UNIVERSIDAD DE GRANADA

The Daugavet property

of C*-algebras and von Neumann preduals

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The Daugavet equation

X Banach space, $T \in L(X)$

$$\|Id + T\| = 1 + \|T\| \qquad \text{(DE)}$$

- **Daugavet, 1963:**
  Every compact operator on $C[0,1]$ satisfies (DE).

- **Lozanoskii, 1966:**
  Every compact operator on $L_1[0,1]$ satisfies (DE).

- **Abramovich, Holub and more, 80's:**
  $X = C(K)$, K perfect compact space
  or $X = L_1(\mu)$, μ atomless measure,
  $\Longrightarrow$ every weakly compact $T \in L(X)$ satisfies (DE).

The Daugavet property

A Banach space X is said to have the Daugavet property if every rank-one operator $T \in L(X)$ satisfies (DE).

* Then, all weakly compact operators also satisfy (DE).

(Kadets–Shvidkoy–Sirotkin–Werner, 1997 & 2000)

* X^* Daugavet property $\Longrightarrow$ X Daugavet property

Examples

- K perfect, μ atomeless, X arbitrary Banach space $\Longrightarrow$ $C(K,X)$, $L_1(\mu,X)$, and $L_\infty(\mu,X)$ have the Daugavet property.

(Kadets, 1996; Nazarenko, –; M.–Villena, 2003)

- K arbitrary compact space. If X has the Daugavet property, then so does $C(K, X)$.

(M.–Payá, 2000)

- $A(\mathbb{D})$ and H^∞ have the Daugavet property.

(Wojtaszczyk, 1992)

- A C^*-algebra has the Daugavet property if and only if it is non-atomic.

- The predual of a von Neumann algebra has the Daugavet property if and only if the algebra is non-atomic.

(Oikhberg, 2002)

Some known properties

Let X be a Banach space with the Daugavet property. Then

- X contains a copy of ℓ_1.
- X does not embed into a space with unconditional basis.

(Kadets–Shvidkoy–Sirotkin–Werner, 2000)

- X does not have the Radon-Nikodým property.

(Wojtaszczyk, 1992)

- Every weakly-open subset of B_X has diameter 2.

(Shvidkoy, 2000)

PROPOSITION (KSSW, 2000)
X Banach space. TFAE:

(i) X has the Daugavet property.

(ii) For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y \in S_X$ such that

$$\mathrm{Re}\, x^*(y) > 1 - \varepsilon \quad \text{and} \quad \|x + y\| \geqslant 2 - \varepsilon.$$

(iii) For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there exists $y^* \in S_{X^*}$ such that

$$\mathrm{Re}\, y^*(x) > 1 - \varepsilon \quad \text{and} \quad \|x^* + y^*\| \geqslant 2 - \varepsilon.$$

(iv) For every $x \in S_X$ and every $\varepsilon > 0$, we have

$$B_X = \overline{\mathrm{co}}\big(\{y \in B_X \ : \ \|x - y\| \geqslant 2 - \varepsilon\}\big).$$

New sufficient conditions

Theorem

Let X be a Banach space such that

$$X^* = Y \oplus_1 Z$$

with Y and Z 1-norming subspaces. Then, X has the Daugavet property.

Corollary

- X L-embedded without extreme points. Then, X^* (and hence X) has the Daugavet property.
- $Y \subseteq L_1[0,1]$, Y L-embedded. Then $\big(L_1[0,1]/Y\big)^*$ has the Daugavet property.

Von Neumann preduals

Let X_* be the predual of the von Neumann algebra X.

- X_* is L-embedded.
- Therefore, if $\mathrm{ex}\,(B_{X_*})$ is empty, then X and X_* have the Daugavet property.

Actually, more can be proved:

THEOREM

X_* the predual of the von Neumann algebra X. TFAE:

(i) X has the Daugavet property.

(ii) X_* has the Daugavet property.

(iii) Every relative weak open subset of B_{X_*} has diameter 2.

(iv) B_{X_*} has no strongly exposed points.

(v) B_{X_*} has no extreme points.

(vi) X is non-atomic, i.e., there is no $p \in X$ such that

$$p^2 = p^* = p \qquad \text{and} \qquad p\,X\,p = \mathbb{C}p.$$

Let X be a von Neumann algebra.

- X decomposes as $\mathcal{A} \oplus_\infty \mathcal{N}$, where $\mathcal{A}$ is purely atomic and $\mathcal{N}$ has no atoms.
- Then, X_* decomposes as $A \oplus_1 N$, where A is generated by its extreme points and N has no extreme points.

Corollary

In the natural decomposition $X_* = A \oplus_1 N$, we have

- N has the Daugavet property and
- A has the RNP.

C^*-algebras

Let X be a C^*-algebra. Then, X^{**} is a von Neumann algebra and, as before,

$$X^* = \left(X^{**}\right)_* = A \oplus_1 N$$

- A is generated by the extreme points of X^*
- B_N has no extreme points

COROLLARY

- The dual of a C^*-algebra does not have the Daugavet property.
- A C^*-algebra $X = Z^{**}$ does not have the Daugavet property.

Let X be a C^*-algebra. Write $X^* = A \oplus_1 N$.

- A is 1-norming for X (Krein-Milman Theorem)
- What's about N?

Proposition

If X is non-atomic, then N is 1-norming for X.
Therefore, X has the Daugavet property.

Actually, more can be proved:

Theorem

Let X be a C^*-algebra. TFAE:

(i) X has the Daugavet property.

(ii) X is non-atomic.

(iii) The norm of X is extremely rough, i.e.,

$$\limsup_{\|h\|\to 0} \frac{\|x+h\| + \|x-h\| - 2}{\|h\|} = 2$$

for every $x \in S_X$.

(iv) The norm of X is not Fréchet-smooth at any point.

Remark

- If X is an arbitrary infinite-dimensional C^*-algebra, then every relative weak-open subset of B_X has diameter 2.
- If X is an arbitrary infinite-dimensional von Neumann algebra, then the norm of X_* is extremely rough.

(Becerra–López–Rodríguez-Palacios, 2003)

The uniform Daugavet property

A Banach space X is said to have the Uniform Daugavet property (UDP) if, for every $\varepsilon > 0$,

$$\inf\{n \in \mathbb{N} \ : \ \mathrm{conv}_n(l^+(x,\varepsilon)) \supset S_X \ \forall x \in S_X\} < \infty$$

where conv_n denotes the set of convex combinations of n-point collections and

$$l^+(x,\varepsilon) = \big\{y \in (1+\varepsilon)B_X \ : \ \|x+y\| > 2-\varepsilon\big\}.$$

(Bilik–Kadets–Shvidkoy–Werner, 2004)

- X has the UDP iff $X_{\mathcal{U}}$ has the Daugavet property for every free ultrafilter $\mathcal{U}$ of $\mathbb{N}$.

Examples

- If K is perfect, $C(K)$ has the UDP.
- $L_1[0,1]$ has the UDP.

(Bilik–Kadets–Shvidkoy–Werner, 2004)

- There exists X having the Daugavet property but not the UDP.

(Kadets–Werner, 2004)

Theorem

The UDP and the Daugavet property are equivalent for C^*-algebras and for von Neumann preduals.

Sketch of the proof

- For C^*-algebras:
  - The ultrapower of a C^*-algebra is a C^*-algebra.
  - The roughness of the norm passes to ultrapower.

- For von Neumann preduals:
  - We do not know if the ultrapower of a von Neumann predual is again a von Neumann predual.
  - But, it is the predual of a JBW^*-triple.
  - The geometrical characterization is valid for preduals of JBW^*-triples.
  - If all the slices of B_X have diameter 2, then the unit ball of $X_{\mathcal{U}}$ has no strongly exposed points.

The alternative Daugavet property

X Banach space, $T \in L(X)$

$$\max_{\omega\in\mathbb{T}} \|Id + \omega\, T\| = 1 + \|T\| \qquad \text{(aDE)}$$

- X is said to have the alternative Daugavet property if every rank-one operator $T \in L(X)$ satisfies (aDE).
  $*$ Then, all weakly compact operators also satisfy (aDE).

(M.–Oikhberg, 2004)

- If **all** the operators $T \in L(X)$ satisfy (aDE), X is said to have numerical index 1.

(Lumer, 1968)

$*$ For a C^*-algebra X:

- The Daugavet property is equivalent to:
  - X does not have any atomic projection, or
  - B_{X^*} has no w^*-strongly exposed points.
- The numerical index 1 is equivalent to:
  - X is commutative, or
  - $|x^{**}(x^*)| = 1$ for $x^{**} \in \mathrm{ex}\,(B_{X^{**}})$ and $x^* \in \mathrm{ex}\,(B_{X^*})$.

(Huruya, 1977)

- The alternative Daugavet property is equivalent to:
  - the atomic projections of X are central, or

    (M.–Oikhberg, 2004)
  - $|x^{**}(x^*)| = 1$ for every $x^{**} \in \operatorname{ex}(B_{X^{**}})$ and every w^*-strongly exposed point x^* of B_{X^*}, or
  - There is a commutative ideal Y of X such that X/Y has the Daugavet property.

$*$ For the predual V_* of a von Neumann algebra V:

- The Daugavet property of V_* is equivalent to:
  - V has the Daugavet property, or
  - V_* has no extreme points.

- The numerical index 1 of V_* is equivalent to:
  - V has numerical index 1, or

    (Kaidi–Morales–Rodríguez-Palacios, 2001)
  - $|v^*(v)| = 1$ for $v^* \in \operatorname{ex}(B_{V^*})$ and $v \in \operatorname{ex}(B_V)$.

- The alternative Daugavet property of V_* is equivalent to:
  - V has the alternative Daugavet property, or
  - $|v(v_*)| = 1$ for $v \in \operatorname{ex}(B_V)$ and $v_* \in \operatorname{ex}(B_{V_*})$, or
  - $V = C \oplus_\infty N$, where C has numerical index 1 and N has the Daugavet property.