The alternative Daugavet property.

Characterizations for C^*-algebras and von Neumann preduals.

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The talk is based on these papers

M. Martín and T. Oikhberg,
An alternative Daugavet property.
Journal of Mathematical Analysis and applications (2004)

J. Becerra Guerrero and M. Martín,
The Daugavet Property of C^*-algebras, JB^*-triples, and of their isometric preduals.
Journal of Functional Analysis (2005)

M. Martín,
The alternative Daugavet property of C^*-algebras and JB^*-triples.
Preprint

The alternative Daugavet equation

X Banach space, $T \in L(X)$, we say that T satisfies the alternative Daugavet equation iff

$$\max_{|\omega|=1} \|Id + \omega\, T\| = 1 + \|T\| \qquad \text{(aDE)}$$

(Duncan–McGregor–Pryce–White, 1970; Holub, Abramovich..., 80's)

Two equivalent reformulations:

- There exists $\omega \in \mathbb{T}$ such that $S = \omega\, T$ satisfies "usual" Daugavet equation, namely

$$\|Id + S\| = 1 + \|S\| \qquad \text{(DE)}$$

- $v(T) = \|T\|$, where

$$v(T) = \sup\{|x^*(Tx)| \ : \ x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1\}.$$

  is the numerical radius of T.

(Duncan–McGregor–Pryce–White, 1970)

The alternative Daugavet property

A Banach space X is said to have the alternative Daugavet property (ADP) iff every rank-one operator on X satisfies (aDE).

(Abramovich, 1991; M.–Oikhberg, 2004)

Two sufficient conditions for the ADP

- If every rank-one operator on X satisfies (DE), i.e. if the space X has the Daugavet property.
- If every operator on X satisfies (aDE), i.e. if numerical radius and norm coincides in the whole $L(X)$.
  In this case, X is said to have numerical index 1.

Outline

The Daugavet property

The Daugavet equation

X Banach space, $T \in L(X)$

$$\|Id + T\| = 1 + \|T\| \qquad \text{(DE)}$$

Classical examples

1. **Daugavet, 1963:**
   Every compact operator on $C[0,1]$ satisfies (DE).
2. **Lozanoskii, 1966:**
   Every compact operator on $L_1[0,1]$ satisfies (DE).
3. **Abramovich, Holub, and more, 80's:**
   $X = C(K)$, K perfect compact space
   or $X = L_1(\mu)$, μ atomless measure
   $\Longrightarrow$ every weakly compact $T \in L(X)$ satisfies (DE).

The Daugavet property

- A Banach space X is said to have the Daugavet property if every rank-one operator on X satisfies (DE).
- Then, every weakly compact operator also satisfies (DE).
- If X^* has the Daugavet property, so does X. The converse is not true.

(Kadets–Shvidkoy–Sirotkin–Werner, 1997 & 2000)

Some examples...

1. K perfect, μ atomeless, X arbitrary Banach space
$\Longrightarrow C(K,X)$, $L_1(\mu,X)$, and $L_\infty(\mu,X)$ have the Daugavet property.
(Kadets, 1996; Nazarenko, –; Shvidkoy, 2001)
2. K arbitrary. If X has the Daugavet property, then so does $C(K,X)$.
(M.–Payá, 2000)

More examples...

3. The c_0, ℓ_1, and ℓ_∞ sums of Banach spaces with the Daugavet property have the Daugavet property.
4. $A(\mathbb{D})$ and H^∞ have the Daugavet property.

(Wojtaszczyk, 1992)

5. $R \subset L_1[0,1] =: L_1$ reflexive, then L_1/R has the Daugavet property.

(Kadets–Shvidkoy–Sirotkin–Werner, 2000)

6. A C^*-algebra has the Daugavet property if and only if it is non-atomic.
7. The predual of a von Neumann algebra has the Daugavet property if and only if the algebra is non-atomic.

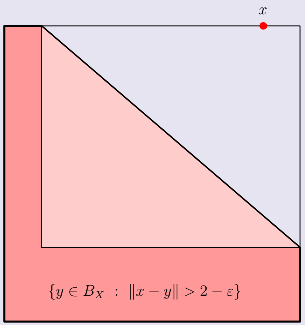
(Oikhberg, 2002)

Geometric characterizations

Theorem [KSSW]

TFAE:

- X has the Daugavet property.
- For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y \in S_X$ such that
$$\operatorname{Re} x^*(y) > 1 - \varepsilon \quad \text{and} \quad \|x - y\| \geqslant 2 - \varepsilon.$$
- For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y^* \in S_{X^*}$ such that
$$\operatorname{Re} y^*(x) > 1 - \varepsilon \quad \text{and} \quad \|x^* - y^*\| \geqslant 2 - \varepsilon.$$
- For every $x \in S_X$ and every $\varepsilon > 0$, we have
$$B_X = \overline{\text{co}}\big(\{y \in B_X \ : \ \|x - y\| \geqslant 2 - \varepsilon\}\big).$$



A new sufficient condition

A new sufficient condition

Theorem

Let X be a Banach space such that

$$X^* = Y \oplus_1 Z$$

with Y and Z norming subspaces. Then, X has the Daugavet property.

A closed subspace $W \subseteq X^*$ is norming if

$$\|x\| = \sup \{|w^*(x)| \ : \ w^* \in W, \ \|w^*\| = 1\}$$

or, equivalently, if B_W is w^*-dense in B_{X^*}.

Proof of the theorem

We have...

$X^* = Y \oplus_1 Z$,
B_Y, B_Z w^*-dense in B_{X^*}.

$\overset{?}{\Longrightarrow}$

We need...

fixed $x_0 \in S_X$, $x_0^* \in S_{X^*}$, $\varepsilon > 0$, find $y^* \in S_{X^*}$ such that

$$\|x_0^* + y^*\| > 2 - \varepsilon \quad \text{and} \quad \operatorname{Re} y^*(x_0) > 1 - \varepsilon.$$

- Write $x_0^* = y_0^* + z_0^*$ with $y_0^* \in Y$, $z_0^* \in Z$, $\|x_0^*\| = \|y_0^*\| + \|z_0^*\|$, and write
$$U = \{x^* \in B_{X^*} \ : \ \operatorname{Re} x^*(x_0) > 1 - \varepsilon\}.$$
- Take $z^* \in B_Z \cap U$ and a net (y_λ^*) in $B_Y \cap U$, such that $(y_\lambda^*) \xrightarrow{w^*} z^*$.
- $(y_\lambda^* + y_0^*) \longrightarrow z^* + y_0^*$ and the norm is w^*-lower semi-continuous, therefore
$$\liminf \|y_\lambda^* + y_0^*\| \geqslant \|z^* + y_0^*\| = \|z^*\| + \|y_0^*\| > 1 + \|y_0^*\| - \varepsilon.$$
- Then, we may find μ such that $\|y_\mu^* + y_0^*\| \geqslant 1 + \|y_0^*\| - \varepsilon/2$.
- Finally, observe that
$$\|x_0^* + y_\mu^*\| = \|(y_0^* + y_\mu^*) + z_0^*\| =$$
$$= \|y_0^* + y_\mu^*\| + \|z_0^*\| > 1 + \|y_0^*\| - \varepsilon + \|z_0^*\| = 2 - \varepsilon,$$

and that $\operatorname{Re} y_\mu^*(x_0) > 1 - \varepsilon$ (since $y_\mu^* \in U$).

Some immediate consequences

Corollary

Let X be an L-embedded space with $\operatorname{ext}(B_X) = \emptyset$. Then, X^* (and hence X) has the Daugavet property.

Corollary

If Y is an L-embedded space which is a subspace of $L_1 \equiv L_1[0,1]$, then $(L_1/Y)^*$ has the Daugavet property.

It was already known that...

- If $Y \subset L_1$ is reflexive, then L_1/Y has the Daugavet property.
  (Kadets–Shvidkoy–Sirotkin–Werner, 2000)
- If $Y \subset L_1$ is L-embedded, then L_1/Y does not have the RNP.
  (Harmand–Werner–Werner, 1993)

von Neumann preduals

von Neumann preduals

- A C^*-algebra X is a von Neumann algebra if it is a dual space.
- In such a case, X has a unique predual X_*.
- X_* is always L-embedded.
- Therefore, if $\operatorname{ext}(B_{X_*})$ is empty, then X and X_* have the Daugavet property.

Actually, much more can be proved:

Theorem

Let X_* be the predual of the von Neumann algebra X. Then, TFAE:

- X has the Daugavet property.
- X_* has the Daugavet property.
- Every weakly open subset of B_{X_*} has diameter 2.
- B_{X_*} has no strongly exposed points.
- B_{X_*} has no extreme points.
- X is non-atomic (i.e. it has no atomic projections).

An atomic projection is an element $p \in X$ such that

$$p^2 = p^* = p \qquad \text{and} \qquad p\,X\,p = \mathbb{C}p.$$

C^*-algebras

Let X be a C^*-algebra. Then, X^{**} is a von Neumann algebra.
Write $X^* = \left(X^{**}\right)_* = A \oplus_1 N$, where

- A is the atomic part,
- N is the non-atomic part.

- Every extreme point of B_{X^*} is in B_A.
- Therefore, A is norming.
- What's about N **?**

Theorem

If X is non-atomic, then N is norming. Therefore, X has the Daugavet property.

Actually, much more can be proved:

Theorem

Let X be a C^*-algebra. Then, TFAE:

- X has the Daugavet property.
- The norm of X is extremely rough, i.e.,

$$\limsup_{\|h\|\to 0} \frac{\|x+h\| + \|x-h\| - 2}{\|h\|} = 2$$

  for every $x \in S_X$ (equivalently, every w^*-slice of B_{X^*} has diameter 2).
- The norm of X is not Fréchet-smooth at any point.
- X is non-atomic.

The numerical index of a Banach space

Numerical range of an operator

- H Hilbert space, $T \in L(H)$,

$$W(T) := \{(Tx|x) \ : \ x \in H, \ \|x\| = 1\}.$$

(Toeplitz, 1918)

- X Banach space, $T \in L(X)$,

$$W(T) := \{x^*(Tx) \ : \ \|x\| = \|x^*\| = x^*(x) = 1\}.$$

(Lumer, 1961; Bauer, 1962)

Numerical radius of an operator

X Banach space, $T \in L(X)$,

$$v(T) := \sup\{|\lambda| \ : \ \lambda \in W(T)\}.$$

Numerical index of a Banach space

X Banach space,

$$n(X) := \max\{k \geqslant 0 \ : \ k\|T\| \leqslant v(T) \quad \forall T \in L(X)\}$$
$$= \inf\{v(T) \ : \ T \in L(X), \ \|T\| = 1\}.$$

(Lumer, 1968; Duncan-McGregor-Pryce-White, 1970)

Immediate properties

- $0 \leqslant n(X) \leqslant 1$.
- $n(X^*) \leqslant n(X)$.

Numerical index 1

X has numerical index 1 if $v(T) = \|T\|$ for every $T \in L(X)$.
Equivalently, if EVERY operator T on X satisfies

$$\max_{|\omega|=1} \|Id + \omega T\| = 1 + \|T\| \qquad \text{(aDE)}$$

Some examples

1. $n\big(L_1(\mu)\big) = 1$ for every positive measure μ.
2. Therefore, $X^* \equiv L_1(\mu) \quad \Rightarrow \quad n(X) = 1$.
3. For instance, $n\big(C(K)\big) = 1$ for every compact space K.

   (Duncan-McGregor-Pryce-White, 1970)

4. The disk algebra $A(\mathbb{D})$ has numerical index 1.

   (Crabb-Duncan-McGregor, 1972)

5. Every space "nicely embedded" into some $C_b(\Omega)$ has numerical index 1.

   (Werner, 1997)

More examples

6. If $\dim(X) < \infty$, then X has numerical index 1 iff
$$|x^*(x)| = 1 \qquad \big(x^* \in \operatorname{ext}(B_{X^*}),\ x \in \operatorname{ext}(B_X)\big).$$
(McGregor, 1971)
7. The c_0, ℓ_1, and ℓ_∞ sums of Banach spaces with numerical index 1 have numerical index 1.
8. X Banach space, K compact space, μ positive measure. Then $C(K,X)$, $L_1(\mu,X)$, and $L_\infty(\mu,X)$ have numerical index 1 iff X does.

(M.–Payá, 2000; M.–Villena, 2003)

C^*-algebras and preduals

Theorem

Let X be a C^*-algebra. Then, TFAE:

- X is commutative.
- $|x^{**}(x^*)| = 1$ for every extreme points x^{**} of $B_{X^{**}}$ and x^* of B_{X^*}.
- X has numerical index 1.
- X^* has numerical index 1.

(Huruya, 1977; Kaidi–Morales–Rodriguez-Palacios, 2001)

Theorem

Let X be a von Neumann algebra. Then, TFAE:

- X is commutative (meaning $n(X) = 1$).
- $|x^*(x)| = 1$ for every extreme points x^* of B_{X^*} and x of B_X.
- X_* has numerical index 1.

(Kaidi–Morales–Rodriguez-Palacios, 2001)

The alternative Daugavet property

The alternative Daugavet equation

X Banach space, $T \in L(X)$

$$\max_{|\omega|=1} \|Id + \omega\, T\| = 1 + \|T\| \qquad \text{(aDE)}$$

(Duncan–McGregor–Pryce–White, 1970; Holub, Abramovich..., 80's)

The alternative Daugavet property

- A Banach space X is said to have the alternative Daugavet property (ADP) iff every rank-one operator on X satisfies (aDE).
- Then, every weakly compact operator also satisfies (aDE).
- If X^* has the ADP, so does X. The converse is not true.

(Abramovich, 1991; M.–Oikhberg, 2004)

Some examples

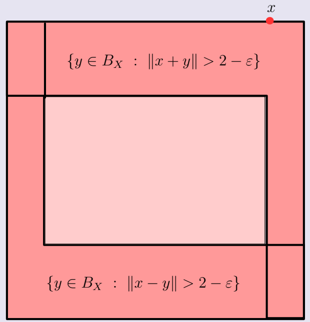
1. Banach spaces with the Daugavet property and Banach spaces with numerical index 1 have the ADP.
2. The c_0, ℓ_1, and ℓ_∞ sums of Banach spaces with the ADP have the ADP.
3. The space $C([0,1],\ell_2) \oplus_\infty c_0$ has the ADP but not the Daugavet property neither numerical index 1.
4. Every Banach space with the ADP can be renormed still having the ADP but lacking the Daugavet property.
5. X Banach space, K compact space, μ positive measure. Then:
   - $C(K,X)$ has the ADP iff K is perfect of X has the ADP.
   - $L_1(\mu,X)$ and $L_\infty(\mu,X)$ have the ADP iff μ is atomless or X has the ADP.
6. X real Banach space, $\dim(X)=\infty$.
   If X has the RNP and the ADP, then $X \supseteq \ell_1$.

Geometric characterizations

Theorem

TFAE:

- X has the ADP.
- For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y \in S_X$ such that
$$|x^*(y)| > 1 - \varepsilon \quad \text{and} \quad \|x - y\| \geqslant 2 - \varepsilon.$$
- For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y^* \in S_{X^*}$ such that
$$|y^*(x)| > 1 - \varepsilon \quad \text{and} \quad \|x^* - y^*\| \geqslant 2 - \varepsilon.$$
- For every $x \in S_X$ and every $\varepsilon > 0$, we have
$$B_X = \overline{\text{co}}\big(\mathbb{T}\,\{y \in B_X \ : \ \|x - y\| \geqslant 2 - \varepsilon\}\big).$$



von Neumann preduals

Proposition

H Hilbert space. If $K(H)$ has the ADP, then $H = \mathbb{C}$.

Let X be a von Neumann algebra.

- $X_* = A \oplus_1 N$ decomposition into atomic and non-atomic part.
- N has the Daugavet property
- $A = \left[\bigoplus_{i\in I} \mathcal{L}_1(H_i)\right]_{\ell_1}$.
- Therefore, X_* has the ADP iff A is commutative.

Actually, much more can be proved:

Theorem

Let X_* be the predual of the von Neumann algebra X. Then, TFAE:

- X has the ADP.
- X_* has the ADP.
- $|x(x_*)| = 1$ for every $x \in \text{ext}(B_X)$ and every $x_* \in \text{ext}(B_{X_*})$.
- $X = \ell_\infty(\Gamma) \oplus_\infty \mathcal{N}$, where $\mathcal{N}$ has the Daugavet property.

C^*-algebras

Let X be a C^*-algebra. Then, X^{**} is a von Neumann algebra.
Write

$$X^* = \left(X^{**}\right)_* = A \oplus_1 N \qquad X^{**} = \mathcal{A} \oplus_\infty \mathcal{N}.$$

- Take $Y = X \cap \mathcal{A}$.
- Then Y is an M-ideal of X and X/Y has the Daugavet property.
- Therefore, X has the ADP iff Y does.
- But Y is an Asplund space, where ADP implies numerical index 1, and Y should be commutative.

Actually, the following result can be proved:

Theorem

Let X be a C^*-algebra. Then, TFAE:

- X has the ADP.
- $|x^{**}(x^*)| = 1$ for every $x^{**} \in \mathrm{ext}(B_{X^{**}})$ and every w^*-strongly exposed $x^* \in B_{X^*}$.
- There exists a commutative ideal Y such that X/Y has the Daugavet property.
- All the atomic projections are in the center of the algebra.

Summary of results

Let V_* be the predual of the von Neumann algebra V.

The Daugavet property of V_* is equivalent to:

- V has no atomic projections, or
- the unit ball of V_* has no extreme points.

V_* has numerical index 1 iff:

- V is commutative, or
- $|v^*(v)| = 1$ for $v \in \operatorname{ext}(B_V)$ and $v^* \in \operatorname{ext}(B_{V^*})$.

The alternative Daugavet property of V_* is equivalent to:

- the atomic projections of V are central, or
- $|v(v_*)| = 1$ for $v \in \operatorname{ext}(B_V)$ and $v_* \in \operatorname{ext}(B_{V_*})$, or
- $V = C \oplus_\infty N$, where C is commutative and N has no atomic projections.

Let X be a C^*-algebra.

The Daugavet property of X is equivalent to:

- X does not have any atomic projection, or
- the unit ball of X^* does not have any w^*-strongly exposed point.

X has numerical index 1 iff:

- X is commutative, or
- $|x^{**}(x^*)| = 1$ for $x^{**} \in \operatorname{ext}(B_{X^{**}})$ and $x^* \in \operatorname{ext}(B_{X^*})$.

The alternative Daugavet property of X is equivalent to:

- the atomic projections of X are central, or
- $|x^{**}(x^*)| = 1$, for $x^{**} \in \operatorname{ext}(B_{X^{**}})$, and $x^* \in B_{X^*}$ w^*-strongly exposed, or
- $\exists$ a commutative ideal Y such that X/Y has the Daugavet property.