Renorming and numerical index

Miguel Martín
Universidad de Granada

Numerical range of an operator

X Banach space, $T \in L(X)$

$$V(T) = \{x^*(Tx) \ : \ \|x\| = \|x^*\| = x^*(x) = 1\}$$

(Toeplitz, 1918; Lumer, 1961; Bauer, 1962)

Numerical radius

$$v(T) = \sup\{|\lambda| \ : \ \lambda \in V(T)\}$$

v is a continuous seminorm:

$$v(T) \leq \|T\|$$

Numerical index of a Banach space

$$n(X) = \max\{k \geq 0 \, : \, k\|T\| \leq v(T) \quad \forall T \in L(X)\}$$
$$= \inf\{v(T) \, : \, T \in L(X), \, \|T\| = 1\}$$

$$0 \leq n(X) \leq 1$$
$$n(X^*) \leq n(X)$$

H Hilbert space, $\dim(H) > 1$

$$n(H) = 0 \qquad \text{if } H \text{ is real}$$
$$n(H) = 1/2 \qquad \text{if } H \text{ is complex}$$

X complex $\Rightarrow$ $n(X) \geq 1/e$

(Bohnenblust-Karlin, 1955; Glickfeld, 1970)

$$\{n(X) \, : \, X \text{ complex}, \, \dim(X) = 2\} = [e^{-1}, 1]$$
$$\{n(X) \, : \, X \text{ real}, \, \dim(X) = 2\} = [0, 1]$$

(Duncan-McGregor-Pryce-White, 1970)

“Clasical” examples

$n\big(L_1(\mu)\big) = 1 \quad \mu$ positive measure

$X^* \equiv L_1(\mu) \;\Rightarrow\; n(X) = 1$

In particular, $n\big(C(K)\big) = 1 \quad K$ compact Hausdorff space

(Duncan et al., 1970)

K compact, μ measure, Y Banach

$$n\big(C(K,Y)\big) = n\big(L_1(\mu,Y)\big) = n(Y)$$

(Martín-Payá, to appear)

A function algebra $\Rightarrow n(A) = 1$

(D. Werner, 1997)

$n(l_p) \;\; p \neq 1, 2, \infty$?

Isomorphic point of view

X Banach $\qquad \mathcal{N}(X) = \{n(Y) \,:\, Y \simeq X\}$

$\mathcal{N}(X)$?

$$\mathcal{N}\big(\mathbb{R}^2\big) = [0,1] \qquad \mathcal{N}\big(\mathbb{C}^2\big) = [e^{-1},1]$$

X reflexive real Banach space,
$\dim(X) = \infty \quad \Rightarrow \quad 1 \notin \mathcal{N}(X)$

X infinite-dimensional real, $1 \in \mathcal{N}(X)$

(i) X has Radon-Nikodým property $\Rightarrow$ $X \hookleftarrow l_1$
(ii) X Asplund space $\Rightarrow$ $X^* \hookleftarrow l_1$

As a consequence:

X real, $\dim(X) = \infty$, $1 \in \mathcal{N}(X)$
$\Rightarrow \quad X^{**}/X$ non-separable

(López-Martín-Payá, 1999)

Positive results (Finet-Martín-Payá)

$\mathcal{N}(\mathbb{R}) = \mathcal{N}(\mathbb{C}) = \{1\}$

So, X Banach space with $\dim(X) > 1$

PROPOSITION

(i) X real $\Rightarrow$ $0 \in \mathcal{N}(X)$

(ii) X complex $\Rightarrow$ $e^{-1} \in \mathcal{N}(X)$

PROPOSITION

$\mathcal{N}(X)$ is an interval

COROLLARY

$$1 \in \mathcal{N}(X) \;\Rightarrow\; \mathcal{N}(X) = \begin{cases} [0,1] & \text{real case} \\ [e^{-1},1] & \text{complex case} \end{cases}$$

can $\mathcal{N}(X)$ reduce to one point ? NO

Lindentrauss' properties (α) and (β)

X has property (α) with constant ρ $\quad (0 \leq \rho < 1)$
iff $\exists\{(x_i, x_i^*\}_{i\in I} \subset X \times X^*$ such that

$$(i) \quad \|x_i\| = \|x_i^*\| = x_i^*(x_i) = 1 \qquad (i \in I)$$

$$(ii) \quad |x_i^*(x_j)| \leq \rho \qquad (i \neq j)$$

$$(iii)_\alpha \quad \|x^*\| = \sup\{|x^*(x_i)| \;:\; i \in I\} \qquad (x^* \in X)$$

If $\{(x_i, x_i^*)\}_{i\in I}$ satisfies (i), (ii), and

$$(iii)_\beta \quad \|x\| = \sup\{|x_i^*(x)| \;:\; i \in I\} \qquad (x \in X)$$

then X has property (β) with constant ρ.

PROPOSITION

X Banach space having property (α) or (β) with constant ρ.

If X is real, then $n(X) \geq \dfrac{1-\rho}{1+\rho}$.

If X is complex, then $n(X) \geq 1-\rho$.

IDEA OF THE PROOF

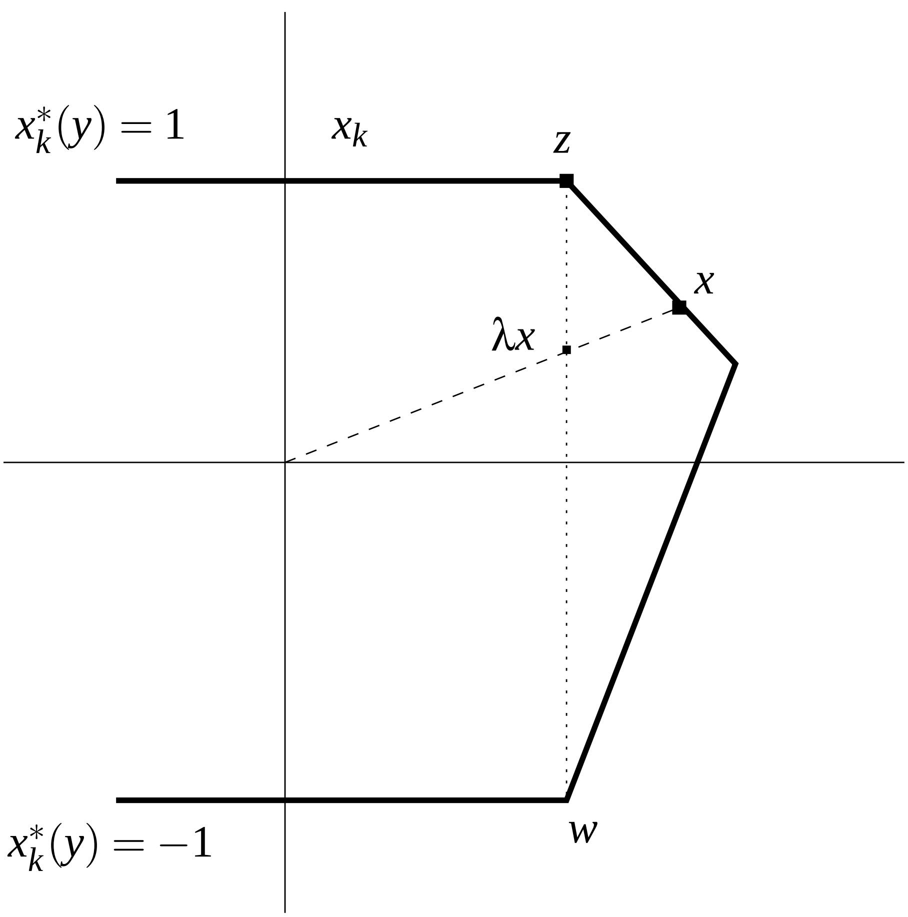
Suppose X has property (β) with constant ρ.

$T \in L(X)$, $\varepsilon > 0$. Take $x \in X$, $\|x\| = 1$ s.t.

$$\|Tx\| > \|T\| - \varepsilon.$$

Take $i \in I$ such that

$$|x_i^*(Tx)| > \|T\| - \varepsilon.$$



Partington 1982:

Every Banach space can be renormed to have property (β) with any constant $\rho \in (1/2, 1)$.

THEOREM

X real Banach space $\Rightarrow \mathcal{N}(X) \supseteq [0, 1/3)$

X complex Banach space $\Rightarrow \mathcal{N}(X) \supseteq [e^{-1}, 1/2)$

Schachermayer 1983; Godun-Troyanski 1993:

"Almost" every Banach space admits equivalent norms verifying property (α) with constant $\rho \in (0, 1)$.

THEOREM

X Banach space with a "large biorthogonal system"
(in particular, X separable or X reflexive)

X real $\Rightarrow \mathcal{N}(X) \supseteq [0, 1)$

X complex $\Rightarrow \mathcal{N}(X) \supseteq [e^{-1}, 1)$

COROLLARY

X real, $\dim(X) = \infty$, X^{**}/X separable $\Rightarrow \mathcal{N}(X) = [0, 1)$