Rango numérico e igualdades de normas para operadores.

Prehistoria, historia y resultados recientes

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Estructura del curso

1. Un poco de prehistoria...
2. Historia
3. C^*-álgebras
4. Igualdades de normas para operadores
5. El grupo de isometrías de un Espacio de Banach y la dualidad

Notación básica

Notación básica

X espacio de Banach real o complejo

- $\mathbb{K}$ cuerpo base (puede ser $\mathbb{R}$ o $\mathbb{C}$),
- $\mathbb{T}$ conjunto de escalares de módulo uno:
- S_X esfera unidad, B_X bola unidad,
- X^* espacio dual topológico,
- $L(X)$ algebra de los operadores lineales y continuos en X,
- $T^* \in L(X^*)$ operador adjunto de $T \in L(X)$,
- $\operatorname{Iso}(X)$ isometrías sobreyectivas.

Capítulo 1
Un poco de prehistoria

F. F. Bonsall and J. Duncan
Numerical Ranges (volúmenes I y II).
London Math. Soc. Lecture Note Series, 1971 – 1973.

I. K. Daugavet
On a property of completely continuous operators in the space C.
Uspekhi Mat. Nauk (1963)

J. Duncan, C. M. McGregor, J. D. Pryce, and A. J. White
The numerical index of a normed space.
J. London Math. Soc. (1970)

D. Werner,
An elementary approach to the Daugavet equation, in: *Interaction between Functional Analysis, Harmonic Analysis and Probability* (N. Kalton, E. Saab and S. Montgomery-Smith editors).
Lecture Notes in Pure and Appl. Math. **175** (1996)

1 Un poco de prehistoria...
- La ecuación de Daugavet
- El rango numérico y la ecuación de Daugavet alternativa
- Relación entre rango numérico y ecuación de Daugavet

2 Historia
- Motivation
- Propaganda
- Geometric characterizations
- From rank-one to other class of operators

3 C^*-álgebras
- The known results
- A new sufficient condition
- Application: C^*-algebras and von Neumann preduals
  - von Neumann preduals
  - C^*-algebras
- The alternative Daugavet equation
  - Definitions and basic results
  - Geometric characterizations
  - C^*-algebras and preduals

4 Igualdades de normas para operadores
- Motivación
- Las ecuaciones
  - $\|\mathrm{Id} + T\| = f(\|T\|)$
  - $\|g(T)\| = f(\|T\|)$
  - $\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$

La ecuación de Daugavet

La ecuación de Daugavet (Daugavet, 1963)

X espacio de Banach, $T \in L(X)$

$$\|\mathrm{Id} + T\| = 1 + \|T\| \qquad \text{(DE)}$$

Ejemplos clásicos

1. **Daugavet, 1963:**
   Todo operador compacto en $C[0,1]$ verifica (DE).
2. **Foias-Singer (Pełczyński), 1965:**
   Todo operador débil compacto en $C(K)$, K perfecto, verifica (DE).
3. **Lozanoskii, 1966:**
   Todo operador compacto en $L_1[0,1]$ verifica (DE).
4. **Abramovich, Holub, y otros, años 80:**
   $X = C(K)$, K compacto perfecto o $X = L_1(\mu)$, μ medida sin átomos
   $\Longrightarrow$ todo operador débilmente compacto $T \in L(X)$ verifica (DE).

Ver otro fichero

¿Por qué la ecuación de Daugavet?

Un motivo...

X espacio de Banach. Si

$$\|\mathrm{Id} + T\| = 1 + \|T\|$$

para todo $T \in \mathcal{A}$, entonces:

- Id está "extremadamente" separada de $\mathcal{A}$.
- Las aproximaciones de Id en $\mathcal{A}$ son "uniformemente" malas.

En el origen...

Si tomamos $\mathcal{A} = K(X)$ o $\mathcal{A} = W(X)$, estamos probando que la mejor aproximación de Id en $\mathcal{A}$ es cero y que con cualquier otra aproximación nos alejamos de la peor manera posible.

Rango numérico: espacios de Hilbert

Rango numérico en espacios de Hilbert (Toeplitz, 1918)

- A $n \times n$ matriz real o compleja

$$W(A) = \Big\{ (Ax \mid x) \ : \ x \in \mathbb{K}^n, \ (x \mid x) = 1 \Big\}.$$

- H espacio de Hilbert real o complejo, $T \in L(H)$,

$$W(T) = \Big\{ (Tx \mid x) \ : \ x \in H, \ \|x\| = 1 \Big\}.$$

Algunas propiedades

H espacio de Hilbert, $T \in L(H)$:

- $W(T)$ es convexo.
- En caso complejo, $\overline{W(T)}$ contiene al espectro de T.
- Si además T es normal, entonces $\overline{W(T)} = \overline{\mathrm{co}}\, \mathrm{Sp}(T)$.

Rango numérico: espacios de Banach

Rango numérico en espacios de Banach (Bauer 1962; Lumer, 1961)

X espacio de Banach, $T \in L(X)$,

$$V(T) = \left\{ x^*(Tx) \ : \ x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1 \right\}$$

Algunas propiedades

X espacio de Banach, $T \in L(X)$:

- $V(T)$ es conexo (no necesariamente convexo).
- En caso complejo, $\overline{W(T)}$ contiene al espectro de T.
- De hecho,

$$\overline{\text{co}}\, Sp(T) = \bigcap \overline{\text{co}}\, V(T),$$

  donde tomamos la intersección sobre los rangos numéricos $V(T)$ correspondientes a todas las posibles normas equivalentes en X.

Rango numérico: álgebras normadas

Rango numérico de álgebra (Stampfli–Williams, 1968; Bonsall, 1969)

A álgebra normada con unidad e, $a \in A$:

$$V_A(a) = \left\{\varphi(a) \ : \ \varphi \in A^*, \ \|\varphi\| = \varphi(\mathrm{e}) = 1\right\}$$

Relación

X espacio de Banach, $T \in L(X)$, entonces

$$\overline{\mathrm{co}}\big(V(T)\big) = V_{L(X)}(T).$$

¿Por qué el Rango numérico?

Motivos en espacios de Hilbert

- En espacios de Hilbert complejos permite trabajar de forma más cómoda con el espectro: si un conjunto no contiene al rango numérico, tampoco al espectro y en su complementario podemos hacer cálculo funcional holomorfo, continuo. . .
- Permite trabajar de forma cómoda con conceptos como operador hermitiano, disipativo. . .
- Permite hacer simulaciones numéricas sobre el espectro.

Motivos en espacios de Banach y álgebras normadas

- Permite definir sin dificultad operador (o elemento) hermitiano, disipativo. . .
- Su mayor éxito es la demostración de parte del Teorema de Vidav-Palmer: toda C^*-álgebra es una B^*-álgebra y se caracteriza por descomponer como hermitianos + i hermitianos.

La ecuación de Daugavet alternativa

La ecuación de Daugavet alternativa (Duncan-McGregor-Pryce-White, 1970)

X espacio de Banach, $T \in L(X)$

$$\max_{\omega \in \mathbb{T}} \|\mathrm{Id} + \omega\, T\| = 1 + \|T\| \qquad \text{(aDE)}$$

Ejemplos clásicos

1. **Duncan-McGregor-Pryce-White, 1970:**
   $X = C(K)$, K compacto arbitrario o $X = L_1(\mu)$, μ medida arbitraria,
   $\Longrightarrow$ todo operador $T \in L(X)$ satisface (aDE).
2. Por tanto, lo mismo ocurre en los espacios de Lindenstrauss, esto es, si X^* es isométrico a un espacio $L_1(\mu)$.
3. **Crab-Duncan-McGregor, 1972:**
   Todo operador en el álgebra del disco $A(\mathbb{D})$ o en H^∞ satisface (aDE).
4. **Werner, 1997:**
   Todo operador en un álgebra de funciones satisface (aDE). Un álgebra de funciones es una subalgebra de $C(K)$ que contiene a las constante y separa los puntos de K.

Relación entre rango numérico y ecuación de Daugavet

Proposición (Duncan-McGregor-Pryce-White, 1970)

X espacio de Banach, $T \in L(X)$. Entonces

$$\sup \operatorname{Re} V(T) = \|T\| \iff T \text{ satisface (DE)}$$
$$\sup |V(T)| = \|T\| \iff T \text{ satisface (aDE)}$$

Capítulo 2
Un poco de historia.

Aparece la propiedad de Daugavet

Y. Abramovich, and C. Aliprantis,
An invitation to operator theory.
Graduate Studies in Math. **50**, AMS, 2002.

Y. Abramovich, and C. Aliprantis,
Problems in operator theory.
Graduate Studies in Math. **51**, AMS, 2002.

V. Kadets, R. Shvidkoy, G. Sirotkin, and D. Werner,
Banach spaces with the Daugavet property.
Trans. Amer. Math. Soc. (2000)

D. Werner,
Recent progress on the Daugavet property.
Irish Math. Soc. Bulletin (2001)

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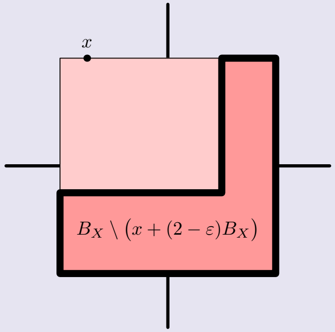
Introduction

- In a Banach space X with the Radon-Nikodým property the unit ball has many denting points.
- $x \in S_X$ is a denting point of B_X if for every $\varepsilon > 0$ one has
$$x \notin \overline{\text{co}}\big(B_X \setminus (x + \varepsilon B_X)\big).$$
- $C[0,1]$ and $L_1[0,1]$ have an extremely opposite property: for every $x \in S_X$ and every $\varepsilon > 0$
$$\overline{\text{co}}\Big(B_X \setminus \big(x + (2-\varepsilon)B_X\big)\Big) = B_X\,.$$
- This geometric property is equivalent to a property of operators on the space.



The Daugavet property

- A Banach space X is said to have the Daugavet property if every rank-one operator on X satisfies (DE).
- If X^* has the Daugavet property, so does X. The converse is not true:

$$C[0,1] \text{ has it but } C[0,1]^* \text{ not.}$$

(Kadets–Shvidkoy–Sirotkin–Werner, 1997 & 2000)

Prior versions of: *Chauveheid, 1982; Abramovich–Aliprantis–Burkinshaw, 1991*

Some examples...

1. K perfect, μ atomeless, E arbitrary Banach space
   $\Longrightarrow C(K,E)$, $L_1(\mu,E)$, and $L_\infty(\mu,E)$ have the Daugavet property.
   (Kadets, 1996; Nazarenko, –; Shvidkoy, 2001)
2. $A(\mathbb{D})$ and H^∞ have the Daugavet property.
   (Wojtaszczyk, 1992)

More examples...

3. A function algebra whose Choquet boundary is perfect has the Daugavet property.

   (Werner, 1997)

4. "Large" subspaces of $C[0,1]$ and $L_1[0,1]$ have the Daugavet property (in particular, this happends for finite-codimensional subspaces).

   (Kadets–Popov, 1997)

5. A C^*-algebra has the Daugavet property if and only if it is non-atomic.
6. The predual of a von Neumann algebra has the Daugavet property if and only if the algebra is non-atomic.

   (Oikhberg, 2002)

Some *propaganda*...

Let X be a Banach space with the Daugavet property. Then

- X does not have the Radon-Nikodým property.

  (Wojtaszczyk, 1992)
- Every slice of B_X and every w^*-slice of B_{X^*} have diameter 2.

  (Kadets–Shvidkoy–Sirotkin–Werner, 2000)
- Actually, every weakly-open subset of B_X has diameter 2.

  (Shvidkoy, 2000)
- X contains a copy of ℓ_1. X^* contains a copy or $L_1[0,1]$.

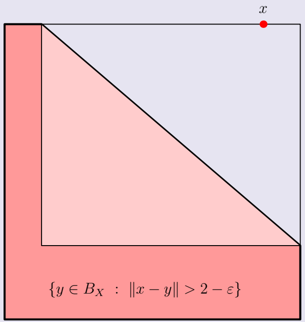
  (Kadets–Shvidkoy–Sirotkin–Werner, 2000)

Ver otro fichero

Geometric characterizations

Theorem [KSSW]

- X has the Daugavet property.
- For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y \in S_X$ such that
$$\operatorname{Re} x^*(y) > 1 - \varepsilon \quad \text{and} \quad \|x - y\| \geqslant 2 - \varepsilon.$$
- For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y^* \in S_{X^*}$ such that
$$\operatorname{Re} y^*(x) > 1 - \varepsilon \quad \text{and} \quad \|x^* - y^*\| \geqslant 2 - \varepsilon.$$
- For every $x \in S_X$ and every $\varepsilon > 0$, we have
$$B_X = \overline{\text{co}}\Big(\{y \in B_X \ : \ \|x - y\| \geqslant 2 - \varepsilon\}\Big).$$



Theorem

Let X be a Banach space with the Daugavet property.

- Every weakly compact operator on X satisfies (DE).

  (Kadets–Shvidkoy–Sirotkin–Werner, 2000)

- Actually, every operator on X which does not fix a copy of ℓ_1 satisfies (DE).

  (Sirotkin, 2000)

Consequences

1. X does not have unconditional basis.

   (Kadets, 1996)

2. Moreover, X does not embed into any space with unconditional basis.

   (Kadets–Shvidkoy–Sirotkin–Werner, 2000)

3. Actually, X does not embed into an unconditional sum of Banach spaces without a copy of ℓ_1.

   (Shvidkoy, 2000)

Capítulo 3
C^*-álgebras

J. Becerra Guerrero and M. Martín,
The Daugavet Property of C^*-algebras, JB^*-triples, and of their isometric preduals.
Journal of Functional Analysis (2005)

M. Martín,
The alternative Daugavet property of C^*-algebras and JB^*-triples.
Mathematische Nachrichten (to appear)

M. Martín and T. Oikhberg,
An alternative Daugavet property.
Journal of Mathematical Analysis and Applications (2004)

T. Oikhberg,
The Daugavet property of C^*-algebras and non-commutative L_p-spaces.
Positivity (2002)

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The known results

Oikhberg, 2002

Let A be a C^*-algebra and V a von Neumann algebra. Then:

1. A has the Daugavet property iff A has no atoms;
2. The predual V_* of V has the Daugavet property iff V does.

Our objectives here

- To give "geometrical" proofs of the above results;
- At the same time, to give characterizations of the Daugavet property for C^*-algebras and von Neumann preduals in terms of the "geometry" of the underlining Banach space (without any algebra).

A new sufficient condition

Theorem

Let X be a Banach space such that

$$X^* = Y \oplus_1 Z$$

with Y and Z norming subspaces. Then, X has the Daugavet property.

A closed subspace $W \subseteq X^*$ is norming if

$$\|x\| = \sup \{|w^*(x)| \ : \ w^* \in W, \ \|w^*\| = 1\}$$

or, equivalently, if B_W is w^*-dense in B_{X^*}.

Proof of the theorem

We have...

$X^* = Y \oplus_1 Z$,
B_Y, B_Z w^*-dense in B_{X^*}.

$\overset{?}{\Longrightarrow}$

We need...

fixed $x_0 \in S_X$, $x_0^* \in S_{X^*}$, $\varepsilon > 0$, find $y^* \in S_{X^*}$ such that

$$\|x_0^* + y^*\| > 2 - \varepsilon \quad \text{and} \quad \operatorname{Re} y^*(x_0) > 1 - \varepsilon.$$

- Write $x_0^* = y_0^* + z_0^*$ with $y_0^* \in Y$, $z_0^* \in Z$, $\|x_0^*\| = \|y_0^*\| + \|z_0^*\|$, and write
$$U = \{x^* \in B_{X^*} \ : \ \operatorname{Re} x^*(x_0) > 1 - \varepsilon/2\}.$$
- Take $z^* \in B_Z \cap U$ and a net (y_λ^*) in $B_Y \cap U$, such that $(y_\lambda^*) \overset{w^*}{\longrightarrow} z^*$.
- $(y_\lambda^* + y_0^*) \longrightarrow z^* + y_0^*$ and the norm is w^*-lower semi-continuous, therefore
$$\liminf \|y_\lambda^* + y_0^*\| \geqslant \|z^* + y_0^*\| = \|z^*\| + \|y_0^*\| > 1 + \|y_0^*\| - \varepsilon/2.$$
- Then, we may find μ such that $\|y_\mu^* + y_0^*\| \geqslant 1 + \|y_0^*\| - \varepsilon/2$.
- Finally, observe that
$$\|x_0^* + y_\mu^*\| = \|(y_0^* + y_\mu^*) + z_0^*\| =$$
$$= \|y_0^* + y_\mu^*\| + \|z_0^*\| > 1 + \|y_0^*\| - \varepsilon + \|z_0^*\| = 2 - \varepsilon,$$

and that $\operatorname{Re} y_\mu^*(x_0) > 1 - \varepsilon$ (since $y_\mu^* \in U$).

Some immediate consequences

Corollary

Let X be an L-embedded space with $\operatorname{ext}(B_X) = \emptyset$. Then, X^* (and hence X) has the Daugavet property.

Corollary

If Y is an L-embedded space which is a subspace of $L_1 \equiv L_1[0,1]$, then $(L_1/Y)^*$ has the Daugavet property.

It was already known that...

- If $Y \subset L_1$ is reflexive, then L_1/Y has the Daugavet property.

  (Kadets–Shvidkoy–Sirotkin–Werner, 2000)
- If $Y \subset L_1$ is L-embedded, then L_1/Y does not have the RNP.

  (Harmand–Werner–Werner, 1993)

von Neumann preduals

von Neumann preduals

- A C^*-algebra X is a von Neumann algebra if it is a dual space.
- In such a case, X has a unique predual X_*.
- X_* is always L-embedded.
- Therefore, if $\operatorname{ext}(B_{X_*})$ is empty, then X and X_* have the Daugavet property. Example: $L_\infty[0,1]$ and $L_1[0,1]$.

Actually, much more can be proved:

Theorem

Let X_* be the predual of the von Neumann algebra X. Then, TFAE:

- X has the Daugavet property.
- X_* has the Daugavet property.
- Every weakly open subset of B_{X_*} has diameter 2.
- B_{X_*} has no strongly exposed points.
- B_{X_*} has no extreme points.
- X is non-atomic (i.e. it has no atomic projections).

An atomic projection is an element $p \in X$ such that

$$p^2 = p^* = p \qquad \text{and} \qquad p\,X\,p = \mathbb{C}p.$$

C^*-algebras

Let X be a C^*-algebra. Then, X^{**} is a von Neumann algebra.
Write $X^* = (X^{**})_* = A \oplus_1 N$, where

- A is the atomic part,
- N is the non-atomic part.

- Every extreme point of B_{X^*} is in B_A.
- Therefore, A is norming.
- What's about N **?**

Theorem

If X is non-atomic, then N is norming. Therefore, X has the Daugavet property.
Example: $C[0,1]$

sketch of the proof of the theorem

We have...

X non-atomic C^*-algebra,
$X^* = A \oplus_1 N$.

$\overset{?}{\Longrightarrow}$

We need...

N to be norming for X, i.e.
$\|x\| = \sup\{|f(x)| \ : \ f \in B_N\} \quad (x \in X)$.

- Write $X^{**} = \mathcal{A} \oplus_\infty \mathcal{N}$ and $Y = \mathcal{A} \cap X$.
- Y is an ideal of X, so Y has no atomic projections.
- Therefore, the norm of Y has no point of Fréchet-smoothness.
- But Y is an Asplund space, so $Y = 0$.
- Now, the mapping
$$X \hookrightarrow X^{**} = \mathcal{A} \oplus_\infty \mathcal{N} \twoheadrightarrow \mathcal{N}$$
in injective. Since it is an homomorphism, it is an isometry.
- But $N^* \equiv \mathcal{N}$, so N is norming for $\mathcal{N}$ and now, also for X.

Theorem

Let X be a C^*-algebra. Then, TFAE:

- X has the Daugavet property.
- The norm of X is extremely rough, i.e.

$$\limsup_{\|h\|\to 0} \frac{\|x+h\|+\|x-h\|-2}{\|h\|} = 2$$

for every $x \in S_X$ (equivalently, every w^*-slice of B_{X^*} has diameter 2).

- The norm of X is not Fréchet-smooth at any point.
- X is non-atomic.

The alternative Daugavet equation

The alternative Daugavet equation

The alternative Daugavet equation

X Banach space, $T \in L(X)$

$$\max_{|\omega|=1} \|\mathrm{Id} + \omega\, T\| = 1 + \|T\| \qquad \text{(aDE)}$$

(Duncan–McGregor–Pryce–White, 1970; Holub, Abramovich..., 80's)

Two equivalent formulations

- There exists $\omega \in \mathbb{T}$ such that $\omega\, T$ satisfies (DE).
- The numerical radius of T, $v(T)$, coincides with $\|T\|$, where

$$v(T) := \sup\{|x^*(Tx)| \ : \ x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1\}.$$

Two possible properties

Let X be a Banach space.

- X is said to have the alternative Daugavet property (ADP) iff every rank-one operator on X satisfies (aDE).
  - Then, every weakly compact operator also satisfies (aDE).
  - If X^* has the ADP, so does X. The converse is not true: $C([0,1],\ell_2)$.

(M.–Oikhberg, 2004; briefly appearance: Abramovich, 1991)

- X is said to have numerical index 1 iff $v(T) = \|T\|$ for every operator on X. Equivalently, if EVERY operator on X satisfies (aDE).

(Lumer, 1968; Duncan–McGregor–Pryce–White, 1970)

Observation

No analogous property is possible for the Daugavet equation:

$$\|\mathrm{Id} + (-\mathrm{Id})\| = 0 \neq 1 + \|-\mathrm{Id}\|.$$

Numerical index 1

- $C(K)$ and $L_1(\mu)$ have numerical index 1.

  (Duncan–McGregor–Pryce–White, 1970)
- All function algebras have numerical index 1.

  (Werner, 1997)
- A C^*-algebra has numerical index 1 iff it is commutative.

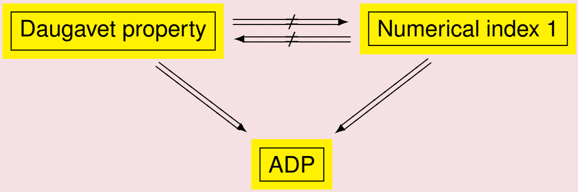
  (Huruya, 1977; Kaidi–Morales–Rodríguez-Palacios, 2000)
- In case $\dim(X) < \infty$, X has numerical index 1 iff

  $$|x^*(x)| = 1 \qquad x^* \in \operatorname{ext}(B_{X^*}), \ x \in \operatorname{ext}(B_X).$$

  (McGregor, 1971)
- In case $\dim(X) = \infty$, if X has numerical index 1 and the RNP, then $X \supseteq \ell_1$.

  (López–M.–Payá, 1999)

The alternative Daugavet property

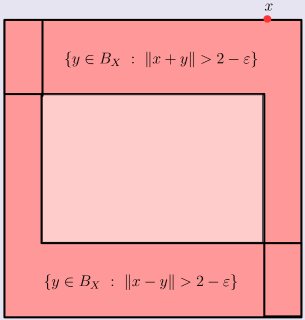


- $c_0 \oplus_\infty C([0,1], \ell_2)$ has the ADP, but neither the Daugavet property, nor numerical index 1.
- For RNP or Asplund spaces, the ADP implies numerical index 1.
- Every Banach space with the ADP can be renormed still having the ADP but failing the Daugavet property.

Geometric characterizations

Theorem

- X has the ADP.
- For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y \in S_X$ such that
$$|x^*(y)| > 1 - \varepsilon \quad \text{and} \quad \|x - y\| \geqslant 2 - \varepsilon.$$
- For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y^* \in S_{X^*}$ such that
$$|y^*(x)| > 1 - \varepsilon \quad \text{and} \quad \|x^* - y^*\| \geqslant 2 - \varepsilon.$$
- For every $x \in S_X$ and every $\varepsilon > 0$, we have
$$B_X = \overline{\text{co}}\big(\mathbb{T}\,\{y \in B_X \ : \ \|x - y\| \geqslant 2 - \varepsilon\}\big).$$



Let V_* be the predual of the von Neumann algebra V.

The Daugavet property of V_* is equivalent to:

- V has no atomic projections, or
- the unit ball of V_* has no extreme points.

V_* has numerical index 1 iff:

- V is commutative, or
- $|v^*(v)| = 1$ for $v \in \operatorname{ext}(B_V)$ and $v^* \in \operatorname{ext}(B_{V^*})$.

The alternative Daugavet property of V_* is equivalent to:

- the atomic projections of V are central, or
- $|v(v_*)| = 1$ for $v \in \operatorname{ext}(B_V)$ and $v_* \in \operatorname{ext}(B_{V_*})$, or
- $V = C \oplus_\infty N$, where C is commutative and N has no atomic projections.

Let X be a C^*-algebra.

The Daugavet property of X is equivalent to:

- X does not have any atomic projection, or
- the unit ball of X^* does not have any w^*-strongly exposed point.

X has numerical index 1 iff:

- X is commutative, or
- $|x^{**}(x^*)| = 1$ for $x^{**} \in \operatorname{ext}(B_{X^{**}})$ and $x^* \in \operatorname{ext}(B_{X^*})$.

The alternative Daugavet property of X is equivalent to:

- the atomic projections of X are central, or
- $|x^{**}(x^*)| = 1$, for $x^{**} \in \operatorname{ext}(B_{X^{**}})$, and $x^* \in B_{X^*}$ w^*-strongly exposed, or
- $\exists$ a commutative ideal Y such that X/Y has the Daugavet property.

Capítulo 4

Igualdades de normas para operadores

V. Kadets, M. Martín y J. Merí,
Norm equalities for operators on Banach spaces.
Indiana U. Math. J. (2007)

P. Koszmider, M. Martín y J. Merí,
Extremely non-complex $C(K)$ spaces.
En preparación

1 Un poco de prehistoria...
- La ecuación de Daugavet
- El rango numérico y la ecuación de Daugavet alternativa
- Relación entre rango numérico y ecuación de Daugavet

2 Historia
- Motivation
- Propaganda
- Geometric characterizations
- From rank-one to other class of operators

3 C^*-álgebras
- The known results
- A new sufficient condition
- Application: C^*-algebras and von Neumann preduals
  - von Neumann preduals
  - C^*-algebras
- The alternative Daugavet equation
  - Definitions and basic results
  - Geometric characterizations
  - C^*-algebras and preduals

4 Igualdades de normas para operadores
- Motivación
- Las ecuaciones
  - $\|\mathrm{Id} + T\| = f(\|T\|)$
  - $\|g(T)\| = f(\|T\|)$
  - $\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$

El problema que nos planteamos

Dadas las importantes consecuencias que la propiedad de Daugavet tiene sobre la geometría de un espacio de Banach, nos planteamos el siguiente problema:

El problema

Estudiar la posibilidad de encontrar igualdades de normas para operadores que puedan ser válidas para todos los operadores de rango uno en un espacio de Banach.

Estudiamos tres casos:

1. $\|\mathrm{Id} + T\| = f(\|T\|)$ para f arbitraria,
2. $\|g(T)\| = f(\|T\|)$ para g entera y f arbitraria,
3. $\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$ para g entera y f *continua*.

Antecedentes: desigualdades tipo Daugavet

Algunos ejemplos

- **Benyamini–Lin, 1985:**
  Para cada $1 < p < \infty$, $p \neq 2$, existe una función $\psi_p : (0, \infty) \longrightarrow (0, \infty)$ tal que
  $$\|\mathrm{Id} + T\| \geqslant 1 + \psi_p(\|T\|)$$
  para todo operador compacto T en $L_p[0, 1]$.
- Si $p = 2$, existe un operador compacto T en $L_2[0, 1]$ con $\|T\| = 1$ y tal que
  $$\|\mathrm{Id} + T\| = 1.$$
- **Boyko–Kadets, 2004:**
  Si llamamos ψ_p a la mejor función posible arriba, entonces
  $$\lim_{p \to 1^+} \psi_p(t) = t \qquad (t > 0).$$
- **Oikhberg, 2005:**
  En cualquier espacio $K(\ell_2) \subseteq X \subseteq L(\ell_2)$ se tiene que
  $$\|\mathrm{Id} + T\| \geqslant 1 + \frac{1}{8\sqrt{2}}\|T\|$$
  para todo operador compacto T en X.

Igualdades de la forma $\|g(T)\| = f(\|T\|)$

Nota

Si X tiene la propiedad de Daugavet, entonces $\|\mathrm{Id} + T\|$ depende sólo de $\|T\|$.

Proposición

X real o complejo, $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ arbitraria, $a, b \in \mathbb{K}$. Si la igualdad

$$\|a\,\mathrm{Id} + b\,T\| = f(\|T\|)$$

se verifica para todo operador de rango uno T en X, entonces

$$f(t) = |a| + |b|\,t \qquad \big(t \in \mathbb{R}_0^+\big).$$

Si $a \neq 0$, $b \neq 0$, entonces X tiene la propiedad de Daugavet.

Demostración

Tenemos...

$\|a\,\mathrm{Id} + b\,T\| = f(\|T\|)\ \forall T \in L(X)$ de rango uno

$\overset{?}{\Longrightarrow}$

Queremos probar...

$f(t) = |a| + |b|\,t \qquad \big(t \in \mathbb{R}_0^+\big).$

- Trivial si $a \cdot b = 0$. Suponemos $a \neq 0$ y $b \neq 0$ y escribimos $\omega_0 = \frac{\overline{b}}{|b|}\frac{a}{|a|} \in \mathbb{T}$.
- Ahora fijamos $x_0 \in S_X$, $x_0^* \in S_{X^*}$ con $x_0^*(x_0) = \omega_0$ y consideramos

$$T_t = t\, x_0^* \otimes x_0 \in L(X) \qquad \big(t \in \mathbb{R}_0^+\big).$$

- Obsérvese que $\|T_t\| = t$ y, por tanto,

$$f(t) = \|a\mathrm{Id} + b\,T_t\| \qquad \big(t \in \mathbb{R}_0^+\big).$$

- Se sigue que

$$|a| + |b|\,t \geqslant f(t) = \|a\mathrm{Id} + b\,T_t\| \geqslant \Big\|[a\mathrm{Id} + b\,T_t](x_0)\Big\|$$

$$= \|a\,x_0 + b\,\omega_0 t\,x_0\| = |a + b\,\omega_0 t|\,\|x_0\| = \left|a + b\,\frac{\overline{b}}{|b|}\frac{a}{|a|}\,t\right| = |a| + |b|\,t.$$

- Finalmente, fijado un operador de rango uno $T \in L(X)$, llamamos $S = \frac{a}{b}\,T$ y tenemos

$$|a|\big(1 + \|T\|\big) = |a| + |b|\,\|S\| = \|a\mathrm{Id} + b\,S\| = |a|\,\|\mathrm{Id} + T\|.$$

Igualdades de la forma $\|g(T)\| = f(\|T\|)$

Teorema

X real o complejo con $\dim(X) \geqslant 2$.
Supongamos que la igualdad

$$\|g(T)\| = f(\|T\|)$$

se verifica para todo operador de rango uno T en X, donde

- $g : \mathbb{K} \longrightarrow \mathbb{K}$ es entera,
- $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ es arbitraria.

Entonces, tres casos son posibles:

- $\|\mathrm{Id}\| = 1$,
- $\|T\| = \|T\|$,

(casos triviales)

- $\|a\,\mathrm{Id} + b\,T\| = |a| + |b|\,\|T\|$, con $a \neq 0$, $b \neq 0$

(Propiedad de Daugavet)

Demostración (caso complejo)

Tenemos...

$\|g(T)\| = f(\|T\|)\ \forall T \in L(X)$ de rango uno

$\overset{?}{\Longrightarrow}$

Queremos probar...

g es afín.

Escribimos $g(\zeta) = \sum_{k=0}^{\infty} a_k \zeta^k$ y $\widetilde{g} = g - a_0$.

Tomamos $x_0, x_1 \in S_X$ y $x_0^*, x_1^* \in S_{X^*}$ tales que

$$x_0^*(x_0) = 0 \qquad \text{y} \qquad x_1^*(x_1) = 1,$$

y definimos los operadores $T_0 = x_0^* \otimes x_0$ y $T_1 = x_1^* \otimes x_1$, que verifican

$$g(\lambda T_0) = a_0 \mathrm{Id} + a_1 \lambda\, T_0 \qquad \text{y} \qquad g(\lambda T_1) = a_0 \mathrm{Id} + \widetilde{g}(\lambda)\, T_1 \qquad (\lambda \in \mathbb{C}).$$

Luego para cada $\lambda \in \mathbb{C}$ tenemos

$$\left\| a_0 \mathrm{Id} + \widetilde{g}(\lambda) T_1 \right\| = \|g(\lambda T_1)\| = f(|\lambda|) = \|g(\lambda T_0)\| = \|a_0 \mathrm{Id} + a_1 \lambda T_0\|.$$

Usamos la desigualdad triangular para obtener que

$$|\widetilde{g}(\lambda)| \leqslant 2|a_0| + |a_1||\lambda| \qquad (\lambda \in \mathbb{C}),$$

y por tanto $\widetilde{g}$ es un polinomio de grado uno (desigualdades de Cauchy).

Caso real en otro fichero

Igualdades de la forma $\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$

- **Caso complejo:**

- X complejo, $\dim(X) \geqslant 2$.
- $g : \mathbb{C} \longrightarrow \mathbb{C}$ entera no constante,
- $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ *continua*.

Supongamos que

$$\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$$

se verifica para todo operador de rango uno T en X.

Idea de la demostración en otro fichero

Teorema

- Si $\operatorname{Re} g(0) \neq -\frac{1}{2}$ entonces X tiene la propiedad de Daugavet.
- Si $\operatorname{Re} g(0) = -\frac{1}{2}$ entonces existe $\omega \in \mathbb{T} \setminus \{1\}$ de modo que

$$\|\mathrm{Id} + \omega T\| = \|\mathrm{Id} + T\|$$

para todo $T \in L(X)$ de rango uno.

Ejemplo

$X = C[0,1] \oplus_2 C[0,1]$ verifica:

- $\left\|\mathrm{Id} + \omega\, T\right\| = \|\mathrm{Id} + T\|$ para $\omega \in \mathbb{T}$ y T de rango uno.
- X no tiene la propiedad de Daugavet.

Igualdades de la forma $\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$

- CASO REAL:

Si $g(0) \neq -1/2$:

Observaciones

- La demostración del teorema anterior no es válida (utiliza el Teorema de Picard).
- Es válida si g es sobreyectiva.
- No sabemos lo que ocurre si g no es sobreyectiva, incluso en los casos más sencillos:
  - $\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$,
  - $\|\mathrm{Id} - T^2\| = 1 + \|T^2\|$.

Si $g(0) = -1/2$:

Ejemplo

$X = C[0,1] \oplus_2 C[0,1]$ verifica:

- $\|\mathrm{Id} - T\| = \|\mathrm{Id} + T\|$ para todo T de rango uno.
- X no tiene la propiedad de Daugavet.

Algunos problemas abiertos

- Obtener caracterizaciones geométricas de los espacios de Banach X en los que la igualdad
$$\|\mathrm{Id} + \omega T\| = \|\mathrm{Id} + T\|$$
se verifica para todo operador de rango uno $T \in L(X)$ y todo $\omega \in \mathbb{T}$.
- Caracterizar los espacios de Banach reales para los que
$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$$
para todo operador de rango uno $T \in L(X)$.
- Caracterizar los espacios de Banach reales para los que
$$\|\mathrm{Id} - T^2\| = 1 + \|T^2\|$$
para todo operador de rango uno $T \in L(X)$.

Una pregunta de Godefroy

Pregunta de Godefroy (comunicación privada)

¿ Existe algún espacio de Banach real X (distinto de $\mathbb{R}$) para el que la igualdad

$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$$

se verifique para todo operador $T \in L(X)$ **?**

Respuesta

La respuesta es afirmativa. De hecho, podemos tomar como X algunos espacios de tipo $C(K)$ con “pocos operadores” (construidos por Koszmider en 2004) y otros espacios $C(K)$ con “muchos operadores”.

Motivación: estructura compleja

Estructura compleja

Un espacio de Banach real X admite una estructura compleja si existe $T \in L(X)$ tal que

$$T^2 = -\mathrm{Id}.$$

Dicho de otra forma, X admite una norma equivalente que lo convierte en espacio normado complejo:

$$|||x||| = \max\left\{ \left\|\alpha x + \beta T(x)\right\| \ : \ \alpha^2 + \beta^2 = 1 \right\} \qquad (x \in X)$$

Espacios extremadamente no complejos

Un espacio de Banach real X es extremadamente no complejo sii

$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$$

para todo $T \in L(X)$.

- Por supuesto, $X = \mathbb{R}$ cumple esto.
- Ningún espacio de dimensión finita lo hace.
- Nuestro objetivo es probar que hay ejemplos de dimensión infinita.

Los primeros ejemplos

Multiplicador débil

Sea K un espacio topológico compacto. $T \in L\big(C(K)\big)$ es un multiplicador débil if

$$T^* = g\,\mathrm{Id} + S$$

donde g es una función de Borel y S es débilmente compacto.

Nuestro principal resultado

Si K es perfecto y todos los operadores en $C(K)$ son multiplicadores débiles, entonces $C(K)$ es extremadamente no complejo.

Teorema (Koszmider, 2004; Plebanek, 2004)

Existen espacios compactos perfecto K tales que todos los operadores en $C(K)$ son multiplicadores débiles. De hecho, hay diversos tipos de ejemplos:

- K conexo y tal que todo operador en $L(C(K))$ tiene la forma $g\,\mathrm{Id} + S$ con $g \in C(K)$ y S débilmente compacto (*multiplicación débil*).
- K totalmente disconexo y perfecto.

En particular, existen espacios $C(K)$ no isomorfos y extremadamente no complejos.

Demostrando un caso sencillo...

Hipótesis

Todo $T \in L\big(C(K)\big)$ tiene la forma $g\mathrm{Id} + S$, con $g \in C(K)$ y S débilmente compacto

Necesitamos...

$\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$

- Si $T = g\mathrm{Id} + S$, entonces $T^2 = g^2\,\mathrm{Id} + S'$ con S' débilmente compacto.
- Basta probar que $\|\mathrm{Id} + (g\,\mathrm{Id} + S)\| = 1 + \|g\,\mathrm{Id} + S\|$ para $g \geqslant 0$ y S débilmente compacto.
- **Paso 1:** Supongamos que $\|g\| \leqslant 1$ and $\min g(K) > 0$.
- **Paso 2:** Podemos evitar la hipótesis de que $\min g(K) > 0$.
- **Paso 3:** Por último, para cada $g \geqslant 0$ tenemos

y hemos acabado.

Demostración

- Sabemos que máx $\|\mathrm{Id} + (g\,\mathrm{Id} + S)\| = 1 + \|g\,\mathrm{Id} + S\|$. Luego basta probar que

$$\Big\|\mathrm{Id} + \frac{1}{\|g\|}(g\,\mathrm{Id} + S)\Big\| = 1 + \frac{1}{\|g\|}\|g\,\mathrm{Id} + S\|$$

- $\|\mathrm{Id} + ($ ...

Demostración

Basta recordar que el conjunto de los operadores que satisfacen (DE) es cerrado.

Nota

Si $\|u + v\| = \|u\| + \|v\| \quad \Longrightarrow \quad \|\alpha\, u + \beta\, v\| = \alpha\|u\| + \beta\|v\|$ para $\alpha, \beta \in \mathbb{R}_0^+$.

Más ejemplos

Observación

Podría pensarse que ser extremadamente no complejo obliga a tener "pocos operadores". Los siguientes ejemplos muestran que esto no es así.

Más ejemplos

- Existe (en ZFC) un espacio topológico compacto K_1 verificando:
  - $C(K_1)$ es extremadamente no complejo y
  - $C(K_1)$ contiene una copia isomórfica complementada de $C[0,1]$.
- Existe (en ZFC) un espacio topológico compacto K_2 verificando:
  - $C(K_2)$ es extremadamente no complejo y
  - $C(K_2)$ contiene una copia isométrica (1-complementada) de ℓ_∞.

Capítulo 5
El grupo de isometrías de un Espacio de Banach y la dualidad

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M. Martín, J. Merí, and A. Rodríguez-Palacios.
Finite-dimensional spaces with numerical index zero.
Indiana U. Math. J. (2004).

H. P. Rosenthal
The Lie algebra of a Banach space.
in: *Banach spaces* (Columbia, Mo., 1984), LNM, Springer, 1985.

1 Un poco de prehistoria...
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4 Igualdades de normas para operadores
- Motivación
- Las ecuaciones
  - $\|\mathrm{Id} + T\| = f(\|T\|)$
  - $\|g(T)\| = f(\|T\|)$
  - $\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$

Objetivo principal de este capítulo

Se construye un espacio de Banach real X tal que

- $\mathrm{Iso}(X)$ no contiene semigrupos uniparamétricos uniformemente continuos;
- pero $\mathrm{Iso}(X^*)$ contiene una cantidad infinita de tales semigrupos.

Hilbert spaces

Hilbert space Numerical range (Toeplitz, 1918)

- A $n \times n$ real or complex matrix

$$W(A) = \Big\{(Ax \mid x) \ : \ x \in \mathbb{K}^n, \ (x \mid x) = 1\Big\}.$$

- H real or complex Hilbert space, $T \in L(H)$,

$$W(T) = \Big\{(Tx \mid x) \ : \ x \in H, \ \|x\| = 1\Big\}.$$

Some properties

H Hilbert space, $T \in L(H)$:

- $W(T)$ is convex.
- In the complex case, $\overline{W(T)}$ contains the spectrum of T.
- If, moreover, T is normal, $\overline{W(T)} = \overline{\mathrm{co}}\,\mathrm{Sp}(T)$.

Banach spaces

Banach space numerical range (Bauer 1962; Lumer, 1961)

X Banach space, $T \in L(X)$,

$$V(T) = \left\{ x^*(Tx) \ : \ x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1 \right\}$$

Some properties

X Banach space, $T \in L(X)$:

- $V(T)$ is connected (not necessarily convex).
- In the complex case, $\overline{W(T)}$ contains the spectrum of T.
- Actually,

$$\overline{\mathrm{co}}\, Sp(T) = \bigcap \overline{\mathrm{co}}\, V(T),$$

the intersection taken over all numerical ranges $V(T)$ corresponding to equivalent norms on X.

Numerical radius

X real or complex Banach space, $T \in L(X)$,

$$v(T) = \sup\left\{|\lambda| \ : \ \lambda \in V(T)\right\}.$$

- v is a seminorm with $v(T) \leqslant \|T\|$.
- $v(T) = v(T^*)$ for every $T \in L(X)$.

Numerical index (Lumer, 1968)

X real or complex Banach space,

$$\begin{aligned} n(X) &= \inf\left\{v(T) \ : \ T \in L(X), \ \|T\| = 1\right\} \\ &= \max\left\{k \geqslant 0 \ : \ k\|T\| \leqslant v(T) \ \forall T \in L(X)\right\}. \end{aligned}$$

Remarks

- $n(X) = 1$ iff $v(T) = \|T\|$ for every $T \in L(X)$.
- If there is $T \neq 0$ with $v(T) = 0$, then $n(X) = 0$.
- The converse is not true.

Relationship with semigroups of operators

A motivating example

A real or complex $n \times n$ matrix. TFAE:

- A is skew-adjoint (i.e. $A^* = -A$).
- $\mathrm{Re}(Ax \mid x) = 0$ for every $x \in H$.
- $B = \exp(\rho A)$ is unitary for every $\rho \in \mathbb{R}$ (i.e. $B^*B = \mathrm{Id}$).

In term of Hilbert spaces

H (n-dimensional) Hilbert space, $T \in L(H)$. TFAE:

- $\mathrm{Re}\, W(T) = \{0\}$.
- $\exp(\rho T) \in \mathrm{Iso}(H)$ for every $\rho \in \mathbb{R}$.

For general Banach spaces

X Banach space, $T \in L(X)$. TFAE:

- $\mathrm{Re}\, V(T) = \{0\}$.
- $\exp(\rho T) \in \mathrm{Iso}(X)$ for every $\rho \in \mathbb{R}$.

Characterizing uniformly continuous semigroups of operators

Theorem

X real or complex Banach space, $T \in L(X)$. TFAE:

- Re $V(T) = \{0\}$.
- $\|$ exp
- $\{$ exp
- T be
  $f : [-$
- lím $_{\rho}$ ... Id in the
  direc

Remark

- For every $T \in L(X)$, one has
  $$\big\|\exp(\rho T)\big\| \leqslant e^{v(T)\rho} \qquad \big(\rho \in \mathbb{R}\big)$$
  and $v(T)$ is the smallest possibility.
- Therefore, $n(X) = 1$ is the worst possibility to find semigroups of isometries.

Demostración en otro fichero

If X is a real Banach space with $n(X) > 0$, then Iso(X) is "small":

- it does not contain any uniformly continuous one-parameter semigroups,
- the tangent space of Iso(X) at Id is zero.

The main example

The construction

E separable Banach space. We construct a Banach space $X(E)$ such that

$$n\big(X(E)\big) = 1 \qquad \text{and} \qquad X(E)^* \equiv E^* \oplus_1 L_1(\mu)$$

The main consequence

Take $E = \ell_2$ (real). Then

- $n\big(X(\ell_2)\big) = 1$, so $\mathrm{Iso}\big(X(\ell_2)\big)$ is "small".
- Since $X(\ell_2)^* \equiv \ell_2 \oplus_1 L_1(\mu)$, given $S \in \mathrm{Iso}(\ell_2)$, the operator

$$T = \begin{pmatrix} S & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$$

  is a surjective isometry of $X(\ell_2)^*$.
- Therefore, $\mathrm{Iso}\big(X(\ell_2)^*\big)$ contains infinitely many semigroups of isometries.

Sketch of the construction I

Define (viewing $E \hookrightarrow C[0,1]$)

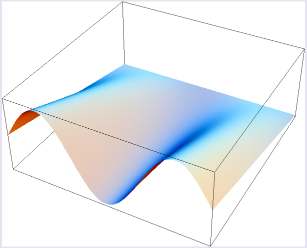
$$Y = \Big\{ f \in C([0,1]\times[0,1]) \ : \ f(\cdot,0) = 0 \Big\}$$
$$X(E) = \Big\{ f \in C([0,1]\times[0,1]) \ : \ f(\cdot,0) \in E \Big\}$$

We need

$$X(E)^* \equiv E^* \oplus_1 L_1(\mu) \quad \& \quad n\big(X(E)\big) = 1$$

Proving that $X(E)^* \equiv E^* \oplus_1 L_1(\mu)$

- Y is an M-ideal of $C([0,1]\times[0,1])$, so Y is an
- This means that $X(E)^* \equiv Y^\perp \oplus_1 Y^*$.
- $Y^* \equiv L_1(\mu)$ for some measure μ; $Y^\perp \equiv (X(E)/Y$
- Define $\Phi : X(E) \longrightarrow E$ by $\Phi(f) = f(\cdot,0)$.
  - $\|\Phi\| \leqslant 1$ and $\ker \Phi = Y$.
  - $\widetilde{\Phi} : X(E)/Y \longrightarrow E$ is a surjective isometry since
  - $\{g \in E \ : \ \|g\| < 1\} \subseteq \Phi\big(\{f \in X(E) \ : \ \|f\| < 1\}\big)$.
- Therefore, $Y^\perp \equiv (X(E)/Y)^* \equiv E^*$.



Sketch of the construction II

Define (viewing $E \hookrightarrow C[0,1]$)

$$Y = \left\{ f \in C([0,1]\times[0,1]) \ : \ f(\cdot,0)=0 \right\}$$
$$X(E) = \left\{ f \in C([0,1]\times[0,1]) \ : \ f(\cdot,0)\in E \right\}$$

We need

$$X(E)^* \equiv E^* \oplus_1 L_1(\mu) \quad \& \quad n\big(X(E)\big)=1$$

Proving that $n\big(X(E)\big)=1$

- Fix $T \in L\big(X(E)\big)$. Find $f_0 \in X(E)$ and $\xi_0 \in]0,1]\times[0,1]$ such that $\big|[Tf_0](\xi_0)\big| \sim \|T\|$.
- Consider the non-empty open set

  If $f_0(\xi_0) \sim 1$, then we were done. This our goal.

  $$V = \big\{\xi \in]0,1]\times[0,1] \ : \ f_0(\xi) \sim f_0(\xi_0)\big\}$$

  and find $\varphi : [0,1]\times[0,1] \longrightarrow [0,1]$ continuous with $\operatorname{supp}(\varphi)\subset V$ and $\varphi(\xi_0)=1$.
- Write $f_0(\xi_0) = \lambda\omega_1 + (1-\lambda)\omega_2$ with $|\omega_i|=1$, and consider the functions
  $$f_i = (1-\varphi)f_0 + \varphi\omega_i \text{ for } i=1,2.$$
- Then, $f_i \in Y \subset X(E)$, $\|f_i\| \leqslant 1$, and
  $$\Big\|f_0 - \big(\lambda f_1 + (1-\lambda)f_2\big)\Big\| = \|\varphi f_0 - \varphi f_0(\xi_0)\| \sim 0.$$
- Therefore, there is $i\in\{1,2\}$ such that $\big|[T(f_i)](\xi_0)\big| \sim \|T\|$, but now $|f_i(\xi_0)|=1$.
- Equivalently,
  $$\Big|\delta_{\xi_0}\big(T(f_i)\big)\Big| \sim \|T\| \qquad \text{and} \qquad |\delta_{\xi_0}(f_i)| = 1,$$
  meaning that $v(T) \sim \|T\|$.

Isometries in finite-dimensional spaces

Theorem

Let X be a finite-dimensional real space. TFAE:

- $\operatorname{Iso}(X)$ is infinite.
- $n(X) = 0$.
- There is $T \in L(X)$, $T \neq 0$, with $v(T) = 0$.

Examples of spaces of this kind

1. Hilbert spaces.
2. $X_{\mathbb{R}}$, the real space subjacent to any complex space X.
3. An absolute sum of any real space and one of the above.
4. Moreover, if $X = X_0 \oplus X_1$ where X_1 is complex and

$$\left\|x_0 + e^{i\theta} x_1\right\| = \|x_0 + x_1\| \qquad \big(x_0 \in X_0,\ x_1 \in X_1,\ \theta \in \mathbb{R}\big).$$

(Note that the other 3 cases are included here)

Question

Can every Banach space X with $n(X) = 0$ be decomposed as in 4 ?

Negative answer I

Infinite-dimensional case

There is an infinite-dimensional real Banach space X with $n(X) = 0$ but X is polyhedral. In particular, X does not contain $\mathbb{C}$ isometrically.

The example is

$$X = \left[\bigoplus_{n \geqslant 2} X_n \right]_{c_0}$$

X_n is the two-dimensional space whose unit ball is the regular polygon of $2n$ vertices.

Note

Such an example is not possible in the finite-dimensional case.

(Quasi affirmative) negative answer II

Finite-dimensional case

X finite-dimensional real space. TFAE:

- $n(X) = 0$.
- $X = X_0 \oplus X_1 \oplus \cdots \oplus X_n$ such that
  - X_0 is a (possible null) real space,
  - $X_1, \ldots, X_n$ are non-null complex spaces,

  there are $\rho_1, \ldots, \rho_n$ rational numbers, such that

  $$\left\| x_0 + e^{i\rho_1\theta} x_1 + \cdots + e^{i\rho_n\theta} x_n \right\| = \|x_0 + x_1 + \cdots + x_n\|$$

  for every $x_i \in X_i$ and every $\theta \in \mathbb{R}$.

Example

$X = (\mathbb{R}^4, \|\cdot\|)$, $\|(a,b,c,d)\| = \dfrac{1}{4}\displaystyle\int_0^{2\pi} \left|\mathrm{Re}\left(e^{2it}(a+ib) + e^{it}(c+id)\right)\right| \, dt.$

Then $n(X) = 0$ but the only decomposition is $X = \mathbb{C} \oplus \mathbb{C}$ with

$$\left\| e^{it} x_1 + e^{2it} x_2 \right\| = \|x_1 + x_2\|.$$

The Lie-algebra of a Banach space

Lie-algebra

X real Banach space, $\mathcal{Z}(X) = \{T \in L(X) \ : \ v(T) = 0\}$.

- When X is finite-dimensional, $\text{Iso}(X)$ is a Lie-group and $\mathcal{Z}(X)$ is the tangent space (i.e. its Lie-algebra).

Remark

If $\dim(X) = n$, then

$$0 \leqslant \dim(\mathcal{Z}(X)) \leqslant \frac{n(n-1)}{2}.$$

An open problem

Given $n \geqslant 3$, which are the possible $\dim\big(\mathcal{Z}(X)\big)$ over all n-dimensional X's?

Observation

When $\dim(X) = 3$, $\dim(\mathcal{Z}(X))$ cannot be 2.

Numerical index of Banach spaces

Numerical index (Lumer, 1968)

X real or complex Banach space,

$$n(X) = \inf\left\{v(T) \ : \ T \in L(X),\ \|T\| = 1\right\} = \max\{k \geqslant 0 \ : \ k\|T\| \leqslant v(T)\ \forall T \in L(X)\}.$$

Some examples

1. $C(K)$, $L_1(\mu)$ have numerical index 1.
2. H Hilbert space, $\dim(H) > 1$, then

$$n(H) = 0 \ \text{ real case} \qquad n(H) = \frac{1}{2} \ \text{ complex case.}$$

3. $n(L_p[0,1]) = n(\ell_p)$ but both are unknown.
4. If X_n is the two-dimensional space such that B_{X_n} is a $2n$-polygon, then

$$n(X_n) = \tan\left(\frac{\pi}{2n}\right) \ \text{ if } n \text{ is even} \qquad n(X_n) = \sin\left(\frac{\pi}{2n}\right) \ \text{ if } n \text{ is odd.}$$

5. If X is a C^*-algebra or the predual of a von Neumann algebra, then $n(X) = 1$ if the algebra is commutative and $n(X) = 1/2$ otherwise.

Numerical index and duality

Proposition

X Banach space.

- $v(T^*) = v(T)$ for every $T \in L(X)$.
- Therefore, $n(X^*) \leqslant n(X)$.

Question

Is it always $n(X) = n(X^*)$ **?**

Another example

- It is known: if X or X^* is a C^*-algebra, then $n(X) = n(X^*)$.
- Consider $Y = X\big(K(\ell_2)\big)$. Then

$$n(Y) = 1 \qquad \text{and} \qquad Y^* \equiv K(\ell_2)^* \oplus_1 L_1(\mu).$$

Then, $Y^{**} \equiv L(\ell_2) \oplus_\infty L_\infty(\mu)$ is a C^*-algebra but $n(Y^*) \leqslant n\big(K(\ell_2)\big) = 1/2$.