Norm equalities for operators

in the spirit of Daugavet equation

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The Daugavet equation

X Banach space, $T \in L(X)$

$$\|\mathrm{Id} + T\| = 1 + \|T\| \qquad \text{(DE)}$$

Classical examples:

- **Daugavet, 1963:**
  Every compact operator on $C[0,1]$ satisfies (DE).

- **Lozanoskii, 1966:**
  Every compact operator on $L_1[0,1]$ satisfies (DE).

- **Abramovich, Holub, and more, 80's:**
  $X = C(K)$, K perfect compact space
  or $X = L_1(\mu)$, μ atomless measure
  $\Longrightarrow$ every weakly compact $T \in L(X)$ satisfies (DE).

The Daugavet property

A Banach space X is said to have the Daugavet property if every rank-one operator on X satisfies (DE).

- Then, every weakly compact operator also satisfies (DE).
- If X^* has the Daugavet property, so does X.
  The converse is not true.

(Kadets–Shvidkoy–Sirotkin–Werner, 1997)

Prior versions of: *Chauveheid, 1982; Abramovich–Aliprantis–Burkinshaw, 1991*

Some propaganda:

X with the Daugavet property. Then

- X does not have the Radon-Nikodým property.

  (Wojtaszczyk, 1992)

- Every weakly-open subset of B_X has diameter 2.

  (Shvidkoy, 2000)

- X contains a copy of ℓ_1. X^* contains a copy or $L_1[0,1]$.

  (Kadets–Shvidkoy–Sirotkin–Werner, 2000)

- X does not embed into a space with unconditional basis.

  (Kadets–Shvidkoy–Sirotkin–Werner, 2000)

Daugavet type inequalities

- **Benyamini–Lin, 1985:**
  If $1 < p < \infty$, $p \neq 2$, $T \in L(L_p[0,1])$ compact, then

$$\|\mathrm{Id} + T\| \geqslant \left(1 + a_p \|T\|^2\right)^{\frac{1}{2}}$$

  for some $a_p \neq 0$.

- If $p = 2$, then there are non-null compact T's such that

$$\|\mathrm{Id} + T\| = 1$$

- **Oikhberg, 2005:**
  If $1 < p < \infty$, $p \neq 2$, $T \in L(\mathcal{L}_p(\tau))$ compact, then

$$\|\mathrm{Id} + T\| \geqslant 1 + k_p \min\{\|T\|, \|T\|^2\}$$

  for some $k_p \neq 0$.

Our main questions

- Is any of the previous inequalities an equality **?**

- Even more, is there **any** norm equality valid for all compact operators on L_p or $\mathcal{L}_p$ spaces **?**

- Actually, we would like to study the possibility of finding norm equalities for operators in the spirit of Daugavet equation, valid for all compact (rank-one) operators on a Banach space. We will study three cases:

  - $\|\mathrm{Id} + T\| = f(\|T\|)$ for arbitrary f.
  - $\|g(T)\| = f(\|T\|)$ for analytic g and arbitrary f.
  - $\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$ for analytic g and *continuous* f.

Equalities of the form $\|\mathrm{Id} + T\| = f(\|T\|)$

PROPOSITION:

X real or complex Banach space, $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ arbitrary, $a, b \in \mathbb{K}$. If the norm equality

$$\|a\,\mathrm{Id} + b\,T\| = f(\|T\|)$$

holds for every rank-one operator T on X, then

$$f(t) = |a| + |b|\,t \qquad \big(t \in \mathbb{R}_0^+\big).$$

If $a \neq 0$, $b \neq 0$, then X has the Daugavet property.

Then, we have to look for Daugavet-type equalities in which $\mathrm{Id} + T$ is replaced by something different.

Equalities of the form $\|g(T)\| = f(\|T\|)$

THEOREM:

X real or complex Banach space with $\dim(X) \geqslant 2$.

Suppose that the norm equality

$$\|g(T)\| = f(\|T\|)$$

holds for every rank-one operator T on X, where

- $g : \mathbb{K} \longrightarrow \mathbb{K}$ is analytic,
- $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ is arbitrary.

Then, there are $a, b \in \mathbb{K}$ such that

$$g(\zeta) = a + b\,\zeta \qquad (\zeta \in \mathbb{K}).$$

COROLLARY:

Only three norm equalities of the form

$$\|g(T)\| = f(\|T\|)$$

are possible:

- (g is constant): $\|a\,\mathrm{Id}\| = |a|$,

- ($g(\zeta) = b\,\zeta$): $\|b\,T\| = |b|\,\|T\|$,

(trivial cases)

- ($g(\zeta) = a + b\,\zeta$ with $a \neq 0$, $b \neq 0$):
  $\|a\,\mathrm{Id} + b\,T\| = |a| + |b|\,\|T\|$, which implies that X has the Daugavet property.

Equalities of the form $\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$

Complex case:

X complex Banach space with $\dim(X) \geqslant 2$.
Suppose that the norm equality

$$\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$$

holds for every rank-one operator on X, where

- $g : \mathbb{C} \longrightarrow \mathbb{C}$ is analytic and non-constant,
- $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ is *continuous*.

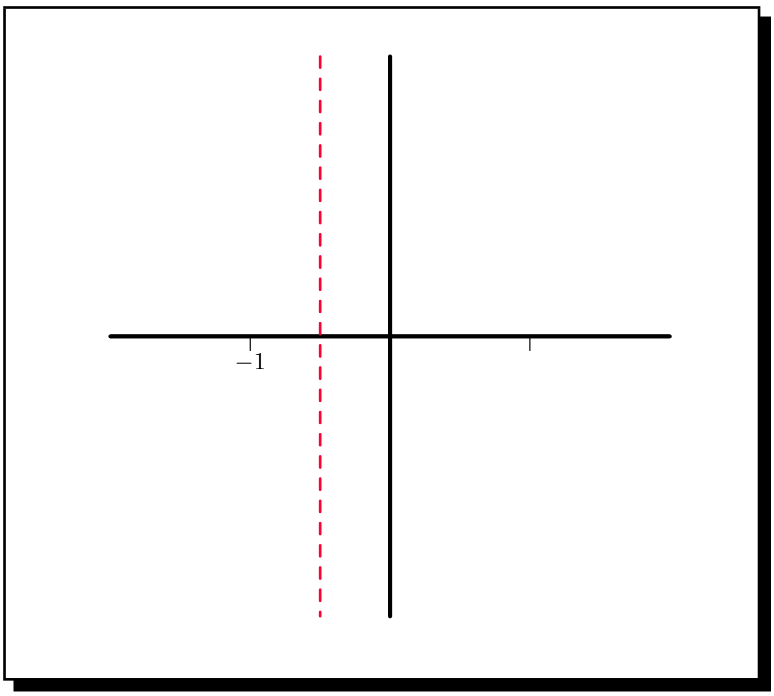
Proposition:

The only possible form of the function f is the following:

$$f(t) = |1 + g(0)| - |g(0)| + t \qquad \big(t \geqslant |g(0)|\big).$$

We obtain two different cases:

- $|1 + g(0)| \neq |g(0)|$ i.e., $\mathrm{Re}\, g(0) \neq -1/2$,
- $|1 + g(0)| = |g(0)|$ i.e., $\mathrm{Re}\, g(0) = -1/2$.



- **Theorem:**

  If $\mathrm{Re}\, g(0) \neq -1/2$ and

  $$\|\mathrm{Id}+g(T)\| = f(\|g(T)\|) \left[= |1+g(0)|-|g(0)|+\|g(T)\|\right]$$

  for every rank-one T, then X has the Daugavet property.

- **Example:**

  If we take $g(\zeta) = -1/2+\zeta$, we obtain the equation $\left\|\mathrm{Id}+(-\frac{1}{2}\mathrm{Id}+T)\right\| = \left\|-\frac{1}{2}\mathrm{Id}+T\right\|$ or, equivalently,

  $$\|\mathrm{Id}+T\| = \|\mathrm{Id}-T\|.$$

  - Every rank-one operator T on $C[0,1]\oplus_2 C[0,1]$ satisfies $\left\|\mathrm{Id}+\mathrm{e}^{i\,\theta}\,T\right\| = \|\mathrm{Id}+T\|$ for every $\theta\in\mathbb{R}$.
  - The space $C[0,1]\oplus_2 C[0,1]$ does not have the Daugavet property.

Real case:

- The proofs of the above results are not valid (we use Picard's Theorem).

- We do not even know if any of the following simple equalities implies the Daugavet property:
  - $\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$ for every rank-one operator,
  - $\|\mathrm{Id} - T^2\| = 1 + \|T^2\|$ for every rank-one operator.

- Every rank-one operator T on the real $C[0,1] \oplus_2 C[0,1]$ satisfies $\|\mathrm{Id} + T\| = \|\mathrm{Id} - T\|$.

Some questions

- Study the real Banach spaces X for which the equality

$$\left\|\mathrm{Id} + T^2\right\| = 1 + \|T^2\|$$

holds for every rank-one operator T on X.

- Is there any real Banach space such that every operator T on X satisfies the equality $\left\|\mathrm{Id} + T^2\right\| = 1 + \|T^2\|$ **?**

- Study the real or complex X for which the equality

$$\|\mathrm{Id} + T\| = \|\mathrm{Id} - T\|$$

holds for every rank-one operator T on X.