Norm equalities for operators

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Introduction

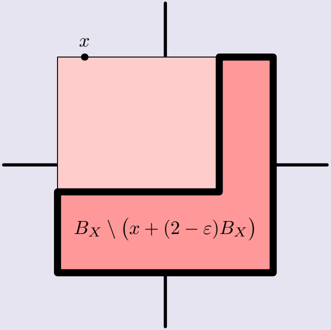
- In a Banach space X with the Radon-Nikodým property the unit ball has many denting points.
- $x \in S_X$ is a denting point of B_X if for every $\varepsilon > 0$ one has

$$x \notin \overline{\text{co}}\big(B_X \setminus (x + \varepsilon B_X)\big).$$

- $C[0,1]$ and $L_1[0,1]$ have an extremely opposite property: for every $x \in S_X$ and every $\varepsilon > 0$

$$\overline{\text{co}}\ \Big(B_X \setminus \big(x + (2-\varepsilon)B_X\big)\Big) = B_X\,.$$

- This geometric property is equivalent to a property of operators on the space.



The Daugavet property

The Daugavet equation

X Banach space, $T \in L(X)$

$$\|\mathrm{Id} + T\| = 1 + \|T\| \qquad \text{(DE)}$$

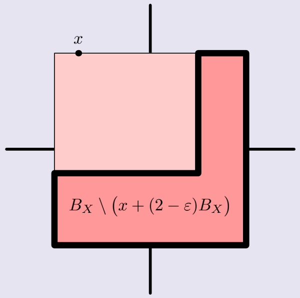
The Daugavet property

A Banach space X is said to have the Daugavet property iff every rank-one operator on X satisfies (DE).

- Then, every weakly compact operator on X satisfies (DE).
- **Geometric characterization**: X has the Daugavet property iff for each $x \in S_X$

$$\overline{\mathrm{co}}\left(B_X \setminus \big(x + (2-\varepsilon)B_X\big)\right) = B_X\,.$$

(Kadets–Shvidkoy–Sirotkin–Werner, 1997 & 2000)



The Daugavet property

Some propaganda

Suppose X has the Daugavet property. Then:

- X does not have the Radon-Nikodým property.

  (Wojtaszczyk, 1992)

- Every weakly-open subset of B_X has diameter 2.

  (Shvidkoy, 2000)

- X contains a copy of ℓ_1. X^* contains a copy of $L_1[0,1]$.

  (Kadets–Shvidkoy–Sirotkin–Werner, 2000)

- X does not have unconditional basis.

  (Kadets, 1996)

- X does not embed into a unconditional sum of Banach spaces without a copy of ℓ_1.

  (Shvidkoy, 2000)

Daugavet type inequalities

Commutative L_p spaces

- **Benyamini–Lin, 1985:**
  For every $1 < p < \infty$, $p \neq 2$, there exists $\psi_p : (0,\infty) \longrightarrow (0,\infty)$ such that
  $$\|\mathrm{Id} + T\| \geqslant \psi_p(\|T\|)$$
  for every compact operator T on $L_p[0,1]$.
- If $p = 2$, then there is a non-null compact T on $L_2[0,1]$ such that
  $$\|\mathrm{Id} + T\| = 1.$$
- **Boyko–Kadets, 2004:**
  If ψ_p is the best possible function above, then
  $$\lim_{p\to 1^+} \psi_p(t) = t \qquad (t > 0).$$

Daugavet type inequalities

Non-commutative $\mathcal{L}_p$ spaces

- **Oikhberg, 2002:**
  For every $1 < p < \infty$, $p \neq 2$, there exists $k_p > 0$ such that
  $$\|\mathrm{Id} + T\| \geqslant 1 + k_p \min\{\|T\|, \|T\|^2\}$$
  for every compact T on $\mathcal{L}_p(\tau)$.

Spaces of operators

- **Oikhberg, 2005:**
  If $K(\ell_2) \subseteq X \subseteq L(\ell_2)$, then
  $$\|\mathrm{Id} + T\| \geqslant 1 + \frac{1}{8\sqrt{2}}\|T\|$$
  for every compact T on X.

The questions

Is any of the previous inequalities an equality **?**

Even more, is there **any** norm equality valid for all compact operators on any of the above spaces **?**

Main question

Study the possibility of finding norm equalities for operators in the spirit of Daugavet equation, valid for all rank-one operators on a Banach space.

We will study three cases:

1. $\|\mathrm{Id} + T\| = f(\|T\|)$ for arbitrary f.
2. $\|g(T)\| = f(\|T\|)$ for analytic g and arbitrary f.
3. $\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$ for analytic g and *continuous* f.

Equalities of the form $\|\mathrm{Id} + T\| = f(\|T\|)$

Proposition

X real or complex, $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ arbitrary, $a, b \in \mathbb{K}$. If the norm equality

$$\|a\,\mathrm{Id} + b\,T\| = f(\|T\|)$$

holds for every rank-one operator $T \in L(X)$, then

$$f(t) = |a| + |b|\,t \qquad \big(t \in \mathbb{R}_0^+\big).$$

If $a \neq 0$, $b \neq 0$, then X has the Daugavet property.

Then, we have to look for Daugavet-type equalities in which $\mathrm{Id} + T$ is replaced by something different.

Equalities of the form $\|g(T)\| = f(\|T\|)$

Theorem

X real or complex with $\dim(X) \geqslant 2$.
Suppose that the norm equality

$$\|g(T)\| = f(\|T\|)$$

holds for every rank-one operator $T \in L(X)$, where

- $g : \mathbb{K} \longrightarrow \mathbb{K}$ is analytic,
- $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ is arbitrary.

Then, there are $a, b \in \mathbb{K}$ such that

$$g(\zeta) = a + b\,\zeta \qquad (\zeta \in \mathbb{K}).$$

Corollary

Only three norm equalities of the form

$$\|g(T)\| = f(\|T\|)$$

are possible:

- $b = 0$: $\|a\,\mathrm{Id}\| = |a|$,
- $a = 0$: $\|b\,T\| = |b|\,\|T\|$,

  (trivial cases)
- $a \neq 0$, $b \neq 0$:
  $\|a\,\mathrm{Id} + b\,T\| = |a| + |b|\,\|T\|$,

  (Daugavet property)

Equalities of the form $\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$

Remark

If X has the Daugavet property and g is analytic, then

$$\|\mathrm{Id} + g(T)\| = |1 + g(0)| - |g(0)| + \|g(T)\|$$

for every rank-one $T \in L(X)$.

- Our aim here is not to show that g has a suitable form,
- but it is to see that for every g another simpler equation can be found.
- From now on, we have to separate the complex and the real case.

Equalities of the form $\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$

- Complex case:

Proposition

X complex, $\dim(X) \geqslant 2$. Suppose that

$$\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$$

for every rank-one $T \in L(X)$, where

- $g : \mathbb{C} \longrightarrow \mathbb{C}$ analytic non-constant,
- $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ *continuous.*

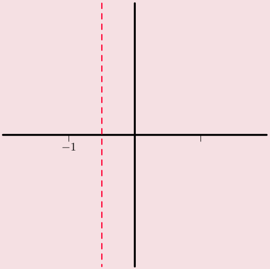
Then

$$\big\|(1 + g(0))\mathrm{Id} + T\big\| = |1 + g(0)| - |g(0)| + \big\|g(0)\mathrm{Id} + T\big\|$$

for every rank-one $T \in L(X)$.

We obtain two different cases:

- $|1 + g(0)| - |g(0)| \neq 0$ or
- $|1 + g(0)| - |g(0)| = 0$.



Equalities of the form $\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$. Complex case

Theorem

If $\mathrm{Re}\, g(0) \neq -1/2$ and

$$\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$$

for every rank-one T, then X has the Daugavet property.

Theorem

If $\mathrm{Re}\, g(0) = -1/2$ and

$$\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$$

for every rank-one T, then exists $\theta_0 \in \mathbb{R}$ s.t.

$$\left\|\mathrm{Id} + \mathrm{e}^{i\,\theta_0}\, T\right\| = \|\mathrm{Id} + T\|$$

for every rank-one $T \in L(X)$.

Example

If $X = C[0,1] \oplus_2 C[0,1]$, then

- $\left\|\mathrm{Id} + \mathrm{e}^{i\,\theta}\, T\right\| = \|\mathrm{Id} + T\|$
  for every $\theta \in \mathbb{R}$, rank-one $T \in L(X)$.
- X does not have the Daugavet property.

Equalities of the form $\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$. Real case

- Real case:

Remarks

- The proofs are not valid (we use Picard's Theorem).
- They work when g is onto.
- But we do not know what is the situation when g is not onto, even in the easiest examples:
  - $\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$,
  - $\|\mathrm{Id} - T^2\| = 1 + \|T^2\|$.

$g(0) = -1/2$:

Example

If $X = C[0,1] \oplus_2 C[0,1]$, then

- $\|\mathrm{Id} - T\| = \|\mathrm{Id} + T\|$ for every rank-one $T \in L(X)$.
- X does not have the Daugavet property.

Some questions

1. Study the real or complex spaces for which the equality

$$\|\mathrm{Id} + T\| = \|\mathrm{Id} - T\|$$

holds for every rank-one operator.

2. Study the real spaces X for which the equality

$$\left\|\mathrm{Id} + T^2\right\| = 1 + \|T^2\|$$

holds for every rank-one operator T on X.

3. Is there any real space X with $\dim(X) > 1$ such that

$$\left\|\mathrm{Id} + T^2\right\| = 1 + \|T^2\|$$

for every operator **?**