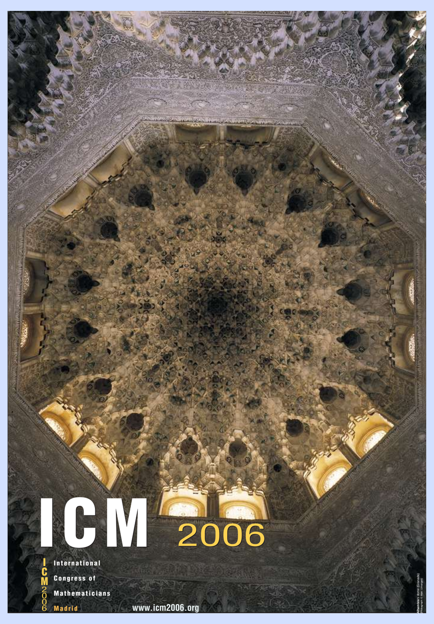

Norm equalities for operators

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Introduction

Our aim is to study equalities involving the norm of operators on Banach spaces, and to discuss the possibility of defining isometric properties for Banach spaces by requiring that all operators of a suitable class satisfy such a norm equality. The interest in this topic goes back to 1963, when I. Daugavet [1] showed that each compact operator T on $C[0,1]$ satisfies the norm equality

$$\text{(DE)} \qquad \|\mathrm{Id} + T\| = 1 + \|T\|.$$

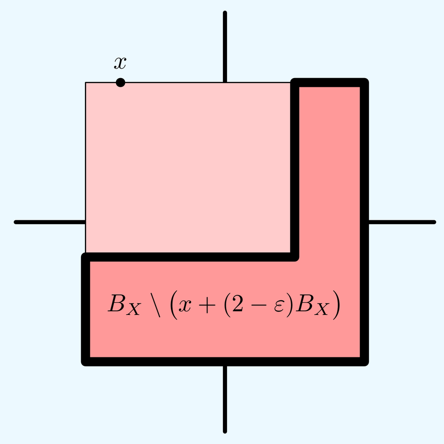
The above equation is nowadays referred to as Daugavet equation.

A Banach space X is said to have the Daugavet property if (DE) holds for every rank-one operator $T \in L(X)$. In such a case, every weakly compact operator on X also satisfies (DE) [2]. Examples of spaces having this property are $C(K)$ and $L_1(\mu)$, provided that K is perfect and μ does not have any atoms (see [3] for an elementary approach), and certain function algebras such as the disk algebra $A(\mathbb{D})$ or the algebra of bounded analytic functions H^∞ [4, 6]. The state-of-the-art on the subject can be found in [5].

- Geometric characterization [2]: X has the Daugavet property if and only if for every $x \in S_X$ and every $\varepsilon > 0$ one has

$$\overline{\mathrm{co}}\left(B_X \setminus \big(x + (2-\varepsilon)B_X\big)\right) = B_X$$

(see figure on the right).



THE PROBLEM: Study the possibility of finding norm equalities for operators in the spirit of Daugavet equation, valid for all rank-one operators on a Banach space. We will study two cases:

- $\|g(T)\| = f(\|T\|)$ for entire g and arbitrary f (when X is a real space g is an entire function which carries the real line into itself).
- $\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$ for entire g and continuous f.

I. Equalities of the form $\|g(T)\| = f(\|T\|)$

REMARK: Let X be Banach space over $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$), $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ an arbitrary function, and $a, b \in \mathbb{K}$. If the norm equality

$$\|a\,\mathrm{Id} + b\,T\| = f(\|T\|)$$

holds for every rank-one operator T on X, then

$$f(t) = |a| + |b|\,t \qquad (t \in \mathbb{R}_0^+).$$

If $a \neq 0$, $b \neq 0$, then X has the Daugavet property. Therefore, in order to find new properties, we have to replace $\mathrm{Id} + T$ by other functions of T.

THEOREM: Let $g : \mathbb{K} \longrightarrow \mathbb{K}$ be an entire function and $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ an arbitrary function. Suppose that there is a Banach space X over $\mathbb{K}$ with $\dim(X) \geqslant 2$ such that the norm equality

$$\|g(T)\| = f(\|T\|)$$

holds for every rank-one operator T on X. Then, only three possibilities may happen:

- g is a constant function (trivial case).
- There is $b \in \mathbb{K} \setminus \{0\}$ so that $g(\zeta) = b\,\zeta$ for every $\zeta \in \mathbb{K}$ (trivial case).
- There are $a, b \in \mathbb{K} \setminus \{0\}$ so that $g(\zeta) = a + b\,\zeta$ for $\zeta \in \mathbb{K}$, and X has the Daugavet property.

Proof (complex case):

Let $g(\zeta) = \sum_{k=0}^{\infty} a_k \zeta^k$ be the power series expansion of g and let $\tilde{g} = g - a_0$. We take $x_0^*, x_1^* \in S_{X^*}$ and $x_0, x_1 \in S_X$ such that $x_0^*(x_0) = 0$ and $x_1^*(x_1) = 1$. We write $T_0 = x_0^* \otimes x_0$ and $T_1 = x_1^* \otimes x_1$, which satisfy $\|T_0\| = \|T_1\| = 1$. For each $\lambda \in \mathbb{C}$ it is straightforward to check that

$$g(\lambda T_1) = a_0 \mathrm{Id} + \tilde{g}(\lambda)\,T_1 \qquad \text{and} \qquad g(\lambda T_0) = a_0 \mathrm{Id} + a_1 \lambda\,T_0\,.$$

Now, fixed $\lambda \in \mathbb{C}$, we have

$$\|a_0 \mathrm{Id} + \tilde{g}(\lambda) T_1\| = \|g(\lambda T_1)\| = f(|\lambda|) = \|g(\lambda T_0)\| = \|a_0 \mathrm{Id} + a_1 \lambda T_0\|.$$

Therefore, using the triangle inequality we obtain

$$|\tilde{g}(\lambda)| \leqslant 2|a_0| + |a_1|\,|\lambda| \qquad (\lambda \in \mathbb{C}).$$

From this, it follows by just using Cauchy's estimates, that $\tilde{g}$ is a polynomial of degree less or equal than one, and the result follows from the above remark. □

II. Equalities of the form $\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$

• Complex case

LEMMA: Let $g : \mathbb{C} \longrightarrow \mathbb{C}$ be a non-constant entire function, let $f : [|g(0)|, +\infty[\longrightarrow \mathbb{R}$ be a continuous function and let X be a Banach space with $\dim(X) \geqslant 2$. Suppose that the norm equality

$$\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$$

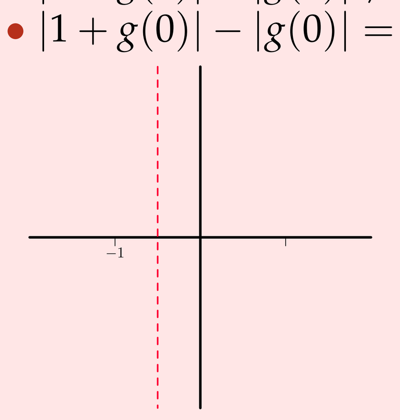
holds for every rank-one operator $T \in L(X)$. Then,

$$\|(1 + g(0))\,\mathrm{Id} + T\| = |1 + g(0)| - |g(0)| + \|g(0)\,\mathrm{Id} + T\|$$

for every rank-one operator $T \in L(X)$.

Two different cases appear:

- $|1 + g(0)| - |g(0)| \neq 0$
- $|1 + g(0)| - |g(0)| = 0.$



If $\mathrm{Re}\, g(0) \neq -1/2$

THEOREM: Assume that we are under the hypothesis of the above lemma and suppose that $\mathrm{Re}\, g(0) \neq -1/2$. Then, X has the Daugavet property.

If $\mathrm{Re}\, g(0) = -1/2$

THEOREM: Assume that we are under the hypothesis of the above lemma and suppose that $\mathrm{Re}\, g(0) = -1/2$. Then, there is $\omega \in \mathbb{T}$ such that $\|\mathrm{Id} + \omega T\| = \|\mathrm{Id} + T\|$ for every rank-one operator $T \in L(X)$.

EXAMPLE: If $X = C[0,1] \oplus_2 C[0,1]$, then

- $\|\mathrm{Id} + e^{i\theta} T\| = \|\mathrm{Id} + T\|$ for every $\theta \in \mathbb{R}$, rank-one T.
- X does not have the Daugavet property.

• Real case

If $g(0) \neq -1/2$

REMARKS:

- The proofs given in complex case are not valid (we use Picard's Theorem).
- They work when g is onto.
- We do not know what is the situation when g is not onto, even in two easy examples:
  - $\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$,
  - $\|\mathrm{Id} - T^2\| = 1 + \|T^2\|$.

If $g(0) = -1/2$

EXAMPLE: If $X = C[0,1] \oplus_2 C[0,1]$, then:

- For every rank-one T we have

$$\|\mathrm{Id} - T\| = \|\mathrm{Id} + T\|.$$

- X does not have the Daugavet property.

Some open questions

- Study the real or complex spaces for which the equality $\|\mathrm{Id} + T\| = \|\mathrm{Id} - T\|$ holds for every rank-one operator.
- Study the real spaces X for which the equality $\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$ holds for every rank-one operator T on X.
- Is there any real space X with $\dim(X) > 1$ such that $\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$ for every operator $T \in L(X)$ **?**

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