Numerical index of Banach spaces

and duality

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Numerical Range of an operator

- **Toeplitz, 1918:**

  H Hilbert space, $T \in L(H)$

$$V(T) = \{(Tx \mid x) \; : \; x \in H, \; \|x\| = 1\}.$$

- **Lumer, 1961; Bauer, 1962:**

  X Banach space, $T \in L(X)$

$$V(T) = \{x^*(Tx) \; : \; \|x\| = \|x^*\| = x^*(x) = 1\}.$$

Numerical radius and numerical index

- **Numerical radius:**

  X Banach space, $T \in L(X)$,

$$v(T) = \sup\{|\lambda| \ : \ \lambda \in V(T)\}$$

  v is a continuous seminorm: $v(T) \leqslant \|T\|$

- **Numerical index:**

  X Banach space,

$$\begin{aligned} n(X) &= \inf\{v(T) \ : \ T \in L(X), \ \|T\| = 1\} \\ &= \max\{k \geqslant 0 \ : \ k\|T\| \leqslant v(T) \ \ \forall T \in L(X)\} \end{aligned}$$

(Lumer, 1968)

Some basic properties

- $n(X) = 1$ iff v and $\|\cdot\|$ coincide.
- $n(X) = 0$ iff v is not an equivalent norm in $L(X)$.
- X real, $\dim(X) < \infty$,

  $n(X) = 0$ iff the group of isometries of X is infinite.

- X complex $\Rightarrow$ $n(X) \geqslant 1/\mathrm{e}$.

  (Bohnenblust–Karlin, 1955; Glickfeld, 1970)

- $\{n(X) \ : \ X \text{ complex}, \ \dim(X) = 2\} = [\mathrm{e}^{-1}, 1]$
  $\{n(X) \ : \ X \text{ real}, \ \dim(X) = 2\} = [0, 1]$

  (Duncan–McGregor–Pryce–White, 1970)

Some examples

- H Hilbert space, $\dim(H) > 1$,

$$n(H) = 0 \qquad \text{if } H \text{ is real,}$$
$$n(H) = 1/2 \qquad \text{if } H \text{ is complex.}$$

- $n\big(L_1(\mu)\big) = 1 \quad \mu$ positive measure.
  $n\big(C(K)\big) = 1 \quad K$ compact Hausdorff space.

(Duncan et al., 1970)

- If $\dim(X) < \infty$, then

$$n(X) = 1 \iff |x^*(x)| = 1 \ \big(x \in \text{ext}(B_X),\ x^* \in \text{ext}(B_{X^*})\big).$$

(McGregor, 1971)

- If A is a C^*-algebra $\Rightarrow \begin{cases} n(A) = 1 & A \text{ commutative,} \\ n(A) = 1/2 & A \text{ not commutative.} \end{cases}$

(Huruya, 1977)

- $A(\mathbb{D})$ disk algebra $\Rightarrow n\big(A(\mathbb{D})\big) = 1.$

(Crabb–Duncan–McGregor, 1972)

- If A is a function algebra $\Rightarrow n(A) = 1.$

(Werner, 1997)

- For $n \geqslant 2$, the unit ball of X_n is a $2n$ regular polygon.

$$n(X_n) = \begin{cases} \tan\left(\dfrac{\pi}{2n}\right) & \text{if } n \text{ is even,} \\ \sin\left(\dfrac{\pi}{2n}\right) & \text{if } n \text{ is odd.} \end{cases}$$

(M.–Merí, ??)

- Direct sums:

$$n\Big([\oplus_{\lambda\in\Lambda}X_\lambda]_{c_0}\Big) = n\Big([\oplus_{\lambda\in\Lambda}X_\lambda]_{l_1}\Big)$$
$$= n\Big([\oplus_{\lambda\in\Lambda}X_\lambda]_{l_\infty}\Big) = \inf_\lambda n(X_\lambda)$$

(M.–Payá, 2000)

- Vector valued function spaces:

$$n\Big(C(K,X)\Big) = n\Big(L_1(\mu,X)\Big) = n\Big(L_\infty(\mu,X)\Big) = n(X)$$

(M.–Payá, 2000; M.–Villena, 2003)

Numerical range and duality

- X Banach space, $T \in L(X)$,
  - $\sup \operatorname{Re} V(T) = \lim\limits_{\alpha \to 0^+} \dfrac{\|Id + \alpha T\| - 1}{\alpha}$.
  - Then, $v(T) = v(T^*)$.
  - Therefore, $n(X^*) \leqslant n(X)$.

(Duncan et al., 1970)

- QUESTION:
  Are $n(X)$ and $n(X^*)$ always equal **?**

- ANSWER:
  No, as we will show in the following.

The counterexample

Let us consider the Banach space

$$X = \left\{(x, y, z) \in c \oplus_\infty c \oplus_\infty c \ : \ \lim x + \lim y + \lim z = 0\right\}.$$

Then, $n(X) = 1$ but $n(X^*) < 1$.

PROOF:

- We write $c^* = \ell_1 \oplus_1 \mathbb{K} \lim$ and we observe that

$$X^* = \left[c^* \oplus_1 c^* \oplus_1 c^*\right] / (\lim, \lim, \lim).$$

- Then, writing $Z = \ell_1^{(3)}/(1,1,1)$, we can identify

$$X^* \equiv \ell_1 \oplus_1 \ell_1 \oplus_1 \ell_1 \oplus_1 Z,$$

$$X^{**} \equiv \ell_\infty \oplus_\infty \ell_\infty \oplus_\infty \ell_\infty \oplus_\infty Z^*.$$

- Let us prove that $n(X) = 1$.

  - $A = \{(e_n, 0, 0, 0)\} \cup \{(0, e_n, 0, 0)\} \cup \{(0, 0, e_n, 0)\}$.
  - Then $B_{X^*} = \overline{\text{aco}}^{w^*}(A)$ and

$$|x^{**}(a)| = 1 \quad \forall\, x^{**} \in \text{ext}(B_{X^{**}})\ \forall\, a \in A.$$

  - Fix $T \in L(X)$ and $\varepsilon > 0$.
  - We find $a \in A$ such that $\|T^*(a)\| > \|T^*\| - \varepsilon$.
  - Then, we find $x^{**} \in \text{ext}(B_{X^{**}})$ such that

$$|x^{**}(T^*(a))| = \|T^*(a)\| > \|T^*\| - \varepsilon.$$

  - Since $|x^{**}(a)| = 1$, this gives that

$$v(T^*) > \|T^*\| - \varepsilon,$$

  so $v(T) = \|T\|$ and $n(X) = 1$.

- Z is an L-summand of X^* so

$$n(X^*) = n(Z).$$

- But $n(Z) < 1$ **!**

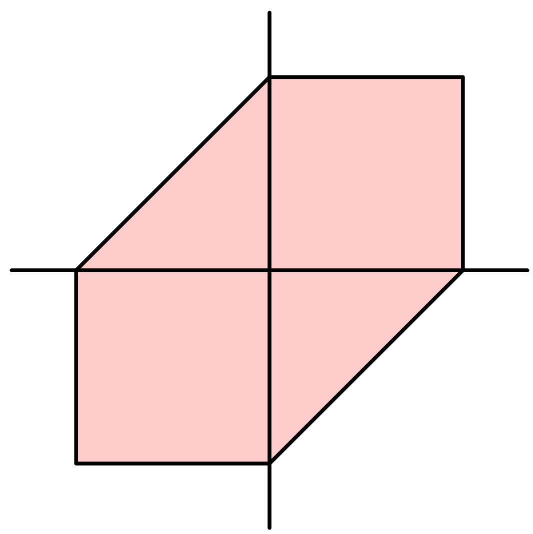

Figure 1: The unit ball of Z

Remarks

- *There is* Z *with two isometric preduals* X_1 *and* X_2 *such that* $n(X_1)$ *and* $n(X_2)$ *are not equal:*

  $$X_1 = \big\{(x, y, z) \in c \oplus_\infty c \oplus_\infty c \ : \ \lim x + \lim y + \lim z = 0\big\},$$

  $$X_2 = \big\{(x, y, z) \in c \oplus_\infty c \oplus_\infty c \ : \ x(1) + y(1) + z(1) = 0\big\}.$$

  - $n(X_1) = 1$, $n(X_2) < 1$.
  - $Z = X_1^* \equiv X_2^*$.

- *There is a real* X *such that* $n(X) = 1$ *and* $n(X^*) = 0$.

Open problems

- Y dual space. Is there a predual X such that $n(X) = n(Y)$ **?**

- Find (isomorphic) properties implying the equality of the numerical index of a space and the one of its dual.
  - Reflexivity is such a property.
  - Asplundness is not.
  - RNP **?**
  - A positive result:
    Let X be a Banach space with the RNP.
    If $n(X) = 1$, then $n(X^*) = 1$.