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CL-spaces and almost-CL-spaces

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Definitions

X Banach space.

- X is a CL-space iff $B_X = \mathrm{aco}(F)$ for every maximal convex subset F of S_X

(Fullerton, 1961)

- X is a almost-CL-space iff $B_X = \overline{\mathrm{aco}}(F)$ for every maximal convex subset F of S_X

(Lindenstrauss, 1964; Lima, 1978)

CL-espacio	$\Longrightarrow$	casi-CL-espacio

Already known examples

- Real $L_1(\mu)$ spaces are CL-spaces
- In the real case, if $X^* \equiv L_1(\mu)$, then X is a CL-space
- In particular, real $C(K)$ spaces are CL-spaces
- If $\dim(X) < \infty$, then X is a CL-space iff

$$|x^*(x)| = 1 \quad \forall x^* \in \text{ex}\,(B_{X^*}) \;\; \forall x \in \text{ex}\,(B_X)$$

Question

Is there any real almost-CL-space
which is not a CL-space **?**

New examples

- Complex $C(K)$ spaces are CL-spaces
- Complex $L_1(\mu)$ spaces are almost-CL-spaces
- The complex spaces ℓ_1 and $L_1[0,1]$ are not CL-spaces

Some isomorphic results

- ☐ X Banach space, then every maximal convex subset of S_X has the form

$$F(x^*) = \{x \in S_X \ : \ x^*(x) = 1\}$$

for a suitable extreme point x^* of B_{X^*}

- ☐ Write

$$\operatorname{mex}(B_{X^*}) = \left\{x^* \in \operatorname{ex}(B_{X^*}) \ : \ F(x^*) \text{ is maximal}\right\}$$

Lemma

X almost-CL-space. Then

$$|x^{**}(x^*)| = 1 \quad \forall x^{**} \in \mathrm{ex}\,(B_{X^{**}})\,,\ \forall x^* \in \mathrm{mex}\,(B_{X^*})$$

Proposition

X real Banach space, $\exists\, A \subset S_X$ with $\#A = \infty$ and

$$|x^*(a)| = 1 \quad \forall x^* \in \mathrm{ex}\,(B_{X^*})\,,\ \forall a \in A.$$

Then, $X \supset c_0$ or $X \supset \ell_1$

(López-Martín-Payá, 1999)

Theorem

X real almost-CL-space, $\dim(X) = \infty \implies X^* \supset \ell_1$.

If in adition X^* is separable $\implies X \supset c_0$

Question

X real almost-CL-space, $\dim(X) = \infty$

$$\overset{?}{\Longrightarrow} \begin{cases} X \supset \ell_1 \\ \quad \text{or} \\ X \supset c_0 \end{cases}$$

Some stability results

□ Sums of Banach spaces

PROPOSITION

- c_0 and ℓ_1 sums of almost-CL-spaces are almost-CL-spaces

- A c_0 sum of CL-spaces is a CL-space

- In the real case, an ℓ_1 sum of CL-spaces is a CL-space

- In the complex case, only finite ℓ_1 sums of CL-spaces are CL-spaces

Corollary

The complex space $L_1(\mu)$ is a CL-space iff $\dim\big(L_1(\mu)\big) < \infty$

□ Vector-valued function spaces

PROPOSITION

K compact Hausdorff space, X Banach space.

- $C(K, X)$ is an almost-CL-space iff X is
- If $C(K, X)$ is a CL-space, then X is a CL-space

QUESTION

$C(K, X)$ is a CL-space whenever X is **?**