UNIVERSIDAD DE GRANADA

Existence of hermitian operators on finite-dimensional Banach spaces: geometrical consequences

Miguel Martín, Javier Merí, Ángel Rodríguez-Palacios

• Hermitian operators

H complex Hilbert space, $T \in L(H)$

$$
\begin{aligned}
T \text{ is } \textbf{Hermitian} \quad &\Longleftrightarrow \quad T = T^* \\
&\Longleftrightarrow \quad \big(Tx \mid x\big) \in \mathbb{R} \quad \forall x \in H \\
&\Longleftrightarrow \quad \big\| \exp(i\rho T) \big\| = 1 \ \ \forall \rho \in \mathbb{R}
\end{aligned}
$$

$$
\begin{aligned}
iT \text{ is Hermitian} \quad &\Longleftrightarrow \quad T^* = -T \\
&\Longleftrightarrow \quad \mathrm{Re}\ \big(Tx \mid x\big) = 0 \quad \forall x \in H \\
&\Longleftrightarrow \quad \big\| \exp(\rho T) \big\| = 1 \ \ \forall \rho \in \mathbb{R}
\end{aligned}
$$

$$iT \text{ is Hermitian} \iff \operatorname{Re}\big(Tx \mid x\big) = 0 \quad \forall x \in H$$

$$\iff \big\|\exp(\rho T)\big\| = 1 \quad \forall \rho \in \mathbb{R}$$

$$\operatorname{Re}\big(Tx \mid x\big) = 0 \quad \forall x \in H$$

$$\Updownarrow$$

$$\operatorname{Re} x^*(Tx) = 0 \;\text{ when } x \in S_H,\; x^* \in S_{H^*},\; x^*(x) = 1$$

$$\Updownarrow$$

$$x^*(Tx) = 0 \;\text{ when } x \in S_X,\; x^* \in S_{X^*},\; x^*(x) = 1$$

(calling $X = H_{\mathbb{R}}$)

• Numerical Range

X Banach space, $T \in L(X)$

□ **Numerical range:**

$$V(T) = \{x^*(Tx) \ : \ x \in S_X, \ x^* \in S_{X^*}, \ x^*(x) = 1\}$$

□ **Numerical radius:**

$$v(T) = \sup\{|\lambda| \ : \ \lambda \in V(T)\}$$

Theorem (Bohnenblust-Karlin, 1955)

□ X real Banach space, $T \in L(X)$

$$v(T) = 0 \quad \Longleftrightarrow \quad \big\| \exp(\rho T) \big\| = 1 \ \forall \rho \in \mathbb{R}$$

$$\big(iT \text{ is hermitian in } L(X)_{\mathbb{C}} \big)$$

☐ **Numerical index:**

$$n(X) = \inf\{v(T)\ :\ \|T\| = 1\}$$
$$= \max\{k \geqslant 0\ :\ k\|T\| \leqslant v(T)\ \forall T\}$$

$$0 \leqslant n(X) \leqslant 1 \text{ if } X \text{ is real}$$
$$\frac{1}{\mathrm{e}} \leqslant n(X) \leqslant 1 \text{ if } X \text{ is complex}$$

☐ $\exists T \neq 0$ with $v(T) = 0 \quad \Rightarrow \quad n(X) = 0$

$\not\Leftarrow$

☐ Of course, if $\dim(X) < \infty$, the equivalence holds

• Real Banach spaces with numerical index zero

Some known examples:

☐ H Hilbert space, $\dim(H) > 1 \implies n(H) = 0$

☐ X complex Banach space $\implies n\big(X_{\mathbb{R}}\big) = 0$

$$\big(Tx = ix \ \forall x \in X, \ \ v(T) = 0\,\big)$$

☐ $n(Z) = 0$, Y arbitrary, $X = Y \oplus Z$ (absolute sum)
$\implies n(X) = 0$

Proposition

X real Banach space, Y, $Z \leqslant X$, $Z \neq 0$.
If Z has a complex structure, $X = Y \oplus Z$, and

$$\left\| y + e^{i\rho} z \right\| = \|y + z\| \quad \big(\rho \in \mathbb{R},\ y \in Y,\ z \in Z\big),$$

then $n(X) = 0$.
Moreover, $T(y, z) = (0, i\,z)$ has numerical radius 0.

Example

Exists a real polyhedral Banach space X such that $n(X) = 0$

- X does not contains $\mathbb{C}$ isometrically
- $v(T) > 0$ for every $T \in L(X) \setminus \{0\}$

• Finite dimension

THEOREM

X finite-dimensional real space. TFAE:

(i) $n(X) = 0$

(ii) Exist nonzero complex spaces $X_1, \ldots, X_n$, a real space X_0, and $q_1, \ldots, q_n \in \mathbb{N}$ such that $X = X_0 \oplus X_1 \oplus \cdots \oplus X_n$ and

$$\left\| x_0 + e^{iq_1\rho}\, x_1 + \cdots + e^{iq_n\rho}\, x_n \right\| = \|x_0 + \cdots + x_n\|$$

for every $\rho \in \mathbb{R}$ and every $x_j \in X_j$ $(j = 0, 1, \ldots, n)$.

Sketch of the proof

☐ Fix $T \in L(X) \setminus \{0\}$ such that $v(T) = 0$

☐ A Theorem by Auerbach: there exists a Hilbert space H with $\dim(H) = \dim(X)$ such that every surjective isometry in $L(X)$ remains isometry in $L(H)$

☐ Apply the above to $\exp(\rho T)$ for every $\rho \in \mathbb{R}$

☐ Use Kronecker's Approximation Theorem to change the roots of the characteristic polynomial of T by rational numbers

Corollary

X real Banach space with $n(X) = 0$

- ☐ If $\dim(X) = 2$, then $X \equiv \mathbb{C}$
- ☐ If $\dim(X) = 3$, then $X \equiv \mathbb{R} \oplus \mathbb{C}$ (absolute sum)

Corollary

X real Banach space, consider the subspace of $L(X)$

$$Z(X) = \{T \in L(X) \ : \ v(T) = 0\}$$

□ $\dim(X) = n \implies \dim\big(Z(X)\big) \leqslant \dfrac{n(n-1)}{2}$

□ $\dim\big(Z(X)\big) = \dfrac{n(n-1)}{2} \iff X$ is a Hilbert space

Example

Exists a real Banach space X such that $\dim(X) = 4$, $n(X) = 0$, and the numbers of complex spaces in the theorem cannot be reduced to one.

$X = (\mathbb{R}^4, \|\cdot\|)$, where

$$\|(a,b,c,d)\| = \frac{1}{4}\int_0^{2\pi} \left|\mathrm{Re}\,\left(\mathrm{e}^{2it}(a+ib) + \mathrm{e}^{it}(c+id)\right)\right|\, dt,$$

which satisfies

$$\left\|\mathrm{e}^{2i\rho}(a,b,0,0) + \mathrm{e}^{i\rho}(0,0,c,d)\right\| = \|(a,b,c,d)\|$$

for every $\rho \in \mathbb{R}$ and every $a,b,c,d \in \mathbb{R}$.

In this case, $\dim\big(Z(X)\big) = 1$ and $Z(X)$ is generated by

$$T \equiv \begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$