On a $C(K)$ space with few operators

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Notation and objective

Basic notation

X real Banach space.

- S_X unit sphere, B_X unit ball,
- X^* dual space, $L(X)$ bounded linear operators
- $T^* \in L(X^*)$ adjoint operator of $T \in L(X)$.

Main Objective

We show that there exists a real Banach space X such that

$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\| \qquad \big(\text{for every } T \in L(X)\big)$$

For topologists...

Actually, we may take as X any $C(K)$ space, K perfect compact space such that

$$T^* = g\,\mathrm{Id} + S \qquad \big(g \text{ Borel function, } S \text{ weakly compact}\big).$$

for every $T \in L(X)$

(existence of such K's proved by Koszmider in 2004).

Outline

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   - Daugavet–type inequalities
   - Norm equalities for operators
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Motivation

The Daugavet equation

What Daugavet did in 1963

The norm equality

$$\|\mathrm{Id} + T\| = 1 + \|T\|$$

holds for every compact T on $C[0,1]$.

The Daugavet equation

X Banach space, $T \in L(X)$, $\|\mathrm{Id} + T\| = 1 + \|T\|$ (DE).

Classical examples

1. **Daugavet, 1963:**
   Every compact operator on $C[0,1]$ satisfies (DE).
2. **Lozanoskii, 1966:**
   Every compact operator on $L_1[0,1]$ satisfies (DE).
3. **Abramovich, Holub, and more, 80's:**
   $X = C(K)$, K perfect compact space
   or $X = L_1(\mu)$, μ atomless measure
   $\Longrightarrow$ every weakly compact $T \in L(X)$ satisfies (DE).

The Daugavet property

The Daugavet property (Kadets–Shvidkoy–Sirotkin–Werner, 1997)

A Banach space X is said to have the Daugavet property iff every rank-one operator on X satisfies (DE).

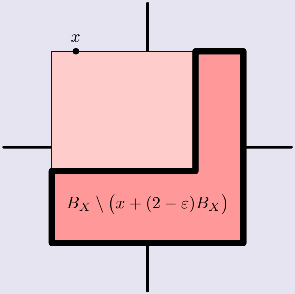
Some results

Let X be a Banach space with the Daugavet property. Then

- Every weakly compact operator on X satisfies (DE).
- X contains ℓ_1.
- X does not embed into a Banach spaces with unconditional basis.
- **Geometric characterization**: X has the Daugavet property iff for each $x \in S_X$

$$\overline{\mathrm{co}}\Big(B_X \setminus \big(x + (2-\varepsilon)B_X\big)\Big) = B_X\,.$$

(Kadets–Shvidkoy–Sirotkin–Werner, 1997 & 2000)



The Daugavet property II

For $C(K)$ spaces

K compact space, $C(K)$ has the Daugavet property if and only if K is perfect.

A related result

For every compact space K and every $T \in L\big(C(K)\big)$,

$$\|\mathrm{Id} + T\| = 1 + \|T\| \qquad \text{or} \qquad \|\mathrm{Id} - T\| = 1 + \|T\|.$$

More examples

The following spaces have the Daugavet property.

- **Wojtaszczyk, 1992:**
  The disk algebra $\mathbb{A}$ and H^∞.
- **Oikhberg, 2005:**
  Non-atomic C^*-algebras and preduals of non-atomic von Neumann algebras.
- **Ivankhno, Kadets, Werner, 2007:**
  $\mathrm{Lip}(K)$ when $K \subseteq \mathbb{R}^n$ is compact and convex.

Daugavet–type inequalities

Some examples

- **Benyamini–Lin, 1985:**
  For every $1 < p < \infty$, $p \neq 2$, there exists $\psi_p : (0, \infty) \longrightarrow (0, \infty)$ such that
  $$\|\mathrm{Id} + T\| \geqslant 1 + \psi_p(\|T\|)$$
  for every compact operator T on $L_p[0, 1]$.
- If $p = 2$, then there is a non-null compact T on $L_2[0, 1]$ such that
  $$\|\mathrm{Id} + T\| = 1.$$
- **Boyko–Kadets, 2004:**
  If ψ_p is the best possible function above, then
  $$\lim_{p \to 1^+} \psi_p(t) = t \qquad (t > 0).$$
- **Oikhberg, 2005:**
  If $K(\ell_2) \subseteq X \subseteq L(\ell_2)$, then
  $$\|\mathrm{Id} + T\| \geqslant 1 + \frac{1}{8\sqrt{2}}\|T\|$$
  for every compact T on X.

Norm equalities for operators

V. Kadets, M. Martín, J. Merí,
Norm equalities for operators.
Indiana U. Math. J. (to appear).

Motivating question

Are there other norm equalities which could define interesting properties of Banach spaces **?**

Concretely

We looked for non-trivial norm equalities of the forms

$$\|g(T)\| = f(\|T\|) \qquad \text{or} \qquad \|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$$

(g analytic, f arbitrary) in such a way that all rank-one operators on a Banach space X satisfy.

Solution

We proved that there are not to many possibilities. . .

Norm equalities for operators: Occlusive results

Theorem

X real or complex with $\dim(X) \geqslant 2$. Suppose that the norm equality

$$\|g(T)\| = f(\|T\|)$$

holds for every rank-one operator $T \in L(X)$, where

- $g : \mathbb{K} \longrightarrow \mathbb{K}$ is analytic,
- $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ is arbitrary.

Then, there are $a, b \in \mathbb{K}$ such that

$$g(\zeta) = a + b\,\zeta \qquad (\zeta \in \mathbb{K}).$$

Corollary

Only three norm equalities of the form

$$\|g(T)\| = f(\|T\|)$$

are possible:

- $b = 0$: $\|a\,\mathrm{Id}\| = |a|$,
- $a = 0$: $\|b\,T\| = |b|\,\|T\|$,

  (trivial cases)
- $a \neq 0,\ b \neq 0$:
  $\|a\,\mathrm{Id} + b\,T\| = |a| + |b|\,\|T\|$,

  (Daugavet property)

Norm equalities for operators: Occlusive results II

Theorem

X complex with $\dim(X) \geqslant 2$. Suppose that the norm equality

$$\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$$

holds for every rank-one operator $T \in L(X)$, where

- $g : \mathbb{K} \longrightarrow \mathbb{K}$ is analytic with $g(0) = 0$,
- $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ is *continuous*.

Then, X has the Daugavet property

Remarks

- We do not know if the result is true in the real case.
- It is true if g is onto.
- Even the simplest case, $g(t) = t^2$, is not known. The only known thing is that, in this case, $f(t) = 1 + t$, leading to the equation

$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$$

The question

Godefroy, private communication

Is there any real Banach space X (with $\dim(X) > 1$) such that

$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$$

for every operator $T \in L(X)$ **?**

Definition

We will call ★ the property defined above.

The examples

The examples

Weak multiplier

Let K be a compact space. $T \in L\big(C(K)\big)$ is a weak multiplier if

$$T^* = g\,\mathrm{Id} + S$$

where g is a Borel function and S is weakly compact.

Our main result

If K is perfect and all operators on $C(K)$ are weak multipliers, then $C(K)$ has ★.

Theorem (Koszmider, 2004; Plebanek, 2004)

There exist perfect compact spaces K such that all operators on $C(K)$ are weak multipliers. There are examples of two kinds:

- K connected where every operator in $L(C(K))$ is of the form $g\,\mathrm{Id} + S$ for $g \in C(K)$ and S weakly compact.
- K totally disconnected and perfect.

In particular, there are nonisomorphic $C(K)$ spaces with ★.

Proving a simple case...

Hypothesis

Every $T \in L\big(C(K)\big)$ is of the form $g\mathrm{Id} + S$, with $g \in C(K)$, S weakly compact.

We need

$\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$

- If $T = g\mathrm{Id} + S$, then $T^2 = g^2\,\mathrm{Id} + S'$ with S' weakly compact.
- We will prove that $\|\mathrm{Id} + g^2\,\mathrm{Id} + S\| = 1 + \|g^2\,\mathrm{Id} + S\|$ for $g \in C(K)$ and S weakly compact.
- **Step 1:** We assume $\|g^2\| \leqslant 1$ and $\min g^2(K) > 0$.
- **Step 2:** We can avoid the assuption that $\min g^2(K) > 0$.
- **Step 3:** Finally, for every g the above gives

$$\left\|\mathrm{Id} + \frac{1}{\|g^2\|}\big(g^2\,\mathrm{Id} + S\big)\right\| = 1 + \frac{1}{\|g^2\|}\|g^2\,\mathrm{Id} + S\|$$

which gives us the result.

Consequences

Consequences I: first results

Proposition

If X has ★, then

- X does not have the RNP.
- X does not have unconditional basis.

For $C(K)$ spaces

This said not too much about $C(K)$...

Consequences II: isometries

Theorem

If X has ★, then every surjective isometry J on X satisfies $J^2 = \mathrm{Id}$.

For $C(K)$ spaces

If all operators on a $C(K)$ space are weak multipliers, then every homeomorphism φ of K satisfies $\varphi^2 = \mathrm{id}$.

Consequences III: complex structure

Complex structure

A comp … mplex
space u …

where J …

Remark

If X has …

For $C(K)$ spaces

- There is a connected compact space K such that
  - $C(K)$ has ★,
  - the only complemented subspaces of $C(K)$ having complex structure are the finite-dimensional ones,
  - In particular, no complemented subspace of $C(K)$ is isomorphic to $Z \oplus Z$ for any Z.
- There is another perfect compact space K' such that
  - $C(K')$ has ★,
  - $C(K')$ contains a complemented copy of c_0.

Theorem

X having ★, Y finite-codimensional subspace of X. Then

- Y does not have any complex structure.
- In particular, Y is not isomorphic to $Z \oplus Z$ for any Z.

Open Problems

Question 1

Find topological characterization of the compact Hausdorff spaces K such that $C(K)$ has ★.

Question 2

Find other Banach spaces having ★.

Question 3

Characterize isometrically and/or isomorphically the property ★.