UNIVERSIDAD DE GRANADA

Departamento de Análisis Matemático

The numerical index of a Banach space

Miguel Martín

Numerical range

H Hilbert space, $T \in L(H)$

$$W(T) = \{(Tx|x) \ : \ x \in H, \ \|x\| = 1\}$$

(Toeplitz, 1918)

X Banach space, $T \in L(X)$

$$W(T) = \{x^*(Tx) \ : \ \|x\| = \|x^*\| = x^*(x) = 1\}$$

(Lumer, 1961; Bauer, 1962)

Numerical radius

$$v(T) = \sup\{|\lambda| \ : \ \lambda \in W(T)\}$$

v is a continuous seminorm; actually

$$v(T) \leq \|T\|$$

Numerical index of a Banach space

X Banach space,

$$\begin{aligned} n(X) &= \max\{k \geq 0 \ : \ k\|T\| \leq v(T) \quad \forall T \in L(X)\} \\ &= \inf\{v(T) \ : \ T \in L(X), \ \|T\| = 1\} \end{aligned}$$

$$0 \leq n(X) \leq 1$$

$$n(X^*) \leq n(X)$$

$$n(X^*) = n(X)$$ **?**

H Hilbert space, $\dim(H) > 1$

$$n(H) = 0 \qquad \text{if } H \text{ is real}$$
$$n(H) = 1/2 \qquad \text{if } H \text{ is complex}$$

X complex space $\Rightarrow$ $n(X) \geq 1/e$

(Bohnenblust-Karlin, 1955; Glickfeld, 1970)

$$\{n(X) \ : \ \text{complex } X,\ \dim(X) = 2\} = [e^{-1}, 1]$$
$$\{n(X) \ : \ \text{real } X,\ \dim(X) = 2\} = [0, 1]$$

(Duncan-McGregor-Pryce-White, 1970)

“Classical” examples

☐ H Hilbert, $\dim(H) > 1$, $n(H) = \begin{cases} 0 & \text{real case} \\ 1/2 & \text{complex case} \end{cases}$

☐ $n\big(L_1(\mu)\big) = 1 \quad \mu$ positive measure

$X^* \equiv L_1(\mu) \quad \Rightarrow \quad n(X) = 1$

For instance, $n\big(C(K)\big) = 1 \quad K$ compact space

(Duncan-McGregor-Pryce-White, 1970)

☐ $A(\mathbb{D})$ disk algebra $\Rightarrow$ $n\big(A(\mathbb{D})\big) = 1$

(Crabb-Duncan-McGregor, 1972)

☐ $n(l_p), \quad p \neq 1, 2, \infty$?

Vector-valued function spaces

(Martín-Payá, 2000; Martín-Villena, preprint)

Proposition

$$n\Big([\oplus_{\lambda\in\Lambda} X_\lambda]_{c_0}\Big) = n\Big([\oplus_{\lambda\in\Lambda} X_\lambda]_{l_1}\Big) = n\Big([\oplus_{\lambda\in\Lambda} X_\lambda]_{l_\infty}\Big) = \inf_\lambda n(X_\lambda)$$

Consequences

☐ The class of Banach spaces with numerical index 1 is stable by c_0, l_1 and l_∞ sums.

☐ X Banach space:

$$n(X) = n\big(c_0(X)\big) = n\big(l_1(X)\big) = n\big(l_\infty(X)\big)$$

□ There exists a Banach space X such that

$$v(T) > 0 \qquad \forall\, T \in L(X) \setminus \{0\}$$

and $n(X) = 0$

□ *Real case:*

For every $t \in [0,1]$, there exist X_t isomorphic to c_0 (or l_1 or l_∞) with $n(X_t) = t$

Complex case:

For every $s \in [e^{-1}, 1]$, there exists X_s isomorphic to c_0 (or l_1 or l_∞) with $n(X_s) = s$

Theorem

$$n\Big(C(K,X)\Big) = n\Big(L_1(\mu,X)\Big) = n\Big(L_\infty(\mu,X)\Big) = n(X)$$

Tensor products

$$C(K, X) = C(K)\widetilde{\otimes}_{\varepsilon} X$$

$$L_1(\mu, X) = L_1(\mu)\widetilde{\otimes}_{\pi} X$$

X, Y Banach spaces:

$$n\big(X\widetilde{\otimes}_{\varepsilon} Y\big) = f\big(n(X), n(Y)\big) \textbf{ ?}$$
$$n\big(X\widetilde{\otimes}_{\pi} Y\big) = g\big(n(X), n(Y)\big) \textbf{ ?}$$

NO: EXAMPLE

$$n\Big(l_1^4\widetilde{\otimes}_{\varepsilon} l_1^4\Big) < 1 = n\Big(l_\infty^4\widetilde{\otimes}_{\varepsilon} l_\infty^4\Big)$$

$$n\Big(l_\infty^4\widetilde{\otimes}_{\pi} l_\infty^4\Big) < 1 = n\Big(l_1^4\widetilde{\otimes}_{\pi} l_1^4\Big)$$

(Lima, 1981)

Numerical index and renorming

(López-Martín-Payá, 1999; Finet-Martín-Payá, 200?)

• The set $\mathcal{N}(X)$

We define the set $\mathcal{N}(X)$ as

$$\mathcal{N}(X) = \{n(Y) \ : \ Y \text{ isomorphic to } X\}$$

EXAMPLES

□ $\mathcal{N}(\mathbb{R}) = \mathcal{N}(\mathbb{C}) = \{1\}$,

□ $\mathcal{N}\big(\mathbb{R}^m\big) = [0,1], \quad \mathcal{N}\big(\mathbb{C}^m\big) = [e^{-1},1] \qquad (m > 1)$

(Duncan et al, 1970; Tillekeratne, 1974)

□ $\mathcal{N}(c_0) = \mathcal{N}(l_1) = \mathcal{N}(l_\infty) = \begin{cases} [0,1] & \text{real case} \\ [e^{-1},1] & \text{complex case} \end{cases}$

(Martín-Payá, 2000)

X Banach space, $\dim(X) = \infty$, $\qquad \mathcal{N}(X)$?

Positive Results:

PROPOSITION

(i) X real space $\Rightarrow$ $0 \in \mathcal{N}(X)$

(ii) X complex space $\Rightarrow$ $e^{-1} \in \mathcal{N}(X)$

PROPOSITION

$\mathcal{N}(X)$ is an interval

COROLLARY

$$1 \in \mathcal{N}(X) \quad \Rightarrow \quad \mathcal{N}(X) = \begin{cases} [0,1] & \text{real case} \\ [e^{-1}, 1] & \text{complex case} \end{cases}$$

Three questions:

- ☐ Is $\mathcal{N}(X)$ a non-trivial interval ?
- ☐ $\sup \mathcal{N}(X) = 1$?
- ☐ $1 \in \mathcal{N}(X)$?

Theorem

X real space $\Rightarrow$ $\mathcal{N}(X) \supseteq [0, 1/3)$

X complex space $\Rightarrow$ $\mathcal{N}(X) \supseteq [e^{-1}, 1/2)$

Theorem

X Banach space having a "long biorthogonal system"

(for example, separable or reflexive or WCG)

X real $\Rightarrow$ $\mathcal{N}(X) \supseteq [0, 1)$

X complex $\Rightarrow$ $\mathcal{N}(X) \supseteq [e^{-1}, 1)$

Negative Results:

X Banach space

$$n(X) = 1 \quad \Leftrightarrow \quad \|T\| = v(T) \quad \forall\, T \in L(X)$$

If $\dim(X) < \infty$, then

$$n(X) = 1 \quad \Leftrightarrow \quad |x^*(x)| = 1 \ \forall x \in \operatorname{ex}(B_X),\ \forall x^* \in \operatorname{ex}(B_{X^*})$$

(McGregor, 1971)

$$|x^{**}(x^*)| = 1 \quad \forall x^* \in \operatorname{ex}(B_{X^*}),\ \forall x^{**} \in \operatorname{ex}(B_{X^{**}})$$

$$\Downarrow \qquad \Uparrow \textbf{?}$$

$$n(X) = 1$$

Lemma

If $n(X) = 1$, then

$$\begin{cases} |x^*(x)| = 1 \quad \forall x^* \in \mathrm{ex}\,(B_{X^*})\,, \ \forall x \in \mathrm{dent}\,(B_X) \\ |x^{**}(x^*)| = 1 \quad \forall x^{**} \in \mathrm{ex}\,(B_{X^{**}})\,, \ \forall x^* \in w^*\!-\!\mathrm{dent}\,(B_{X^*}) \end{cases}$$

Proposition

X real Banach space, A infinite subset of S_X such that $|x^*(a)| = 1$ for all $a \in A$ and $x^* \in \mathrm{ex}\,(B_{X^*})$. Then $X \supseteq c_0$ or $X \supseteq l_1$.

Therefore:

X real Banach space with $n(X) = 1$.

(i) If $\sharp\mathrm{dent}\,(B_X) = \infty \quad \Rightarrow \quad X \supset c_0$ o $X \supset l_1$

(ii) If $\sharp w^*\!-\!\mathrm{dent}\,(B_{X^*}) = \infty \quad \Rightarrow \quad X^* \supset l_1$

CONSEQUENCES

☐ X real space, $\dim(X) = \infty$, RNP, $n(X) = 1$

$\Rightarrow \; X \supset l_1$

☐ X real Asplund space, $\dim(X) = \infty$, $n(X) = 1$

$\Rightarrow \; X^* \supset l_1$

☐ X real Asplund space, RNP, $n(X) = 1$

$\Rightarrow \; \dim(X) < \infty$

☐ X real reflexive or quasi-reflexive space, $n(X) = 1$

$\Rightarrow \; \dim(X) < \infty$.

☐ X real space, $\dim(X) = \infty$, $n(X) = 1$

$\Rightarrow \; X^{**}/X$ is non-separable

☐ X real space, $\dim(X) = \infty$, X^{**}/X separable

$\Rightarrow \; \mathcal{N}(X) = [0, 1)$

SOME OPEN PROBLEMS

(1) Calculate $n(l_p)$

(2) X arbitrary space, $\sup \mathcal{N}(X) = 1$ **?**

(3) Characterize those (real) Banach spaces admiting an equivalent norm with numerical index 1

- Sufficient conditions

- Necessary conditions:

CONJECTURE

X real space, $\dim(X) = \infty$, $n(X) = 1$

$\Rightarrow \; X \supset c_0$ o $X \supset l_1$

(4) $n(X) = n(X^*)$ **?**