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The Daugavet property of C^* -algebras and von Neumann preduals

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The Daugavet equation

X Banach space, $T \in L(X)$

$$\|Id + T\| = 1 + \|T\| \quad (\text{DE})$$

- **Daugavet, 1963:**

Every compact operator on $C[0, 1]$ satisfies (DE).

- **Lozanoskii, 1966:**

Every compact operator on $L_1[0, 1]$ satisfies (DE).

- **Abramovich, Holub and more, 80's:**

$X = C(K)$, K perfect compact space

or $X = L_1(\mu)$, μ atomless measure,

\implies every $T \in L(X)$ weakly compact satisfies (DE).

The Daugavet property

A Banach space X is said to have the **Daugavet property** if every rank-one operator $T \in L(X)$ satisfies (DE).

Then, all weakly compact operators also satisfy (DE).

(Kadets–Shvidkoy–Sirotkin–Werner, 1997 & 2000)

* X^* Daugavet property \implies X Daugavet property

EXAMPLES

- K perfect, μ atomeless, X arbitrary Banach space
 $\implies C(K, X)$, $L_1(\mu, X)$, and $L_\infty(\mu, X)$ have the Daugavet property.

(Kadets, 1996; Nazarenko, –; Martín–Villena, 2003)

THE DAUGAVET PROPERTY

- K arbitrary compact space. If X has the Daugavet property, then so does $C(K, X)$.

(Martín–Payá, 2000)

- $A(\mathbb{D})$ and H^∞ have the Daugavet property.

(Wojtaszczyk, 1992)

- A C^* -algebra has the Daugavet property if and only if it is non-atomic.
- The predual of a von Neumann algebra has the Daugavet property if and only if the algebra is non-atomic.

(Oikhberg, 2002)

THE DAUGAVET PROPERTY

SOME KNOWN PROPERTIES

Let X be a Banach space with the Daugavet property. Then

- X contains a copy of ℓ_1 .
- X does not embed into a space with unconditional basis.

(Kadets–Shvidkoy–Sirotkin–Werner, 2000)

- X does not have the Radon-Nikodým property.

(Wojtaszczyk, 1992)

- Every weakly-open subset of B_X has diameter 2.

(Shvidkoy, 2000)

THE DAUGAVET PROPERTY

PROPOSITION (KSSW, 2000)

X Banach space. TFAE:

- (i) X has the Daugavet property.
- (ii) For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y \in S_X$ such that

$$\operatorname{Re} x^*(y) > 1 - \varepsilon \quad \text{and} \quad \|x + y\| \geq 2 - \varepsilon.$$

- (iii) For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y^* \in S_{X^*}$ such that

$$\operatorname{Re} y^*(x) > 1 - \varepsilon \quad \text{and} \quad \|x^* + y^*\| \geq 2 - \varepsilon.$$

- (iv) For every $x \in S_X$ and every $\varepsilon > 0$, we have

$$B_X = \overline{\operatorname{co}}(\{y \in B_X : \|x - y\| \geq 2 - \varepsilon\}).$$

New sufficient conditions

THEOREM

Let X be a Banach space such that

$$X^* = Y \oplus_1 Z$$

with Y and Z 1-norming subspaces. Then, X has the Daugavet property.

COROLLARY

- X L -embedded without extreme points. Then, X^* (and hence X) has the Daugavet property.
- Y L -embedded, $Y \subseteq L_1[0, 1]$. Then $(L_1[0, 1]/Y)^*$ has the Daugavet property.

Von Neumann preduals

Let X_* be the predual of the von Neumann algebra X .

- X_* is L -embedded.
- Therefore, if $\text{ex}(B_{X_*})$ is empty, then X and X_* have the Daugavet property.

Actually, more can be proved:

THEOREM

X_* the predual of the von Neumann algebra X . TFAE:

- (i) X has the Daugavet property.
- (ii) X_* has the Daugavet property.
- (iii) Every relative weak open subset of B_{X_*} has diameter 2.
- (iv) B_{X_*} has no strongly exposed points.
- (v) B_{X_*} has no extreme points.
- (vi) X is **non-atomic**, i.e., there is no $p \in X$ such that

$$p^2 = p^* = p \quad \text{and} \quad p X p = \mathbb{C}p.$$

Let X be a von Neumann algebra.

- X decomposes as $\mathcal{A} \oplus_{\infty} \mathcal{N}$, where \mathcal{A} is purely atomic and \mathcal{N} has no atoms.
- Then, X_* decomposes as $A \oplus_1 N$, where A is generated by its extreme points and N has no extreme points.

COROLLARY

In the natural decomposition $X_* = A \oplus_1 N$, we have

- N has the Daugavet property and
- A has the RNP.

C^* -algebras

Let X be a C^* -algebra. Then, X^{**} is a von Neumann algebra and, as before,

$$X^* = (X^{**})_* = A \oplus_1 N$$

- A is generated by the extreme points of X^*
- B_N has no extreme points

Therefore,

- A is 1-norming for X (Krein-Milman Theorem)
- What's about N ?

PROPOSITION

If X is non-atomic, then N is 1-norming for X .
Therefore, X has the Daugavet property.

Actually, more can be proved:

THEOREM

Let X be a C*-algebra. TFAE:

- (i) X has the Daugavet property.
- (ii) X is non-atomic.
- (iii) The norm of X is extremely rough, i.e.,

$$\limsup_{\|h\| \rightarrow 0} \frac{\|x + h\| + \|x - h\| - 2}{\|h\|} = 2$$

for every $x \in S_X$.

- (iv) The norm of X is not Fréchet-smooth at any point.

REMARK

- If X is an arbitrary infinite-dimensional C^* -algebra, then every relative weak-open subset of B_X has diameter 2.
- If X is an arbitrary infinite-dimensional von Neumann algebra, then the norm of X_* is extremely rough.

(Becerra–López–Rodríguez-Palacios, 2003)

The uniform Daugavet property

A Banach space X is said to have the **Uniform Daugavet property (UDP)** if, for every $\varepsilon > 0$,

$$\inf\{n \in \mathbb{N} : \text{conv}_n(l^+(x, \varepsilon)) \supset S_X \ \forall x \in S_X\} < \infty$$

where conv_n denotes the set of convex combinations of n -point collections and

$$l^+(x, \varepsilon) = \{y \in (1 + \varepsilon)B_X : \|x + y\| > 2 - \varepsilon\}.$$

(Bilik–Kadets–Shvidkoy–Werner, 2004)

- X has the UDP iff $X_{\mathcal{U}}$ has the Daugavet property for every free ultrafilter \mathcal{U} of \mathbb{N} .

THE UNIFORM DAUGAVET PROPERTY

EXAMPLES

- If K is perfect, $C(K)$ has the UDP.
- $L_1[0, 1]$ has the UDP.

(Bilik–Kadets–Shvidkoy–Werner, 2004)

- There exists X having the Daugavet property but not the UDP.

(Kadets–Werner, 2004)

THEOREM

The UDP and the Daugavet property are equivalent for C^* -algebras and von Neumann preduals.

SKETCH OF THE PROOF

- For C^* -algebras:
 - The ultrapower of a C^* -algebra is a C^* -algebra.
 - The roughness of the norm passes to ultrapower.

- For von Neumann preduals:
 - We do not know if the ultrapower of a von Neumann predual is again a von Neumann predual.
 - But, it is the predual of a JBW^* -triple.
 - The geometrical characterization is valid for preduals of JBW^* -triples.
 - If all the slices of B_X have diameter 2, then the unit ball of $X_{\mathcal{U}}$ has no strongly exposed points.

The alternative Daugavet property

X Banach space, $T \in L(X)$

$$\max_{\omega \in \mathbb{T}} \|Id + \omega T\| = 1 + \|T\| \quad (\text{aDE})$$

- X is said to have the **alternative Daugavet property** if every rank-one operator $T \in L(X)$ satisfies (aDE).
Then, all weakly compact operators also satisfy (aDE).

(Martin–Oikhberg, 2004)

- If **all** the operators $T \in L(X)$ satisfy (aDE), X is said to have **numerical index 1**.

(Lumer, 1968)

THE ALTERNATIVE DAUGAVET PROPERTY

For a C^* -algebra X :

- X has the Daugavet property iff it does not have any atomic projection.
- X has numerical index 1 iff it is commutative.

(Huruya, 1977)

- X has the alternative Daugavet property iff the atomic projections are central. In such a case:
 - If X is a von Neumann algebra, then $X = C \oplus_{\infty} D$ with C commutative and D having the Daugavet property.
 - No such decomposition is possible for general C^* -algebras.

(Martín–Oikhberg, 2004)