space admit an equivalent norm to have numerical index 1 **?**

Negative answer (López–M.–Payá, 1999)

Not every real Banach space can be renormed to have numerical index 1.
Concretely:

- If X is real, reflexive, and $\dim(X) = \infty$, then $n(X) < 1$.
- Actually, if X is real, $\dim(X) = \infty$ and $n(X) = 1$, then X^{**}/X is non-separable.
- Moreover, if X is real, RNP, $\dim(X) = \infty$, and $n(X) = 1$, then $X \supset \ell_1$.

A very recent result (Avilés–Kadets–M.–Merí–Shepelska)

If X is real, $\dim(X) = \infty$ and $n(X) = 1$, then $X^* \supset \ell_1$.

Isomorphic properties (positive results)

A renorming result (Boyko–Kadets–M.–Merí, 200?)

If X is separable, $X \supset c_0$, then X can be renormed to have numerical index 1.

Consequence

If X is an infinite-dimensional subspace of c_0, then there is $Z \simeq X$ such that

$$n(Z) = 1 \qquad \text{and} \qquad \begin{cases} n(Z^*) = 0 & \text{real case} \\ n(Z^*) = \mathrm{e}^{-1} & \text{complex case} \end{cases}$$

Open questions

- Find isomorphic properties which assures renorming with numerical index 1
- In particular, if $X \supset \ell_1$, can X be renormed to have numerical index 1 **?**

Negative result (Bourgain–Delbaen, 1980)

There is X such that $X^* \simeq \ell_1$ and X has the RNP. Then, X can not be renormed with numerical index 1 (in such a case, $X \supset \ell_1$ **!**)

Isometric properties: finite-dimensional spaces

Finite-dimensional spaces (McGregor, 1971; Lima, 1978)

X real or complex finite-dimensional space. TFAE:

- $n(X) = 1$.
- $|x^*(x)| = 1$ for every $x^* \in \operatorname{ext}(B_{X^*})$, $x \in \operatorname{ext}(B_X)$.
- $B_X = \text{aconv}(F)$ for every maximal convex subset F of S_X (X is a CL-space).

Remark

This shows a rough behavior of the norm of a finite-dimensional space with numerical index 1:

- The space is not smooth.
- The space is not strictly convex.

Question

What is the situation in the infinite-dimensional case **?**

Isometric properties: infinite-dimensional spaces

Theorem (Kadets–M.–Merí–Payá, 20??)

X infinite-dimensional Banach space, $n(X)=1$. Then

- X^* is neither smooth nor strictly convex.
- The norm of X cannot be Fréchet-smooth.
- There is no WLUR points in S_X.

Example without completeness

There is a (non-complete) space X such that

- $X^* \equiv L_1(\mu)$ (so $n(X)=1$ and more),
- and X is strictly convex.

Open question

Is there any infinite-dimensional Banach space X with $n(X)=1$ which is smooth or strictly convex **?**

Asymptotic behavior of the set of spaces with numerical index one

Theorem (Oikhberg, 2005)

There is a universal constant c such that

$$\operatorname{dist}\left(X, \ell_2^{(m)}\right) \geqslant c\, m^{\frac{1}{4}}$$

for every $m \in \mathbb{N}$ and every m-dimensional X with $n(X) = 1$.

Old examples

$$\operatorname{dist}\left(\ell_1^{(m)}, \ell_2^{(m)}\right) = \operatorname{dist}\left(\ell_\infty^{(m)}, \ell_2^{(m)}\right) = m^{\frac{1}{2}}$$

Open questions

- Is there a universal constant c such that

  $$\operatorname{dist}\left(X, \ell_2^{(m)}\right) \geqslant c\, m^{\frac{1}{2}}$$

  for every $m \in \mathbb{N}$ and every m-dimensional X's with $n(X) = 1$.
- What is the diameter of the set of all m-dimensional X's with $n(X) = 1$.

Bibliography

F. F. Bonsall and J. Duncan
Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras.
London Math. Soc. Lecture Note Series, 1971.

F. F. Bonsall and J. Duncan
Numerical Ranges II.
London Math. Soc. Lecture Note Series, 1973.

V. Kadets, M. Martín, and R. Payá.
Recent progress and open questions on the numerical index of Banach spaces.
RACSAM (2006)

H. P. Rosenthal
The Lie algebra of a Banach space.
in: *Banach spaces* (Columbia, Mo., 1984), LNM, Springer, 1985.