The group of isometries of a Banach space and duality

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Notation and objective

Basic notation

X Banach space over $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$).

- S_X unit sphere, B_X unit ball,
- X^* dual space,
- $L(X)$ bounded linear operators,
- $\mathrm{Iso}(X)$ surjective linear isometries,
- $T^* \in L(X^*)$ adjoint operator of $T \in L(X)$.

Main Objective

We construct a real Banach space X such that

- $\mathrm{Iso}(X)$ does not contains uniformly continuous one-parameter semigroups.
- But $\mathrm{Iso}(X^*)$ contains infinitely many uniformly continuous one-parameter semigroups.

The tool: numerical range of operators

F. F. Bonsall and J. Duncan
Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras.
London Math. Soc. Lecture Note Series, 1971.

F. F. Bonsall and J. Duncan
Numerical Ranges II.
London Math. Soc. Lecture Note Series, 1973.

H. P. Rosenthal
The Lie algebra of a Banach space.
in: *Banach spaces* (Columbia, Mo., 1984), LNM, Springer, 1985.

Hilbert spaces

Hilbert space Numerical range (Toeplitz, 1918)

- A $n \times n$ real or complex matrix

$$W(A) = \Big\{ (Ax \mid x) \ : \ x \in \mathbb{K}^n, \ (x \mid x) = 1 \Big\}.$$

- H real or complex Hilbert space, $T \in L(H)$,

$$W(T) = \Big\{ (Tx \mid x) \ : \ x \in H, \ \|x\| = 1 \Big\}.$$

Some properties

H Hilbert space, $T \in L(H)$:

- $W(T)$ is convex.
- In the complex case, $\overline{W(T)}$ contains the spectrum of T.
- If, moreover, T is normal, $\overline{W(T)} = \overline{\mathrm{co}}\, Sp(T)$.

Banach spaces

Banach space numerical range (Bauer 1962; Lumer, 1961)

X Banach space, $T \in L(X)$,

$$V(T) = \left\{ x^*(Tx) \ : \ x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1 \right\}$$

Some properties

X Banach space, $T \in L(X)$:

- $V(T)$ is connected (not necessarily convex).
- In the complex case, $\overline{W(T)}$ contains the spectrum of T.
- Actually,

$$\overline{\mathrm{co}}\, Sp(T) = \bigcap \overline{\mathrm{co}}\, V(T),$$

  the intersection taken over all numerical ranges $V(T)$ corresponding to equivalent norms on X.

Numerical radius

X real or complex Banach space, $T \in L(X)$,

$$v(T) = \sup\left\{|\lambda| \ : \ \lambda \in V(T)\right\}.$$

- v is a seminorm with $v(T) \leqslant \|T\|$.
- $v(T) = v(T^*)$ for every $T \in L(X)$.

Numerical index (Lumer, 1968)

X real or complex Banach space,

$$\begin{aligned} n(X) &= \inf\left\{v(T) \ : \ T \in L(X),\ \|T\| = 1\right\} \\ &= \max\{k \geqslant 0 \ : \ k\|T\| \leqslant v(T)\ \forall T \in L(X)\right\}. \end{aligned}$$

Remarks

- $n(X) = 1$ iff $v(T) = \|T\|$ for every $T \in L(X)$.
- If there is $T \neq 0$ with $v(T) = 0$, then $n(X) = 0$.
- The converse is not true.

Relationship with semigroups of operators

A motivating example

A real or complex $n \times n$ matrix. TFAE:

- A is skew-adjoint (i.e. $A^* = -A$).
- $\operatorname{Re}(Ax \mid x) = 0$ for every $x \in H$.
- $B = \exp(\rho A)$ is unitary for every $\rho \in \mathbb{R}$ (i.e. $B^* B = \mathrm{Id}$).

In term of Hilbert spaces

H (n-dimensional) Hilbert space, $T \in L(H)$. TFAE:

- $\operatorname{Re} W(T) = \{0\}$.
- $\exp(\rho T) \in \operatorname{Iso}(H)$ for every $\rho \in \mathbb{R}$.

For general Banach spaces

X Banach space, $T \in L(X)$. TFAE:

- $\operatorname{Re} V(T) = \{0\}$.
- $\exp(\rho T) \in \operatorname{Iso}(X)$ for every $\rho \in \mathbb{R}$.

Characterizing uniformly continuous semigroups of operators

Theorem

X real or complex Banach space, $T \in L(X)$. TFAE:

- $\operatorname{Re} V(T) = \{0\}$.
- $\|\exp(\rho T)\| \leqslant 1$ for every $\rho \in \mathbb{R}$.
- $\left\{\exp(\rho T) \ : \ \rho \in \mathbb{R}_0^+\right\} \subset \operatorname{Iso}(X)$.
- T belongs to the tangent space of $\operatorname{Iso}(X)$ at Id, i.e. exists a function $f : [-1,1] \longrightarrow \operatorname{Iso}(X)$ with $f(0) = \mathrm{Id}$ and $f'(0) = T$.
- $\displaystyle\lim_{\rho\to 0} \frac{\|\mathrm{Id} + \rho\, T\| - 1}{\rho} = 0$, i.e. the derivative or the norm of $L(X)$ at Id in the direction of T is null.

Main consequence for us

If X is a real Banach space with $n(X) > 0$, then $\operatorname{Iso}(X)$ is “small”:

- it does not contain any uniformly continuous one-parameter semigroups,
- the tangent space of $\operatorname{Iso}(X)$ at Id is zero.

The example

M. Martín
The group of isometries of a Banach space and duality.
preprint.

The main example

The construction

E separable Banach space. We construct a Banach space $X(E)$ such that

$$n\big(X(E)\big) = 1 \qquad \text{and} \qquad X(E)^* \equiv E^* \oplus_1 L_1(\mu)$$

The main consequence

Take $E = \ell_2$ (real). Then

- $n\big(X(\ell_2)\big) = 1$, so $\mathrm{Iso}\big(X(\ell_2)\big)$ is "small".
- Since $X(\ell_2)^* \equiv \ell_2 \oplus_1 L_1(\mu)$, given $S \in \mathrm{Iso}(\ell_2)$, the operator

$$T = \begin{pmatrix} S & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$$

  is a surjective isometry of $X(\ell_2)^*$.
- Therefore, $\mathrm{Iso}\big(X(\ell_2)^*\big)$ contains infinitely many semigroups of isometries.

Sketch of the construction I

Define (viewing $E \hookrightarrow C[0,1]$)

$$Y = \Big\{ f \in C([0,1]\times[0,1]) \ : \ f(\cdot,0) = 0 \Big\}$$
$$X(E) = \Big\{ f \in C([0,1]\times[0,1]) \ : \ f(\cdot,0) \in E \Big\}$$

We need

$$X(E)^* \equiv E^* \oplus_1 L_1(\mu) \quad \& \quad n\big(X(E)\big) = 1$$

Proving that $X(E)^* \equiv E^* \oplus_1 L_1(\mu)$

- Y is an M-ideal of $C([0,1]\times[0,1])$, so Y is an M-ideal of $X(E)$.
- This means that $X(E)^* \equiv Y^\perp \oplus_1 Y^*$.
- $Y^* \equiv L_1(\mu)$ for some measure μ; $Y^\perp \equiv \big(X(E)/Y\big)^*$.
- Define $\Phi : X(E) \longrightarrow E$ by $\Phi(f) = f(\cdot,0)$.
    - $\|\Phi\| \leqslant 1$ and $\ker\Phi = Y$.
    - $\widetilde{\Phi} : X(E)/Y \longrightarrow E$ is a surjective isometry since:
    - $\{g \in E \ : \ \|g\| < 1\} \subseteq \Phi\big(\{f \in X(E) \ : \ \|f\| < 1\}\big)$.
- Therefore, $Y^\perp \equiv \big(X(E)/Y\big)^* \equiv E^*$.

Sketch of the construction II

Define (viewing $E \hookrightarrow C[0,1]$)

$$Y = \Big\{ f \in C([0,1]\times[0,1]) \ : \ f(\cdot,0) = 0 \Big\}$$
$$X(E) = \Big\{ f \in C([0,1]\times[0,1]) \ : \ f(\cdot,0) \in E \Big\}$$

We need

$$X(E)^* \equiv E^* \oplus_1 L_1(\mu) \quad \& \quad n\big(X(E)\big) = 1$$

Proving that $n\big(X(E)\big) = 1$

- Fix $T \in L\big(X(E)\big)$. Find $f_0 \in X(E)$ and $\xi_0 \in]0,1]\times[0,1]$ such that $\big|[Tf_0](\xi_0)\big| \sim \|T\|$.
- Consider the non-empty open set
$$V = \big\{\xi \in]0,1]\times[0,1] \ : \ f_0(\xi) \sim f_0(\xi_0)\big\}$$
and find $\varphi : [0,1]\times[0,1] \longrightarrow [0,1]$ continuous with $\mathrm{supp}(\varphi) \subset V$ and $\varphi(\xi_0) = 1$.
- Write $f_0(\xi_0) = \lambda\omega_1 + (1-\lambda)\omega_2$ with $|\omega_i| = 1$, and consider the functions
$$f_i = (1-\varphi)f_0 + \varphi\omega_i \text{ for } i = 1,2.$$
- Then, $f_i \in Y \subset X(E)$, $\|f_i\| \leqslant 1$, and
$$\Big\| f_0 - \big(\lambda f_1 + (1-\lambda)f_2\big)\Big\| = \|\varphi f_0 - \varphi f_0(\xi_0)\| \sim 0.$$
- Therefore, there is $i \in \{1,2\}$ such that $\big|[T(f_i)](\xi_0)\big| \sim \|T\|$, but now $|f_i(\xi_0)| = 1$.
- Equivalently,
$$\Big|\delta_{\xi_0}\big(T(f_i)\big)\Big| \sim \|T\| \qquad \text{and} \qquad |\delta_{\xi_0}(f_i)| = 1,$$
meaning that $v(T) \sim \|T\|$.

Some related results

K. Boyko, V. Kadets, M. Martín, and D. Werner.
Numerical index of Banach spaces and duality.
Math. Proc. Cambridge Philos. Soc. (2007).

M. Martín, J. Merí, and A. Rodríguez-Palacios.
Finite-dimensional spaces with numerical index zero.
Indiana U. Math. J. (2004).

H. P. Rosenthal
The Lie algebra of a Banach space.
in: *Banach spaces* (Columbia, Mo., 1984), LNM, Springer, 1985.

Isometries in finite-dimensional spaces

Theorem

Let X be a finite-dimensional real space. TFAE:

- $\operatorname{Iso}(X)$ is infinite.
- $n(X) = 0$.
- There is $T \in L(X)$, $T \neq 0$, with $v(T) = 0$.

Examples of spaces of this kind

1. Hilbert spaces.
2. $X_{\mathbb{R}}$, the real space subjacent to any complex space X.
3. An absolute sum of any real space and one of the above.
4. Moreover, if $X = X_0 \oplus X_1$ where X_1 is complex and

$$\left\| x_0 + e^{i\theta} x_1 \right\| = \|x_0 + x_1\| \qquad \left(x_0 \in X_0,\ x_1 \in X_1,\ \theta \in \mathbb{R}\right).$$

   (Note that the other 3 cases are included here)

Question

Can every Banach space X with $n(X) = 0$ be decomposed as in 4 ?

Negative answer I

Infinite-dimensional case

There is an infinite-dimensional real Banach space X with $n(X) = 0$ but X is polyhedral. In particular, X does not contain $\mathbb{C}$ isometrically.

The example is

$$X = \left[\bigoplus_{n \geqslant 2} X_n \right]_{c_0}$$

X_n is the two-dimensional space whose unit ball is the regular polygon of $2n$ vertices.

Note

Such an example is not possible in the finite-dimensional case.

(Quasi affirmative) negative answer II

Finite-dimensional case

X finite-dimensional real space. TFAE:

- $n(X) = 0$.
- $X = X_0 \oplus X_1 \oplus \cdots \oplus X_n$ such that
  - X_0 is a (possible null) real space,
  - $X_1, \ldots, X_n$ are non-null complex spaces,

  there are $\rho_1, \ldots, \rho_n$ rational numbers, such that

$$\left\| x_0 + e^{i\rho_1\theta} x_1 + \cdots + e^{i\rho_n\theta} x_n \right\| = \left\| x_0 + x_1 + \cdots + x_n \right\|$$

  for every $x_i \in X_i$ and every $\theta \in \mathbb{R}$.

Remark

- The theorem is due to Rosenthal, but with real ρ's.
- The fact that the ρ's may be chosen as rational numbers is due to M.–Merí–Rodríguez-Palacios.

The Lie-algebra of a Banach space

Lie-algebra

X real Banach space, $\mathcal{Z}(X) = \big\{T \in L(X) \ : \ v(T) = 0\big\}$.

- When X is finite-dimensional, $\mathrm{Iso}(X)$ is a Lie-group and $\mathcal{Z}(X)$ is the tangent space (i.e. its Lie-algebra).

Remark

If $\dim(X) = n$, then

$$0 \leqslant \dim(\mathcal{Z}(X)) \leqslant \frac{n(n-1)}{2}.$$

An open problem

Given $n \geqslant 3$, which are the possible $\dim\big(\mathcal{Z}(X)\big)$ over all n-dimensional X's?

Observation

When $\dim(X) = 3$, $\dim(\mathcal{Z}(X))$ cannot be 2.

Numerical index of Banach spaces

Numerical index (Lumer, 1968)

X real or complex Banach space,

$$n(X) = \inf\left\{v(T) \ : \ T \in L(X),\ \|T\| = 1\right\} = \max\{k \geqslant 0 \ : \ k\|T\| \leqslant v(T)\ \forall T \in L(X)\big\}.$$

Some examples

1. $C(K)$, $L_1(\mu)$ have numerical index 1.
2. H Hilbert space, $\dim(H) > 1$, then
$$n(H) = 0 \ \text{ real case} \qquad n(H) = \frac{1}{2} \ \text{ complex case.}$$
3. $n(L_p[0,1]) = n(\ell_p)$ but both are unknown.
4. If X_n is the two-dimensional space such that B_{X_n} is a $2n$-polygon, then
$$n(X_n) = \tan\left(\frac{\pi}{2n}\right) \ \text{ if } n \text{ is even} \qquad n(X_n) = \sin\left(\frac{\pi}{2n}\right) \ \text{ if } n \text{ is odd.}$$
5. If X is a C^*-algebra or the predual of a von Neumann algebra, then $n(X) = 1$ if the algebra is commutative and $n(X) = 1/2$ otherwise.

Numerical index and duality

Proposition

X Banach space.

- $v(T^*) = v(T)$ for every $T \in L(X)$.
- Therefore, $n(X^*) \leqslant n(X)$.

Question

Is is always $n(X) = n(X^*)$ **?**

Answer (Boyko-Kadets-M.-Werner, 2007)

The answer is **NO**.

With our construction is easy to give an example

Example

Take $X(\ell_2)$. Then $n\big(X(\ell_2)\big) = 1$ and $X(\ell_2)^* \equiv \ell_2 \oplus_1 L_1(\mu)$.
Since there is $S \in L\big(X(\ell_2)^*\big)$, $S \neq 0$ with $v(S) = 0$, then $n\big(X(\ell_2)^*\big) = 0$.