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Finite-dimensional Banach spaces with numerical index zero

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Hermitian operators

H complex Hilbert space, $T \in L(H)$

$$
\begin{aligned}
T \text{ is } \textbf{Hermitian} \quad &\Longleftrightarrow \quad T = T^* \\
&\Longleftrightarrow \quad \big(Tx \mid x\big) \in \mathbb{R} \quad \forall x \in H \\
&\Longleftrightarrow \quad \big\|\exp(i\rho T)\big\| = 1 \;\; \forall \rho \in \mathbb{R}
\end{aligned}
$$

$$
\begin{aligned}
iT \text{ is Hermitian} \quad &\Longleftrightarrow \quad T^* = -T \\
&\Longleftrightarrow \quad \mathrm{Re}\ \big(Tx \mid x\big) = 0 \quad \forall x \in H \\
&\Longleftrightarrow \quad \big\|\exp(\rho T)\big\| = 1 \;\; \forall \rho \in \mathbb{R}
\end{aligned}
$$

$$\mathrm{Re}\ \big(Tx \mid x\big) = 0 \quad \forall x \in H$$

$$\Updownarrow$$

$$\mathrm{Re}\ x^*(Tx) = 0 \;\; x \in S_H,\; x^* \in S_{H^*},\; x^*(x) = 1$$

Numerical Range of operators

- H Hilbert space, $T \in L(H)$

$$V(T) = \{(Tx \mid x) \ : \ x \in H, \ \|x\| = 1\}$$

(Toeplitz, 1918)

- X Banach space, $T \in L(X)$

$$V(T) = \{x^*(Tx) \ : \ \|x\| = \|x^*\| = x^*(x) = 1\}$$

(Lumer, 1961; Bauer, 1962)

- **Numerical radius:**

  X Banach space, $T \in L(X)$,

$$v(T) = \sup\{|\lambda| \ : \ \lambda \in V(T)\}$$

v is a continuous seminorm:

$$v(T) \leqslant \|T\|$$

- **Numerical index:**

  X Banach space,

$$\begin{aligned} n(X) &= \inf\{v(T) \ : \ T \in L(X), \ \|T\| = 1\} \\ &= \max\{k \geqslant 0 \ : \ k\|T\| \leqslant v(T) \ \ \forall T \in L(X)\} \end{aligned}$$

(Lumer, 1970)

- $n(X) = 0 \iff v$ is not an equivalent norm

- $0 \leqslant n(X) \leqslant 1$ if X is real

  $\frac{1}{\mathrm{e}} \leqslant n(X) \leqslant 1$ if X is complex

(Bohnenblust-Karlin, 1955; Glickfeld, 1970)

Some examples and results

- H Hilbert space, $\dim(H) > 1$
  $$\Longrightarrow \begin{cases} n(H) = 0 & \text{if } H \text{ is real} \\ n(H) = 1/2 & \text{if } H \text{ is complex} \end{cases}$$

- $n\big(L_1(\mu)\big) = 1$
  $X^* \equiv L_1(\mu) \implies n(X) = 1$
  In particular, $n\big(C(K)\big) = 1$

  (Duncan-McGregor-Pryce-White, 1970)

- X real, $\dim(X) = \infty$, RNP, $n(X) = 1 \implies X \supset \ell_1$

  (López-Martín-Payá, 1999)

- $(X, \|\cdot\|)$ Banach space, $\dim(X) > 1$, write

$$\mathcal{N}(X) = \{n(X, |||\cdot|||) \ : \ |||\cdot||| \equiv \|\cdot\|\}$$

Then

☐ $0 \in \mathcal{N}(X)$ in the real case
$1/\mathrm{e} \in \mathcal{N}(X)$ in the complex case

☐ $\mathcal{N}(X)$ is a non-trivial interval

☐ If X has a "long biorthogonal system"
(v.g. if X is WCG),

$$\Longrightarrow \begin{cases} \mathcal{N}(X) \supset [0,1) & \text{real case} \\ \mathcal{N}(X) \supset [\mathrm{e}^{-1},1) & \text{complex case} \end{cases}$$

(Finet-Martín-Payá, 2003)

Real Banach spaces with numerical index zero

EXAMPLES:

☐ H real Hilbert space, $\dim(H) > 1 \implies n(H) = 0$

☐ X complex Banach space $\implies n\big(X_{\mathbb{R}}\big) = 0$

$$\big(Tx = ix \ \forall x \in X, \quad v(T) = 0\,\big)$$

☐ $n(Z) = 0$, Y arbitrary, $X = Y \oplus Z$ (absolute sum)

$\implies n(X) = 0$

Proposition

X real Banach space, Y, $Z \leqslant X$, $Z \neq 0$.

If Z has a complex structure, $X = Y \oplus Z$, and

$$\left\| y + \mathrm{e}^{i\rho}\, z \right\| = \| y + z \| \quad \big(\rho \in \mathbb{R},\ y \in Y,\ z \in Z\big),$$

then $n(X) = 0$.

Moreover, $T(y,z) = (0, i\,z)$ has numerical radius 0.

Example

Exists a real polyhedral Banach space X such that $n(X) = 0$

□ X does not contains $\mathbb{C}$ isometrically

□ $v(T) > 0$ for every $T \in L(X) \setminus \{0\}$

Finite-dimensional spaces

THEOREM

X finite-dimensional real space. TFAE:

(i) $n(X) = 0$

(ii) Exist nonzero complex $X_1, \dots, X_n$, a real X_0, and $q_1, \dots, q_n \in \mathbb{N}$ such that $X = X_0 \oplus X_1 \oplus \cdots \oplus X_n$ and

$$\left\| x_0 + \mathrm{e}^{iq_1\rho} x_1 + \cdots + \mathrm{e}^{iq_n\rho} x_n \right\| = \|x_0 + \cdots + x_n\|$$

for every $\rho \in \mathbb{R}$ and every $x_j \in X_j$ $(j = 0, 1, \dots, n)$.

Sketch of the proof

☐ Fix $T \in L(X) \setminus \{0\}$ such that $v(T) = 0$

☐ A Theorem by Bohnenblust & Karlin:

$$v(T) = 0 \quad \Longleftrightarrow \quad \big\| \exp(\rho T) \big\| = 1 \ \ \forall \rho \in \mathbb{R}$$

☐ A Theorem by Auerbach: there exists a Hilbert space H with $\dim(H) = \dim(X)$ such that every surjective isometry in $L(X)$ remains isometry in $L(H)$

☐ Apply the above to $\exp(\rho T)$ for every $\rho \in \mathbb{R}$

☐ You get iT is hermitian in $L(H)$, so $T^* = -T$ and T^2 is self-adjoint. The X_j's are the eigenspaces of T^2.

☐ Use Kronecker's Approximation Theorem to change the roots of the characteristic polynomial of T^2 by rational numbers

A simple case

Let $X = X_0 \oplus X_1 \oplus X_2$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ s.t.

$$\left\| x_0 + \mathrm{e}^{i\rho}\, x_1 + \mathrm{e}^{i\alpha\rho}\, x_2 \right\| = \| x_0 + x_1 + x_2 \| \quad \forall \rho,\ \forall x_0, x_1, x_2$$

Then

$$\| x_0 + x_1 + x_2 \| = \left\| x_0 + \mathrm{e}^{i\rho} \left(x_1 + \mathrm{e}^{i(\alpha-1)\rho}\, x_2 \right) \right\| \quad \forall \rho$$

Take $\rho = \dfrac{2\pi k}{\alpha - 1}$ with $k \in \mathbb{Z}$. Then

$$\| x_0 + (x_1 + x_2) \| = \left\| x_0 + \mathrm{e}^{i\,\frac{2\pi k}{\alpha-1}} (x_1 + x_2) \right\| \quad \forall k \in \mathbb{Z}$$

But $\left\{ \dfrac{2\pi k}{\alpha - 1} \ : \ k \in \mathbb{Z} \right\}$ is dense in $\mathbb{T}$, so

$$\| x_0 + (x_1 + x_2) \| = \left\| x_0 + \mathrm{e}^{i\rho} (x_1 + x_2) \right\| \quad \forall \rho \in \mathbb{R}$$

and $X = X_0 \oplus Z$ where $Z = X_1 \oplus X_2$ is a complex space

Corollary

X real Banach space with $n(X) = 0$

☐ If $\dim(X) = 2$, then $X \equiv \mathbb{C}$

☐ If $\dim(X) = 3$, then $X \equiv \mathbb{R} \oplus \mathbb{C}$ (absolute sum)

Corollary

X real Banach space, consider the subspace of $L(X)$

$$Z(X) = \big\{ T \in L(X) \ : \ v(T) = 0 \big\}$$

$\square$ $\dim(X) = n \implies \dim\big(Z(X)\big) \leqslant \dfrac{n(n-1)}{2}$

$\square$ $\dim\big(Z(X)\big) = \dfrac{n(n-1)}{2} \iff X$ is a Hilbert space

Example

Exists a real Banach space X such that $\dim(X) = 4$, $n(X) = 0$, and the numbers of complex spaces in the theorem cannot be reduced to one.

$X = (\mathbb{R}^4, \|\cdot\|)$ where

$$\big\|(a,b,c,d)\big\| = \frac{1}{4}\int_0^{2\pi} \left|\mathrm{Re}\,\left(\mathrm{e}^{2it}(a+ib) + \mathrm{e}^{it}(c+id)\right)\right|\ dt,$$

which satisfies

$$\left\|\mathrm{e}^{2i\rho}(a,b,0,0) + \mathrm{e}^{i\rho}(0,0,c,d)\right\| = \|(a,b,c,d)\|$$

for every $\rho \in \mathbb{R}$ and every $a,b,c,d \in \mathbb{R}$.

In this case, $\dim\big(Z(X)\big) = 1$ and $Z(X)$ is generated by

$$T \equiv \begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$