The Daugavet equation for polynomials

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The Daugavet equation for operators

X Banach space, $T \in L(X)$

$$\|\mathrm{Id} + T\| = 1 + \|T\| \qquad \text{(DE)}$$

Classical examples

1. **Daugavet, 1963:**
   Every compact operator on $C[0,1]$ satisfies (DE).
2. **Lozanoskii, 1966:**
   Every compact operator on $L_1[0,1]$ satisfies (DE).
3. **Abramovich, Holub, and more, 80's:**
   $X = C(K)$, K perfect compact space
   or $X = L_1(\mu)$, μ atomless measure
   $\Longrightarrow$ every weakly compact $T \in L(X)$ satisfies (DE).

The Daugavet property

A Banach space X is said to have the Daugavet property if every compact operator on X satisfies (DE).

(Kadets–Shvidkoy–Sirotkin–Werner, 1997 & 2000)

Prior versions of: *Chauveheid, 1982; Abramovich–Aliprantis–Burkinshaw, 1991*

Some examples

1. K perfect, μ atomeless, E arbitrary Banach space $\Longrightarrow C(K,E)$, $L_1(\mu,E)$, and $L_\infty(\mu,E)$ have the Daugavet property.

   (Kadets, 1996; Nazarenko, –; Shvidkoy, 2001)

2. $A(\mathbb{D})$ and H^∞ have the Daugavet property.

   (Wojtaszczyk, 1992)

3. "Large" subspaces of $C[0,1]$ and $L_1[0,1]$ have the Daugavet property (in particular, this happends for finite-codimensional subspaces).

   (Kadets–Popov, 1997)

Some consequences

Let X be a Banach space with the Daugavet property. Then

- X does not have the Radon-Nikodým property.

  (Wojtaszczyk, 1992)

- Every slice of B_X and every w^*-slice of $B_{X^\star}$ have diameter 2.
- X contains a copy of ℓ_1. $X^\star$ contains a copy or $L_1[0,1]$.

  (Kadets–Shvidkoy–Sirotkin–Werner, 2000)

- X does not have unconditional basis.

  (Kadets, 1996)

- Moreover, X does not embed into any space with unconditional basis.

  (Kadets–Shvidkoy–Sirotkin–Werner, 2000)

- Actually, X does not embed into an unconditional sum of Banach spaces without a copy of ℓ_1.

  (Shvidkoy, 2000)

The alternative Daugavet equation for operators

X Banach space, $T \in L(X)$

$$\max_{|\omega|=1} \|\mathrm{Id} + \omega\, T\| = 1 + \|T\| \qquad \text{(aDE)}$$

(Duncan–McGregor–Pryce–White, 1970; Holub, Abramovich. . . , 80’s)

Examples

1. **Duncan–McGregor–Pryce–White, 1970:**
   Every operator on $C(K)$ or $L_1(\mu)$ satisfies (aDE) for arbitrary K and μ.
2. **Crabb–Duncan–McGregor, 1972:**
   Every operator on $A(\mathbb{D})$ satisfies (aDE).
3. **Werner, 1997:**
   Every operator on an arbitrary function algebra satisfies (aDE).
4. **M.–Oikhberg, 2004:**
   Every compact operator (but not all operators) on $C([0,1],\ell_2) \oplus_1 c_0$ satisfies (aDE).

The alternative Daugavet property

A Banach space X is said to have the alternative Daugavet property iff every compact operator on X satisfies (aDE).

(M.–Oikhberg, 2004; briefly appearance: Abramovich, 1991)

Consequences

Let X be a Banach space with the alternative Daugavet property.

- If $\dim(X) = n < \infty$, then the unit ball of X can be viewed as the absolutely closed convex hull of some vertices of the n-cube.

  (McGregor, 1971)

- If X is real and $\dim(X) = \infty$, then $X^{\star\star}/X$ is non-separable.
- Actually, X real, $\dim(X) = \infty$ and with the RNP, then $X \supset \ell_1$.

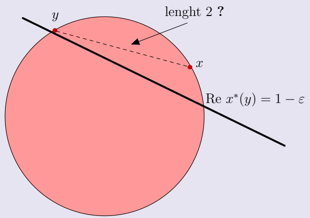
(Lopez–M.–Payá, 1999)

Geometric characterizations

Theorem [Kadets and others]. TFAE:

- X has the Daugavet property.
- For every $x \in S_X$, $x^\star \in S_{X^\star}$, and $\varepsilon > 0$, there exists $y \in S_X$ such that

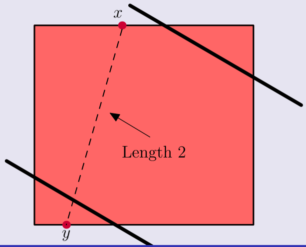
$$\operatorname{Re} x^\star(y) > 1 - \varepsilon \quad \text{and} \quad \|x - y\| \geqslant 2 - \varepsilon.$$



Theorem [M.–Oikhberg]. TFAE:

- X has the alternative Daugavet property.
- For every $x \in S_X$, $x^\star \in S_{X^\star}$, and $\varepsilon > 0$, there exists $y \in S_X$ such that

$$|x^\star(y)| > 1 - \varepsilon \quad \text{and} \quad \|x - y\| \geqslant 2 - \varepsilon.$$



Let X be a C^*-algebra.

The Daugavet property of X is equivalent to:

- X does not have any atomic projection, or
- the unit ball of $X^\star$ does not have any w^*-strongly exposed point.

All operators on X satisfy (aDE) iff:

- X is commutative, or
- $|x^{\star\star}(x^\star)| = 1$ for $x^{\star\star} \in \operatorname{ext}(B_{X^{\star\star}})$ and $x^\star \in \operatorname{ext}(B_{X^\star})$.

The alternative Daugavet property of X is equivalent to:

- the atomic projections of X are central, or
- $|x^{\star\star}(x^\star)| = 1$, for $x^{\star\star} \in \operatorname{ext}(B_{X^{\star\star}})$, and $x^\star \in B_{X^\star}$ w^*-strongly exposed, or
- $\exists$ a commutative ideal Y such that X/Y has the Daugavet property.

(Huruya, 1977; Oikhberg, 2002; M.–Oikhberg, 2004; Becerra–M., 2005; M.,200?)

The Daugavet equation for polynomials

Notation

Notation

X Banach space.

- $\mathcal{P}(X;X)$: polynomials from X to X
- $\mathcal{P}(X)$: scalar polynomials on X

- The norm of $P \in \mathcal{P}(X;X)$ is given by

$$\|P\| = \sup\left\{\|P(x)\| \ : \ x \in B_X\right\}$$

- The norm of $p \in \mathcal{P}(X)$ is given by

$$\|p\| = \sup\left\{|P(x)| \ : \ x \in B_X\right\}$$

By a compact polynomial on X we mean an element $P \in \mathcal{P}(X;X)$ such that $P(B_X)$ is relatively compact.

Definitions

The Daugavet equation for polynomials

X Banach space, $P \in \mathcal{P}(X;X)$

$$\|\mathrm{Id} + P\| = 1 + \|P\| \qquad \text{(DE)}$$

The alternative Daugavet equation for polynomials

X Banach space, $P \in \mathcal{P}(X;X)$

$$\max_{|\omega|=1} \|\mathrm{Id} + \omega\, P\| = 1 + \|P\| \qquad \text{(aDE)}$$

Equivalently: there exists $\omega \in \mathbb{T}$ such that $\omega\, P$ satisfies (DE).

Some easy examples...

1. There are polynomials on $\mathbb{C}$ which does not satisfies (DE):

$$P(z) = i\,z, \qquad \|P\| = 1, \quad \|\mathrm{Id} + P\| = \sqrt{2}.$$

2. But every polynomial on $\mathbb{C}$ satisfies (aDE) (this is an easy consequence of the Maximum Modulus Theorem).
3. This is not true in the real case:

$$P(t) = 1 - t^2, \qquad \|P\| = 1, \quad \|\mathrm{Id} \pm P\| = \frac{5}{4}.$$

- Our aim is to study (DE) and (aDE) for polynomials, mainly the following properties:

The polynomial Daugavet property

We say that a Banach space X has the polynomial Daugavet property if every compact polynomial on X satisfies (DE).

The polynomial alternative Daugavet property

We say that a Banach space X has the polynomial alternative Daugavet property if every compact polynomial on X satisfies (aDE).

- Of course, the first step should be to present examples of spaces having these properties.

Examples

$C(K)$ spaces

Theorem

Let K be a perfect compact space. Then $C(K)$ has the polynomial Daugavet property. Actually, for every Banach space E, every finite-codimensional subspace of $C(K, E)$ has the polynomial Daugavet property.

The main tool is the following characterization:

Theorem

Let X be a Banach space. TFAE:

- X has the polynomial Daugavet property.
- For every $p \in \mathcal{P}(X)$, $x \in S_X$, and $\varepsilon > 0$, there exists $\omega \in \mathbb{T}$ and $y \in S_X$ such that

$$\operatorname{Re} \omega\, p(y) > 1 - \varepsilon \quad \text{and} \quad \|x + \omega\, y\| \geqslant 2 - \varepsilon.$$

Examples

When K is not perfect,

- $C(K)$ does not have the polynomial Daugavet property,
- $C(K)$ does have the linear alternative Daugavet property, and...

Theorem

The complex space $C(K)$ has always the alternative Daugavet property.

Example

If K is non-perfect, the real space $C(K)$ does not have the alternative Daugavet property. Indeed, consider an isolated point $t_1 \in K$ and $t_2 \in K \setminus \{t_1\}$; then the 2-homogeneous polynomial on $C(K)$

$$P(f) = \left(f(t_2)^2 - \frac{1}{2} f(t_1)^2 \right) \chi_{\{t_1\}}$$

does not satisfy the (aDE).

Examples

$L_1(\mu)$ spaces

When μ has atoms, $L_1(\mu)$ does not have the polynomial Daugavet property; even more:

Example

The real or complex space ℓ_1 does not have the polynomial alternative Daugavet property: the compact (2-homogeneous) polynomial

$$P(x_1, x_2, x_3, x_4, \ldots) = \left(\frac{1}{2}x_1^2 + 2x_1x_2,\ -\frac{1}{2}x_2^2 - x_1x_2, 0, 0, \ldots\right)$$

does not satisfy (aDE).

(Choi–Kim, 1996)

Observation

Let us observe that the linear case does not distinguish between $C(K)$ and $L_1(\mu)$ spaces but, in the complex case, the polynomial case does:

- Complex ℓ_∞ has the polynomial alternative Daugavet property,
- Complex ℓ_1 does not have the polynomial alternative Daugavet property.

Open problems

Open problems

About examples

1. Does $L_1[0,1]$ have the polynomial Daugavet property in the real or in the complex case **?**
2. What's about $A(\mathbb{D})$ and H^∞ **?**
3. Actually, we do not know of any example of space with the linear Daugavet property which does not have the polynomial Daugavet property.

About geometry

Let X be a Banach space with the (alternative) polynomial Daugavet property. It would be interesting to studty the geometry of X. For instance:

4. How similar is X to a $C(K)$ space **?**
5. Does X contain a copy of c_0 **?**