The Daugavet property

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The talk is based on these papers

J. Becerra Guerrero and M. Martín,
The Daugavet Property of C^*-algebras, JB^*-triples, and of their isometric preduals.
Journal of Functional Analysis (2005)

M. Martín,
The alternative Daugavet property of C^*-algebras and JB^*-triples.
Mathematische Nachrichten (to appear)

M. Martín and T. Oikhberg,
An alternative Daugavet property.
Journal of Mathematical Analysis and Applications (2004)

Outline

Introduction

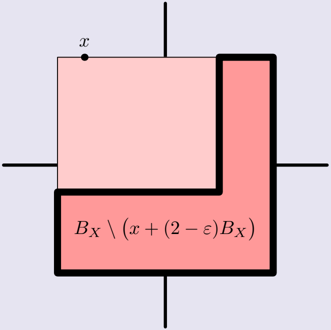
- In a Banach space X with the Radon-Nikodým property the unit ball has many denting points.
- $x \in S_X$ is a denting point of B_X if for every $\varepsilon > 0$ one has

$$x \notin \overline{\mathrm{co}}\Big(B_X \setminus (x + \varepsilon B_X)\Big).$$

- $C[0,1]$ and $L_1[0,1]$ have an extremely opposite property: for every $x \in S_X$ and every $\varepsilon > 0$

$$\overline{\mathrm{co}}\,\Big(B_X \setminus \Big(x + (2-\varepsilon)B_X\Big)\Big) = B_X\,.$$

- This geometric property is equivalent to a property of operators on the space.



The Daugavet equation

X Banach space, $T \in L(X)$

$$\|Id + T\| = 1 + \|T\| \qquad \text{(DE)}$$

Classical examples

1. **Daugavet, 1963:**
   Every compact operator on $C[0,1]$ satisfies (DE).
2. **Lozanoskii, 1966:**
   Every compact operator on $L_1[0,1]$ satisfies (DE).
3. **Abramovich, Holub, and more, 80's:**
   $X = C(K)$, K perfect compact space
   or $X = L_1(\mu)$, μ atomless measure
   $\Longrightarrow$ every weakly compact $T \in L(X)$ satisfies (DE).

The Daugavet property

- A Banach space X is said to have the Daugavet property if every rank-one operator on X satisfies (DE).
- If X^* has the Daugavet property, so does X. The converse is not true:

$$C[0,1] \text{ has it but } C[0,1]^* \text{ not.}$$

(Kadets–Shvidkoy–Sirotkin–Werner, 1997 & 2000)

Prior versions of: *Chauveheid, 1982; Abramovich–Aliprantis–Burkinshaw, 1991*

Some examples...

1. K perfect, μ atomeless, E arbitrary Banach space
$\Longrightarrow C(K,E)$, $L_1(\mu,E)$, and $L_\infty(\mu,E)$ have the Daugavet property.
(Kadets, 1996; Nazarenko, –; Shvidkoy, 2001)
2. $A(\mathbb{D})$ and H^∞ have the Daugavet property.
(Wojtaszczyk, 1992)

More examples...

3. A function algebra whose Choquet boundary is perfect has the Daugavet property.

   (Werner, 1997)

4. "Large" subspaces of $C[0,1]$ and $L_1[0,1]$ have the Daugavet property (in particular, this happends for finite-codimensional subspaces).

   (Kadets–Popov, 1997)

5. A C^*-algebra has the Daugavet property if and only if it is non-atomic.
6. The predual of a von Neumann algebra has the Daugavet property if and only if the algebra is non-atomic.

   (Oikhberg, 2002)

Some *propaganda*...

Let X be a Banach space with the Daugavet property. Then

- X does not have the Radon-Nikodým property.

  (Wojtaszczyk, 1992)
- Every slice of B_X and every w^*-slice of B_{X^*} have diameter 2.

  (Kadets–Shvidkoy–Sirotkin–Werner, 2000)
- Actually, every weakly-open subset of B_X has diameter 2.

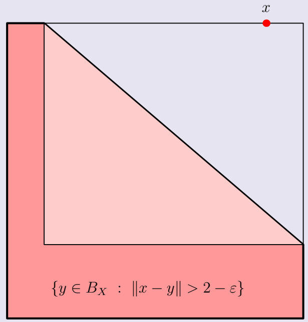
  (Shvidkoy, 2000)
- X contains a copy of ℓ_1. X^* contains a copy or $L_1[0,1]$.

  (Kadets–Shvidkoy–Sirotkin–Werner, 2000)

Geometric characterizations

Theorem [KSSW]

- X has the Daugavet property.
- For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y \in S_X$ such that
$$\operatorname{Re} x^*(y) > 1 - \varepsilon \quad \text{and} \quad \|x - y\| \geqslant 2 - \varepsilon.$$
- For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y^* \in S_{X^*}$ such that
$$\operatorname{Re} y^*(x) > 1 - \varepsilon \quad \text{and} \quad \|x^* - y^*\| \geqslant 2 - \varepsilon.$$
- For every $x \in S_X$ and every $\varepsilon > 0$, we have
$$B_X = \overline{\text{co}}\Big(\{y \in B_X \ : \ \|x - y\| \geqslant 2 - \varepsilon\}\Big).$$



Theorem

Let X be a Banach space with the Daugavet property.

- Every weakly compact operator on X satisfies (DE).

  (Kadets–Shvidkoy–Sirotkin–Werner, 2000)

- Actually, every operator on X which does not fix a copy of ℓ_1 satisfies (DE).

  (Sirotkin, 2000)

Consequences

1. X does not have unconditional basis.

   (Kadets, 1996)

2. Moreover, X does not embed into any space with unconditional basis.

   (Kadets–Shvidkoy–Sirotkin–Werner, 2000)

3. Actually, X does not embed into an unconditional sum of Banach spaces without a copy of ℓ_1.

   (Shvidkoy, 2000)

A new sufficient condition

A new sufficient condition

Theorem

Let X be a Banach space such that

$$X^* = Y \oplus_1 Z$$

with Y and Z norming subspaces. Then, X has the Daugavet property.

A closed subspace $W \subseteq X^*$ is norming if

$$\|x\| = \sup\{|w^*(x)| \ : \ w^* \in W, \ \|w^*\| = 1\}$$

or, equivalently, if B_W is w^*-dense in B_{X^*}.

Proof of the theorem

We have...

$X^* = Y \oplus_1 Z$,
B_Y, B_Z w^*-dense in B_{X^*}.

$$\overset{?}{\Longrightarrow}$$

We need...

fixed $x_0 \in S_X$, $x_0^* \in S_{X^*}$, $\varepsilon > 0$, find $y^* \in S_{X^*}$ such that

$$\|x_0^* + y^*\| > 2 - \varepsilon \quad \text{and} \quad \operatorname{Re} y^*(x_0) > 1 - \varepsilon.$$

- Write $x_0^* = y_0^* + z_0^*$ with $y_0^* \in Y$, $z_0^* \in Z$, $\|x_0^*\| = \|y_0^*\| + \|z_0^*\|$, and write
$$U = \{x^* \in B_{X^*} \ : \ \operatorname{Re} x^*(x_0) > 1 - \varepsilon/2\}.$$
- Take $z^* \in B_Z \cap U$ and a net (y_λ^*) in $B_Y \cap U$, such that $(y_\lambda^*) \xrightarrow{w^*} z^*$.
- $(y_\lambda^* + y_0^*) \longrightarrow z^* + y_0^*$ and the norm is w^*-lower semi-continuous, therefore
$$\liminf \|y_\lambda^* + y_0^*\| \geqslant \|z^* + y_0^*\| = \|z^*\| + \|y_0^*\| > 1 + \|y_0^*\| - \varepsilon/2.$$
- Then, we may find μ such that $\|y_\mu^* + y_0^*\| \geqslant 1 + \|y_0^*\| - \varepsilon/2$.
- Finally, observe that
$$\begin{aligned}\|x_0^* + y_\mu^*\| &= \|(y_0^* + y_\mu^*) + z_0^*\| = \\ &= \|y_0^* + y_\mu^*\| + \|z_0^*\| > 1 + \|y_0^*\| - \varepsilon + \|z_0^*\| = 2 - \varepsilon,\end{aligned}$$

and that $\operatorname{Re} y_\mu^*(x_0) > 1 - \varepsilon$ (since $y_\mu^* \in U$).

Some immediate consequences

Corollary

Let X be an L-embedded space with $\operatorname{ext}(B_X) = \emptyset$. Then, X^* (and hence X) has the Daugavet property.

Corollary

If Y is an L-embedded space which is a subspace of $L_1 \equiv L_1[0,1]$, then $(L_1/Y)^*$ has the Daugavet property.

It was already known that...

- If $Y \subset L_1$ is reflexive, then L_1/Y has the Daugavet property.

  (Kadets–Shvidkoy–Sirotkin–Werner, 2000)
- If $Y \subset L_1$ is L-embedded, then L_1/Y does not have the RNP.

  (Harmand–Werner–Werner, 1993)

Application:

The Daugavet property of

C^*-algebras and von Neumann preduals

von Neumann preduals

von Neumann preduals

- A C^*-algebra X is a von Neumann algebra if it is a dual space.
- In such a case, X has a unique predual X_*.
- X_* is always L-embedded.
- Therefore, if $\mathrm{ext}\,(B_{X_*})$ is empty, then X and X_* have the Daugavet property. Example: $L_\infty[0,1]$ and $L_1[0,1]$.

Actually, much more can be proved:

Theorem

Let X_* be the predual of the von Neumann algebra X. Then, TFAE:

- X has the Daugavet property.
- X_* has the Daugavet property.
- Every weakly open subset of B_{X_*} has diameter 2.
- B_{X_*} has no strongly exposed points.
- B_{X_*} has no extreme points.
- X is non-atomic (i.e. it has no atomic projections).

An atomic projection is an element $p \in X$ such that

$$p^2 = p^* = p \qquad \text{and} \qquad p\,X\,p = \mathbb{C}p.$$

C^*-algebras

Let X be a C^*-algebra. Then, X^{**} is a von Neumann algebra.
Write $X^* = (X^{**})_* = A \oplus_1 N$, where

- A is the atomic part,
- N is the non-atomic part.

- Every extreme point of B_{X^*} is in B_A.
- Therefore, A is norming.
- What's about N **?**

Theorem

If X is non-atomic, then N is norming. Therefore, X has the Daugavet property.
Example: $C[0,1]$

sketch of the proof of the theorem

We have...

X non-atomic C^*-algebra,
$X^* = A \oplus_1 N$.

$$\overset{?}{\Longrightarrow}$$

We need...

N to be norming for X, i.e.
$\|x\| = \sup\{|f(x)| \ : \ f \in B_N\} \quad (x \in X)$.

- Write $X^{**} = \mathcal{A} \oplus_\infty \mathcal{N}$ and $Y = \mathcal{A} \cap X$.
- Y is an ideal of X, so Y has no atomic projections.
- Therefore, the norm of Y has no point of Fréchet-smoothness.
- But Y is an Asplund space, so $Y = 0$.
- Now, the mapping
$$X \hookrightarrow X^{**} = \mathcal{A} \oplus_\infty \mathcal{N} \twoheadrightarrow \mathcal{N}$$
in injective. Since it is an homomorphism, it is an isometry.
- But $N^* \equiv \mathcal{N}$, so N is norming for $\mathcal{N}$ and now, also for X.

Theorem

Let X be a C^*-algebra. Then, TFAE:

- X has the Daugavet property.
- The norm of X is extremely rough, i.e.

$$\limsup_{\|h\|\to 0} \frac{\|x+h\|+\|x-h\|-2}{\|h\|} = 2$$

  for every $x \in S_X$ (equivalently, every w^*-slice of B_{X^*} has diameter 2).
- The norm of X is not Fréchet-smooth at any point.
- X is non-atomic.

The alternative Daugavet equation

The alternative Daugavet equation

The alternative Daugavet equation

X Banach space, $T \in L(X)$

$$\max_{|\omega|=1} \|Id + \omega\, T\| = 1 + \|T\| \qquad \text{(aDE)}$$

(Duncan–McGregor–Pryce–White, 1970; Holub, Abramovich. . . , 80's)

Two equivalent formulations

- There exists $\omega \in \mathbb{T}$ such that $\omega\, T$ satisfies (DE).
- The numerical radius of T, $v(T)$, coincides with $\|T\|$, where

$$v(T) := \sup\{|x^*(Tx)| \ : \ x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1\}.$$

Two possible properties

Let X be a Banach space.

- X is said to have the alternative Daugavet property (ADP) iff every rank-one operator on X satisfies (aDE).
  - Then, every weakly compact operator also satisfies (aDE).
  - If X^* has the ADP, so does X. The converse is not true: $C([0,1],\ell_2)$.

  (M.–Oikhberg, 2004; briefly appearance: Abramovich, 1991)

- X is said to have numerical index 1 iff $v(T) = \|T\|$ for every operator on X. Equivalently, if EVERY operator on X satisfies (aDE).

  (Lumer, 1968; Duncan–McGregor–Pryce–White, 1970)

Observation

No analogous property is possible for the Daugavet equation:

$$\|Id + (-Id)\| = 0 \neq 1 + \|-Id\|.$$

Numerical index 1

- $C(K)$ and $L_1(\mu)$ have numerical index 1.

  (Duncan–McGregor–Pryce–White, 1970)
- All function algebras have numerical index 1.

  (Werner, 1997)
- A C^*-algebra has numerical index 1 iff it is commutative.

  (Huruya, 1977; Kaidi–Morales–Rodríguez-Palacios, 2000)
- In case $\dim(X) < \infty$, X has numerical index 1 iff

  $$|x^*(x)| = 1 \qquad x^* \in \operatorname{ext}\left(B_{X^*}\right), \ x \in \operatorname{ext}\left(B_X\right).$$

  (McGregor, 1971)
- In case $\dim(X) = \infty$, if X has numerical index 1 and the RNP, then $X \supseteq \ell_1$.

  (López–M.–Payá, 1999)

The alternative Daugavet property

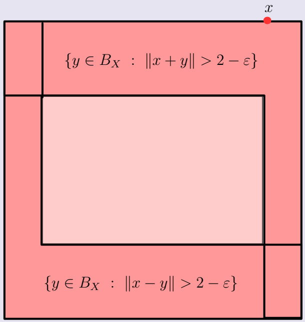
$$\begin{tikzcd}
\boxed{\text{Daugavet property}} \arrow[rr, shift left, Rightarrow, "/" description] \arrow[dr, Rightarrow] & & \boxed{\text{Numerical index 1}} \arrow[ll, shift left, Rightarrow, "/" description] \arrow[dl, Rightarrow] \\
& \boxed{\text{ADP}} &
\end{tikzcd}$$

- $c_0 \oplus_\infty C([0,1], \ell_2)$ has the ADP, but neither the Daugavet property, nor numerical index 1.
- For RNP or Asplund spaces, the ADP implies numerical index 1.
- Every Banach space with the ADP can be renormed still having the ADP but failing the Daugavet property.

Geometric characterizations

Theorem

- X has the ADP.
- For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y \in S_X$ such that
$$|x^*(y)| > 1 - \varepsilon \quad \text{and} \quad \|x - y\| \geqslant 2 - \varepsilon.$$
- For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y^* \in S_{X^*}$ such that
$$|y^*(x)| > 1 - \varepsilon \quad \text{and} \quad \|x^* - y^*\| \geqslant 2 - \varepsilon.$$
- For every $x \in S_X$ and every $\varepsilon > 0$, we have
$$B_X = \overline{\mathrm{co}}\left(\mathbb{T}\,\{y \in B_X \ :\ \|x - y\| \geqslant 2 - \varepsilon\}\right).$$



Let V_* be the predual of the von Neumann algebra V.

The Daugavet property of V_* is equivalent to:

- V has no atomic projections, or
- the unit ball of V_* has no extreme points.

V_* has numerical index 1 iff:

- V is commutative, or
- $|v^*(v)| = 1$ for $v \in \operatorname{ext}(B_V)$ and $v^* \in \operatorname{ext}(B_{V^*})$.

The alternative Daugavet property of V_* is equivalent to:

- the atomic projections of V are central, or
- $|v(v_*)| = 1$ for $v \in \operatorname{ext}(B_V)$ and $v_* \in \operatorname{ext}(B_{V_*})$, or
- $V = C \oplus_\infty N$, where C is commutative and N has no atomic projections.

Let X be a C^*-algebra.

The Daugavet property of X is equivalent to:

- X does not have any atomic projection, or
- the unit ball of X^* does not have any w^*-strongly exposed point.

X has numerical index 1 iff:

- X is commutative, or
- $|x^{**}(x^*)| = 1$ for $x^{**} \in \operatorname{ext}(B_{X^{**}})$ and $x^* \in \operatorname{ext}(B_{X^*})$.

The alternative Daugavet property of X is equivalent to:

- the atomic projections of X are central, or
- $|x^{**}(x^*)| = 1$, for $x^{**} \in \operatorname{ext}(B_{X^{**}})$, and $x^* \in B_{X^*}$ w^*-strongly exposed, or
- $\exists$ a commutative ideal Y such that X/Y has the Daugavet property.

Recommended readings...

Y. Abramovich, and C. Aliprantis,
An invitation to operator theory.
Graduate Studies in Math. **50**, AMS, 2002.

Y. Abramovich, and C. Aliprantis,
Problems in operator theory.
Graduate Studies in Math. **51**, AMS, 2002.

V. Kadets, N. Kalton, and D. Werner,
Remarks on rich subspaces of Banach spaces.
Studia Math. (2003).

V. Kadets, M. Martín, and R. Payá,
Recent progress and open questions on the numerical index of Banach spaces.
RACSAM (2006).

D. Werner,
Recent progress on the Daugavet property.
Irish Math. Soc. Bulletin (2001).