Complete linear Weingarten surfaces of Bryant type. A Plateau problem at infinity.

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Abstract

In this paper we study a large class of Weingarten surfaces which includes the constant mean curvature one surfaces and flat surfaces in the hyperbolic 3-space. We show that these surfaces can be parametrized by holomorphic data like minimal surfaces in the Euclidean 3-space and we use it to study their completeness. We also establish some existence and uniqueness theorems by studing the Plateau problem at infinity: when is a given curve on the ideal boundary the asymptotic boundary of a complete surface in our family? and, how many embedded solutions are there?

Key words and phrases: Hyperbolic 3-space, Weingarten surfaces, Plateau problem, Weierstrass data.

1   Introduction.

During the last two decades there has been a renewed interest in the geometry of surfaces in hyperbolic manifolds. Basic problems such as existence, symmetry, behaviour at infinity, global structure and classification of complete surfaces of constant curvature in the hyperbolic 3-space $\mathbb{H}^3$ have called the attention of several authors (see for example, [5], [11], [12], [18]).

Concerning constant mean curvature one surfaces in $\mathbb{H}^3$, also called *Bryant surfaces*, it was crucial the paper of R. Bryant [2], who showed how to parametrize these surfaces by meromorphic data and began the study of their geometry. Since Bryant's work many properties and examples have been discovered in [3], [15], [22], [25], [26] among others. By considering the complex structure determined by the second fundamental form, the authors proved in [6] that a "holomorphic resolution" like the Bryant representation also holds for

flat surfaces in $\mathbb{H}^3$ and used it to the study of some examples and the behaviour at infinity of complete ends. The most important tool in these representations is that both kinds of surfaces have a conformal hyperbolic Gauss map with respect to a natural complex structure determined by either the first, I, or the second, II, fundamental form.

In this paper we extend the two above mentioned Weierstrass type representations through the investigation of Weingarten surfaces. We shall study surfaces in $\mathbb{H}^3$ whose mean curvature H and Gauss curvature K_I satisfy a linear relation of the form

$$2a(H - 1) + bK_I = 0,$$

for some $a, b \in \mathbb{R}$, of elliptic type, i. e. $a + b \neq 0$. We call them *linear Weingarten surfaces of Bryant type*, in short, BLW-surfaces. These surfaces are a particular case of the special surfaces studied by Sa Earp and Toubiana in [21] (see also [17], [19] and [20]).

A BLW-surface Σ may be considered lying on a straight half-line λ_{ab} in the (H, K_I)-plane, starting at the point $(1, 0)$: $\lambda_{ab} = \{(1 - tb, 2ta) : t \geq 0\}$, see Figure 1 and Remark 3. The geometry of Σ is rather different depending on whether Σ is in one of the following regions,

$$R_1 = \{(H, K_I) : 1 \leq H, 0 \leq K_I\}, \qquad R_2 = \{(H, K_I) : 0 \leq K_I, 2(H - 1) \leq K_I\},$$
$$R_3 = \{(H, K_I) : 2(H - 1) \leq K_I \leq 0\}, \quad R_4 = \{(H, K_I) : 2(H - 1) \geq K_I, 0 \geq K_I\}.$$

If Σ is in λ_{ab}, we shall see that

$$\sigma = aI + bII$$

is a Riemannian metric which induces a natural complex structure on Σ such that its hyperbolic Gauss map is a conformal map. We shall prove that the complex representations of Bryant [2] and the authors [6] extend to BLW-surfaces (see Theorem 2).

The previous complex resolution is applied to the study of completeness. Thus, we shall start by proving (Theorem 3) that a complete BLW-surface in $R_1 \cup R_2$ is a totally umbilical round sphere, a horosphere or a hyperbolic cylinder.

The regions R_3 and R_4 are much richer in global examples. Every complete BLW-surface Σ in $\lambda_{ab} \subset R_4$, with meromorphic data (h, ω) has an associated complete constant mean curvature one surface, Σ_0, with geometry as Σ and with meromorphic data $(\sqrt{a/(a+b)}\, h, \omega)$.

Concerning R_3, we shall see that complete BLW-surfaces in $\lambda_{ab} \subset R_3$, are conformally equivalent to the unit disk, $\mathbb{D}$, and we shall prove (Theorem 4) that they are, up to conformal transformations of $\mathbb{D}$, in correspondence with the set of meromorphic maps $G : \mathbb{D} \longrightarrow \mathbb{C} \cup \{\infty\}$ with bounded Schwarzian derivative. In that way the hyperbolic Gauss map and the standard Euclidean Gauss map of complete BLW-surfaces in the half space model of $\mathbb{H}^3$ lying on R_3 give geometric meaning to classical families of complex functions which have been studied in connection with the Schwarzian derivative (see, for example, [9], [13], [14], [16]).

We also establish some existence and uniqueness theorems for complete BLW-surfaces in R_3. Our main interest shall be the Plateau problem at infinity: when is a given curve on the ideal boundary $\mathbb{S}^2_\infty$ of $\mathbb{H}^3$ the asymptotic boundary of a complete BLW-surface?

If $\lambda_{ab} \subset R_3$, we shall prove (Theorem 5) that a convex Jordan curve $\Gamma \subset \mathbb{S}^2_\infty$ is the asymptotic boundary of a unique embedded complete BLW-surface in λ_{ab} with an "inner" hyperbolic normal. Moreover if $-1/2 < a/(a+b) < 0$ there are exactly two embedded complete BLW-surfaces on λ_{ab} with asymptotic boundary Γ.

Finally, we shall deal with the general case and shall prove (Theorem 6) that if $-1/2 < a/(a+b) < 0$, a Jordan curve Γ on $\mathbb{S}^2_\infty$ is the asymptotic boundary of at least two complete BLW-surfaces on λ_{ab}. If $-1/4 < a/(a+b) < 0$, then there only exist two embedded complete BLW-surfaces on λ_{ab} with asymptotic boundary Γ.

2   Conformal representation.

In the Lorentz-Minkowski model, $\mathbb{L}^4$, the hyperbolic 3-space is described by one connected component of a two-sheeted hyperboloid. More precisely,

$$\mathbb{H}^3 = \left\{ (x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : \; -x_0^2 + x_1^2 + x_2^2 + x_3^2 = -1, \; x_0 > 0 \right\}$$

endowed with the induced metric of $\mathbb{L}^4$ given by the quadratic form $-x_0^2 + x_1^2 + x_2^2 + x_3^2$.

Associated with that metric we shall denote by $\wedge$ the usual exterior product in $\mathbb{H}^3$, that is,

(1) $$u \wedge v = p \times u \times v,$$

for any tangent vectors u, v at a point $p \in \mathbb{H}^3$, where $p \times u \times v$ is the unique vector in $\mathbb{L}^4$ such that $\langle p \times u \times v, w \rangle = \det(p, u, v, w)$, for any $w \in \mathbb{L}^4$. Here $\langle ., . \rangle$ denotes the inner product in $\mathbb{L}^4$ and det the usual determinant.

If we consider

$$\mathbb{N}^3_+ = \left\{ (x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : \; -x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0, \; x_0 > 0 \right\}$$

the positive null cone in $\mathbb{L}^4$, then $\mathbb{N}^3_+/\mathbb{R}^+$ inherits a natural conformal structure and it can be regarded as the ideal boundary $\mathbb{S}^2_\infty$ of $\mathbb{H}^3$.

We shall regard $\mathbb{L}^4$ as the space of 2×2 Hermitian matrices, Herm(2), in the standard way (see [2], [6]), by identifying $(x_0, x_1, x_2, x_3) \in \mathbb{L}^4$ with the matrix

$$\begin{pmatrix} x_0 - x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 + x_3 \end{pmatrix}.$$

Under this identification one has $\langle m, m \rangle = -\det(m)$ for all $m \in \text{Herm}(2)$. Then $\mathbb{H}^3$ is the set of $m \in \text{Herm}(2)$ with $\det(m) = 1$. The action of $\mathbb{SL}(2, \mathbb{C})$ on these Hermitian matrices defined by

$$g \cdot m = gmg^*,$$

where $g \in \mathbb{SL}(2, \mathbb{C})$, $g^* = {}^t\overline{g}$, preserves the inner product and leaves $\mathbb{H}^3$ invariant.

The space $\mathbb{N}_+^3$ is seen as the space of positive semi-definite 2×2 Hermitian matrices of determinant 0 and its elements can be written as $w\,{}^t\overline{w}$, where ${}^tw = (w_1, w_2)$ is a non zero vector in $\mathbb{C}^2$ uniquely defined up to multiplication by a unimodular complex number. The map $w\,{}^t\overline{w} \longrightarrow [(w_1, w_2)] \in \mathbb{CP}^1$ becomes the quotient map of $\mathbb{N}_+^3$ on $\mathbb{S}_\infty^2$ and identifies $\mathbb{S}_\infty^2$ to $\mathbb{CP}^1$. Thus, the natural action of $\mathbb{SL}(2, \mathbb{C})$ on $\mathbb{S}_\infty^2$ is the action of $\mathbb{SL}(2, \mathbb{C})$ on $\mathbb{CP}^1$ by Möbius transformations.

Now, let S be a surface and $\psi : S \longrightarrow \mathbb{H}^3$ an immersion with Gauss map η. Then, ψ is called a *linear Weingarten immersion of Bryant type*, (in short, BLW-surface) if the mean curvature H and the Gauss curvature K_I satisfy a linear relation of the form,

$$(2) \qquad\qquad 2\,a\,(H-1) + b\,K_I = 0,$$

for some $a, b \in \mathbb{R}$, $a + b \neq 0$.

Remark 1 This particular kind of linear Weingarten immersions includes the Bryant surfaces and the flat surfaces.

The immersions satisfying (2) with $a + b = 0$, that is, the non elliptic case, are the ones with a constant principal curvature equal to 1. They are studied in [1]. $\qquad\square$

Lemma 1 *Let $\psi : S \longrightarrow \mathbb{H}^3$ be a BLW-surface. Then, we can consider that $|a + b| = 1$,*

$$(3) \qquad\qquad 2\,a\,(H-1) + b\,(K-1) = 0$$

and $\sigma = a\,I + b\,II$ is a positive definite metric, where K is the Gauss-Kronecker curvature, $I = \langle d\psi, d\psi \rangle$ and $II = \langle d\psi, -d\eta \rangle$, the first and second fundamental form of the immersion, respectively.

Proof: Since $K_I = K - 1$, by scaling a and b in (2), we can assume that (3) is satisfied, with $a + b = 1$.

Let $\{e_1, e_2\}$ be an orthonormal basis at a point p which diagonalizes $d\eta$, that is, $d\eta(e_i) = -k_i e_i$, $i = 1, 2$. Then,

$$(4) \qquad \begin{aligned} \sigma(e_1, e_1)\,\sigma(e_2, e_2) - \sigma(e_1, e_2)^2 &= (a + bk_1)\,(a + bk_2) = \\ &= a^2 + b\,(2aH + bK) = (a+b)^2 = 1 \end{aligned}$$

and so σ is definite. Now, by changing the sign of a and b if necessary, we obtain that σ is positive definite and $|a + b| = 1$, which concludes the proof. $\qquad\square$

From now on, we will suppose that every BLW-surface satisfies the above lemma and we shall regard S as a Riemann surface with the conformal structure induced by $\sigma = a\,I + b\,II$.

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Lemma 2 *Let $\psi : S \longrightarrow \mathbb{H}^3$ be a BLW-surface and z be a conformal parameter for σ. Then*

(5)
$$a\psi_z - b\eta_z = -i\eta \wedge \psi_z,$$

where, for instance, ψ_z denotes the derivative of ψ with respect to z.

Proof: Let z be a conformal parameter such that $\sigma = 2\rho |dz|^2$. From (4),

(6)
$$\langle \psi_z \wedge \psi_{\overline{z}}, \eta \rangle = i\rho.$$

Since $a\psi_z - b\eta_z$ is a tangent vector field, it can be written as

$$a\psi_z - b\eta_z = \alpha \eta \wedge \psi_z + \beta \eta \wedge \psi_{\overline{z}}$$

for some local complex functions α and β.

Now, using that z is conformal and (6),

$$0 = \sigma(\psi_z, \psi_z) = \langle a\psi_z - b\eta_z, \psi_z \rangle = \beta \langle \eta \wedge \psi_{\overline{z}}, \psi_z \rangle = -i\beta\rho,$$
$$\rho = \sigma(\psi_z, \psi_{\overline{z}}) = \langle a\psi_z - b\eta_z, \psi_{\overline{z}} \rangle = \alpha \langle \eta \wedge \psi_z, \psi_{\overline{z}} \rangle = i\alpha\rho$$

and we are done. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

As in Bryant's paper [2], we consider for any immersion $\psi : S \longrightarrow \mathbb{H}^3$ with associated Gauss map η its *hyperbolic Gauss map* as the map given by $G := [\psi + \eta]$. That is, for any $p \in S$, the oriented geodesic emanating from p meets the ideal boundary $\mathbb{S}^2_\infty$ of $\mathbb{H}^3$ at $G(p)$.

Theorem 1 *Let $\psi : S \longrightarrow \mathbb{H}^3$ be a BLW-surface. Then $\psi + \eta$ is a conformal map with respect to the metric $\sigma = aI + bII$ and*

(7)
$$\Delta^\sigma(\psi + \eta) \;=\; \frac{2}{a+b}\{(H-1)\psi + (K-H)\eta\},$$

where Δ^σ denotes the Laplacian of σ. In particular, its hyperbolic Gauss map G is conformal.

Moreover, the immersion lies in a horosphere or $\langle d(\psi + \eta), d(\psi + \eta) \rangle$ is a pseudometric of constant curvature $\frac{a}{a+b}$.

Proof: Let z be a conformal parameter such that $\sigma = 2\rho |dz|^2$. Then we can write

$$\eta_z = -\Lambda \psi_z + \Upsilon \psi_{\overline{z}},$$
$$\eta_{\overline{z}} = \overline{\Upsilon} \psi_z - \Lambda \psi_{\overline{z}}$$

for a local real function Λ and a complex function Υ.

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Thus, $H = \Lambda$ and $K = \Lambda^2 - |\Upsilon|^2 = H^2 - |\Upsilon|^2$, that is,

(8) $$\eta_z = -H\psi_z + \Upsilon\psi_{\bar{z}}.$$

Therefore, using (3)

$$(a + bH)^2 - b^2|\Upsilon|^2 = a^2 + 2abH + b^2K = (a + b)^2 = 1.$$

The cross product of (5) and (8) with ψ_z and $\psi_{\bar{z}}$ gives

$$\begin{cases} -b\eta_z \wedge \psi_z = -i(\eta \wedge \psi_z) \wedge \psi_z = i\langle\psi_z, \psi_z\rangle\eta, \\ -b\eta_z \wedge \psi_z = -b\Upsilon\psi_{\bar{z}} \wedge \psi_z, \end{cases}$$

$$\begin{cases} (a\psi_z - b\eta_z) \wedge \psi_{\bar{z}} = -i(\eta \wedge \psi_z) \wedge \psi_{\bar{z}} = i\langle\psi_z, \psi_{\bar{z}}\rangle\eta, \\ (a\psi_z - b\eta_z) \wedge \psi_{\bar{z}} = a\psi_z \wedge \psi_{\bar{z}} + b(H\psi_z - \Upsilon\psi_{\bar{z}}) \wedge \psi_{\bar{z}} = (a + bH)\psi_z \wedge \psi_{\bar{z}}, \end{cases}$$

and by taking the inner product with η one has from (6)

(9) $$\begin{aligned} \langle\psi_z, \psi_z\rangle &= b\Upsilon\rho, \\ \langle\psi_z, \psi_{\bar{z}}\rangle &= (a + bH)\rho. \end{aligned}$$

In that way, from (5),
(10) $$b(\psi + \eta)_z = (a + b)\psi_z + i\eta \wedge \psi_z$$
and, using (9), one obtains $\langle(\psi + \eta)_z, (\psi + \eta)_z\rangle = 0$ when $b \neq 0$. Moreover, if $b = 0$ then, from (3) and (8), we have that $H = 1$, $(\psi + \eta)_z = \Upsilon\psi_{\bar{z}}$ and, using (9), $\langle(\psi + \eta)_z, (\psi + \eta)_z\rangle = 0$. Therefore, $\psi + \eta$ is a conformal map in any case.

Now, bearing in mind (1) and (6), if we take the derivative of (5) with respect to $\bar{z}$, then

$$a\psi_{z\bar{z}} - b\eta_{z\bar{z}} = \rho\psi - iH\psi_z \wedge \psi_{\bar{z}} - i\psi \times \eta \times \psi_{z\bar{z}}.$$

As the above equality has imaginary part $0 = \psi \times \eta \times \psi_{z\bar{z}}$, we have

(11) $$a\psi_{z\bar{z}} - b\eta_{z\bar{z}} = \rho(\psi + H\eta).$$

On the other hand, (5) also gives $(a\psi_z - b\eta_z) \wedge \eta = -i\psi_z$ and by deriving with respect to $\bar{z}$ we obtain
(12) $$\psi_{z\bar{z}} = \rho\{(a + bH)\psi + (aH + bK)\eta\}.$$

If $b \neq 0$, then (12) minus (11) (respectively, (12) plus (11)) and (6) give the expression (7) when $a + b = 1$ (respectively, when $a + b = -1$).

If $b = 0$, then $H = 1$, $\sigma = I$, $\langle\psi_z, \psi_z\rangle = 0$ and $\langle\psi_z, \psi_{\bar{z}}\rangle = \rho$, hence

$$\psi_{zz} = \frac{\rho_z}{\rho}\psi_z - \Upsilon\rho\eta, \quad \psi_{z\bar{z}} = \rho\eta, \quad \eta_z = -\psi_z + \Upsilon\psi_{\bar{z}}.$$

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Since $(\psi_{zz})_{\bar{z}} = (\psi_{z\bar{z}})_z$, we have $(\rho\Upsilon)_{\bar{z}} = 0$ and, consequently,

$$(\psi + \eta)_{z\bar{z}} = (\Upsilon\psi_{\bar{z}})_{\bar{z}} = -|\Upsilon|^2\rho\eta = \rho(K - 1)\eta.$$

In order to determine the singular points of $\psi + \eta$, we have from (6), (9) and (10) that

(13)
$$b^2\langle(\psi + \eta)_z, (\psi + \eta)_{\bar{z}}\rangle = \langle(a + b)\psi_z + i\eta \wedge \psi_z, (a + b)\psi_{\bar{z}} - i\eta \wedge \psi_{\bar{z}}\rangle =$$
$$= 2(\langle\psi_z, \psi_{\bar{z}}\rangle - (a + b)\rho) = 2b(H - 1)\rho.$$

And we conclude that if $b \neq 0$, $\psi + \eta$ has a singular point at z_0 if and only if $H(z_0) = 1$. Moreover, from (3) and (9), if $H(z_0) = 1$ and $b \neq 0$ then $K(z_0) = 1$ and $a + b = 1$. When $b = 0$ the same conclusion is clear from (8), (with $a = 1$ because σ is positive definite).

Now, from (2) and (7), ψ can be written as

(14)
$$\psi = \frac{a + b}{\rho(2H - K - 1)}(\psi + \eta)_{z\bar{z}} + \frac{2a + b}{2(a + b)}(\psi + \eta),$$

at the regular points of $\psi + \eta$.

Using that $\psi + \eta$ is conformal, we have that there exist holomorphic functions, A, B, such that $[\psi + \eta]$ is represented as $[(A, B)] \in \mathbb{CP}^1 \equiv \mathbb{S}_\infty^2$ and

(15)
$$\psi + \eta = \lambda \begin{pmatrix} A \\ B \end{pmatrix} (\overline{A}, \overline{B}) = \lambda \begin{pmatrix} A\overline{A} & A\overline{B} \\ \overline{A}B & B\overline{B} \end{pmatrix},$$

for some positive regular function λ. Thus, (14) gives

$$\psi = g\Omega g^* \quad \text{and} \quad \eta = g\widetilde{\Omega}g^*,$$

with

$$g = \begin{pmatrix} A & A_z \\ B & B_z \end{pmatrix}, \quad \Omega = \begin{pmatrix} \delta\lambda_{z\bar{z}} - \beta\lambda & \delta\lambda_z \\ \delta\lambda_{\bar{z}} & \delta\lambda \end{pmatrix}, \quad \widetilde{\Omega} = \begin{pmatrix} (1 + \beta)\lambda - \delta\lambda_{z\bar{z}} & -\delta\lambda_z \\ -\delta\lambda_{\bar{z}} & -\delta\lambda \end{pmatrix},$$

$\beta = -\frac{2a+b}{2(a+b)}$ and $\delta = \frac{a+b}{\rho(2H-K-1)}$. From these expressions one gets

$$2 = \langle\eta, \eta\rangle - \langle\psi, \psi\rangle = -\det(\eta) + \det(\psi) = \delta\lambda^2|\det(g)|^2,$$

$$1 = -\langle\psi, \psi\rangle = \det(\psi) = -2\beta + \frac{4}{\lambda^4|\det(g)|^2}(\lambda\lambda_{z\bar{z}} - \lambda_z\lambda_{\bar{z}})$$

and, consequently,

$$(\log(\lambda^2|\det(g)|^2))_{z\bar{z}} = \frac{1 + 2\beta}{2}\lambda^2|\det(g)|^2 = \frac{-a}{2(a + b)}\lambda^2|\det(g)|^2.$$

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That is, $\langle d(\psi + \eta), d(\psi + \eta)\rangle = \lambda^2|\det(g)|^2|dz|^2$ has constant Gauss curvature $\varepsilon = \frac{a}{a+b}$ at the regular points of $\psi + \eta$.

From (7) and (12), the two 2-forms $Q_I = \langle\psi_z, \psi_z\rangle dz^2$ and $Q_{II} = \langle\psi_z, -\eta_z\rangle dz^2$ are holomorphic on S. And, since ψ has an umbilical point at z_0 if and only if $Q_I(z_0) = 0 = Q_{II}(z_0)$, then ψ is totally umbilical or has isolated umbilical points. Hence, either ψ is totally umbilical with $H = 1 = K$, that is, $\psi(S)$ lies on a horosphere or $\langle d(\psi + \eta), d(\psi + \eta)\rangle$ is a pseudometric of constant curvature ε. □

Remark 2 If $\psi : S \longrightarrow \mathbb{H}^3$ is a BLW-surface and S is a topological sphere then, using that Q_I and Q_{II} are holomorphic, they vanish identically and we obtain that $\psi(S)$ is a totally umbilical round sphere. □

Remark 3 The regions R_1, R_2, R_3 and R_4 given in the introduction separate into four parts the exterior of the domain $H^2 - (K_I + 1) < 0$ in the (H, K_I)-plane according to the sign of $a + b$ and $\varepsilon = a/(a + b)$, see Figure 1.

In fact, if $b \neq 0$, from (7) and (13) we obtain

$$\frac{2}{b}(H - 1) = \frac{1}{\rho}\langle(\psi + \eta)_z, (\psi + \eta)_{\overline{z}}\rangle = -\frac{1}{\rho}\langle(\psi + \eta)_{z\overline{z}}, \psi + \eta\rangle = \frac{2(H - 1) - K_I}{a + b}.$$

When $a + b = 1$, from (3),

$$2(H - 1) \geq K_I = (a + b)K_I = aK_I - 2a(H - 1) = a(K_I - 2(H - 1)),$$

that is,

1. if $a \geq 0$ then $2(H - 1) \geq K_I$, $K_I \leq 0$ and we are in the region R_4,

2. if $a \leq 0$ then $2(H - 1) \geq K_I \geq 0$ and we are in the region R_1.

In a similar way, when $a + b = -1$ we are either in R_2 or in R_3.

Moreover, from (13), if $b \geq 0$ it is satisfied that $H \geq 1$ and if $b \leq 0$ then $H \leq 1$. Consequently, from (3), if $a \geq 0$ one gets $K_I \leq 0$ and if $a \leq 0$ then $K_I \geq 0$. □

Theorem 2 (Conformal representation)

i) Let S be a non compact, simply connected surface and $\psi : S \longrightarrow \mathbb{H}^3$ a BLW-surface. Then, there exist a meromorphic curve $g : S \longrightarrow \mathbb{SL}(2, \mathbb{C})$ and a pair (h, ω) consisting of a meromorphic function h and a holomorphic 1-form ω on S, such that the immersion and its Gauss map can be recovered as

(16) $$\psi = g\Omega g^* \quad and \quad \eta = g\widetilde{\Omega}g^*,$$

where

(17) $$\Omega = \begin{pmatrix} \frac{1 + \varepsilon^2|h|^2}{1 + \varepsilon|h|^2} & -\varepsilon\overline{h} \\ -\varepsilon h & 1 + \varepsilon|h|^2 \end{pmatrix} \quad and \quad \widetilde{\Omega} = \begin{pmatrix} \frac{1 - \varepsilon^2|h|^2}{1 + \varepsilon|h|^2} & \varepsilon\overline{h} \\ \varepsilon h & -1 - \varepsilon|h|^2 \end{pmatrix},$$

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with $\varepsilon = \frac{a}{a+b}$ and $1 + \varepsilon|h|^2 > 0$. Moreover, the curve g satisfies

$$(18) \qquad g^{-1}dg = \begin{pmatrix} 0 & \omega \\ dh & 0 \end{pmatrix}.$$

The induced metric and $\sigma = a\,I + b\,II$ are given, respectively, by

$$(19) \qquad I = (1 - \varepsilon)\omega dh + \left(\frac{(1-\varepsilon)^2|dh|^2}{(1+\varepsilon|h|^2)^2} + (1+\varepsilon|h|^2)^2|\omega|^2 \right) + (1-\varepsilon)\bar{\omega}d\bar{h}$$

and

$$(20) \qquad \sigma = (a+b)\left((1+\varepsilon|h|^2)^2|\omega|^2 - \frac{(1-\varepsilon)^2|dh|^2}{(1+\varepsilon|h|^2)^2} \right).$$

ii) *Conversely, let S be a Riemann surface, $g : S \longrightarrow \mathbb{SL}(2,\mathbb{C})$ a meromorphic curve and (h,ω) a pair as above satisfying (18) and such that (20) is a positive definite metric. Then $\psi = g\Omega g^* : S \longrightarrow \mathbb{H}^3$, ($\Omega$ as in (17)), is a BLW-surface satisfying (3) with induced metric and σ given by (19) and (20).*

Proof: Let us assume that $\psi(S)$ is not a piece of a horosphere, otherwise the result is easily obtained with h constant. Since S is non compact and simply connected there exist a global conformal parameter z and as $\langle d(\psi + \eta), d(\psi + \eta) \rangle$ is a pseudometric on S of constant Gauss curvature ε then, from the Frobenius theorem, there exists a meromorphic function h on S (holomorphic if $\varepsilon \le 0$) such that

$$1 + \varepsilon|h|^2 > 0 \qquad \text{and} \qquad \langle d(\psi+\eta), d(\psi+\eta) \rangle = \lambda^2 |\det(g)|^2|dz|^2 = \frac{4|dh|^2}{(1+\varepsilon|h|^2)^2},$$

where we follow the same notation as above.

From this expression, if h has a pole of order m, then dh has a pole of order $m + 1$ and $\det(g)$ has a zero of order $m - 1$. Hence, there exists a meromorphic function R on S (holomorphic if $\varepsilon \le 0$) with

$$(21) \qquad R^2 = \frac{dh}{\det(g)dz} \ne 0 \qquad \text{and} \qquad (AR)(BR)_z - (BR)(AR)_z = h_z.$$

Now, if we replace A and B by AR and BR, respectively, in (15) and choose the matrix

$$(22) \qquad g = \begin{pmatrix} A & \frac{1}{h_z}A_z \\ B & \frac{1}{h_z}B_z \end{pmatrix}$$

with meromorphic coefficients and $\det(g)=1$ on S, then the new λ^2 is $4/(1+\varepsilon|h|^2)^2$. Moreover,

$$g^{-1}\,dg = \begin{pmatrix} 0 & \frac{1}{h_z^2}(A_{zz}B_z - A_zB_{zz}) \\ h_z & 0 \end{pmatrix}dz$$

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and we obtain (18) for $\omega = \frac{A_{zz}B_z - A_z B_{zz}}{h_z^2} dz$.

From (14) one has that (16) is satisfied for

$$\Omega = \begin{pmatrix} \delta\lambda_{z\bar{z}} - \beta\lambda & \delta\lambda_z \overline{h_z} \\ \delta\lambda_{\bar{z}} h_z & \delta\lambda |h_z|^2 \end{pmatrix} \qquad \text{and} \qquad \widetilde{\Omega} = \begin{pmatrix} (1+\beta)\lambda - \delta\lambda_{zz} & -\delta\lambda_z \overline{h_z} \\ -\delta\lambda_{\bar{z}} h_z & -\delta\lambda |h_z|^2 \end{pmatrix}$$

where $2 = \langle \eta, \eta \rangle - \langle \psi, \psi \rangle = \delta\lambda^2 |h_z|^2$. Therefore, (16) and (17) are satisfied.

Besides,

$$\begin{aligned}
d\psi &= d(g\Omega g^*) = g\left(\begin{pmatrix} 0 & \omega \\ dh & 0 \end{pmatrix} \Omega + d\Omega + \Omega \begin{pmatrix} 0 & d\overline{h} \\ \overline{\omega} & 0 \end{pmatrix} \right) g^* = \\
&= g\begin{pmatrix} \star & (1+\varepsilon|h|^2)\omega + \frac{1-\varepsilon}{1+\varepsilon|h|^2} d\overline{h} \\ (1+\varepsilon|h|^2)\overline{\omega} + \frac{1-\varepsilon}{1+\varepsilon|h|^2} dh & 0 \end{pmatrix} g^*,
\end{aligned}$$

$$\begin{aligned}
d\eta &= d(g\widetilde{\Omega}g^*) = g\left(\begin{pmatrix} 0 & \omega \\ dh & 0 \end{pmatrix} \widetilde{\Omega} + d\widetilde{\Omega} + \widetilde{\Omega} \begin{pmatrix} 0 & d\overline{h} \\ \overline{\omega} & 0 \end{pmatrix} \right) g^* = \\
&= g\begin{pmatrix} \star & -(1+\varepsilon|h|^2)\omega + \frac{1+\varepsilon}{1+\varepsilon|h|^2} d\overline{h} \\ -(1+\varepsilon|h|^2)\overline{\omega} + \frac{1+\varepsilon}{1+\varepsilon|h|^2} dh & 0 \end{pmatrix} g^*
\end{aligned}$$

and

$$d(\psi + \eta) = g\begin{pmatrix} \star & \frac{2d\overline{h}}{1+\varepsilon|h|^2} \\ \frac{2dh}{1+\varepsilon|h|^2} & 0 \end{pmatrix} g^*.$$

Hence, one calculates the induced metric I, (19), and the metric $\sigma = aI + bII$, (20), from

$$I = \langle d\psi, d\psi \rangle = -\det(d\psi)$$

and

$$II = \langle d\psi, -d\eta \rangle = \frac{1}{2}\left(\det(d(\psi + \eta)) - \det(d\psi) - \det(d\eta) \right).$$

In particular, $(1 + \varepsilon|h|^2)^2 |\omega|^2$ is a real expression on S and consequently ω must be holomorphic. $\qquad\square$

Following the same notation as in [2] and [6] the pair (h, ω) given by the above theorem will be called the *Weierstrass data* associated with the conformal representation of the BLW-surface ψ. However, from (18) it is clear that the pair (h, g) could be also called the Weierstrass data of the immersion.

Remark 4 If S is a non compact, simply connected surface and $\psi_1, \psi_2 : S \longrightarrow \mathbb{H}^3$ are two BLW-surfaces satisfying (3) with the same Weierstrass data (h, ω), then ψ_1 and ψ_2 are the same immersion, up to a rigid motion in $\mathbb{H}^3$. Indeed, $\psi_1 = g_1 \Omega g_1^*$ and $\psi_2 = g_2 \Omega g_2^*$ where

$$g_1^{-1}\, dg_1 = \begin{pmatrix} 0 & \omega \\ dh & 0 \end{pmatrix} = g_2^{-1}\, dg_2$$

for some meromorphic curves $g_1, g_2 : S \longrightarrow \mathbb{SL}(2, \mathbb{C})$.

Thus, one has

$$d(g_1\, g_2^{-1}) = dg_1\, g_2^{-1} + g_1\, d(g_2^{-1}) = g_1 \begin{pmatrix} 0 & \omega \\ dh & 0 \end{pmatrix} g_2^{-1} + g_1 \begin{pmatrix} 0 & -\omega \\ -dh & 0 \end{pmatrix} g_2^{-1} = 0$$

and there exists a constant matrix $g_0 \in \mathbb{SL}(2, \mathbb{C})$ such that $g_1 = g_0\, g_2$. Consequently, $\psi_1 = g_0\, \psi_2\, g_0^*$ as we wanted to prove. □

It is interesting to observe that if $\psi : S \longrightarrow \mathbb{H}^3$ is an immersion with $H = 1$ then $\varepsilon = 1$,

$$\Omega = \begin{pmatrix} 0 & -i \\ -i & ih \end{pmatrix} \begin{pmatrix} 0 & i \\ i & -i\bar{h} \end{pmatrix}$$

and $\psi = g\Omega g^* = FF^*$, where

$$F = g \begin{pmatrix} 0 & -i \\ -i & ih \end{pmatrix} \in \mathbb{SL}(2, \mathbb{C}),$$

that is, the conformal representation becomes the Bryant's representation (see [2], [25]).

Moreover, if the immersion does not lie in a horosphere and we denote by G its hyperbolic Gauss map then taking $A = G$ and $B = 1$ in (15) one gets from (21) and (22)

$$g = \begin{pmatrix} iG\sqrt{\frac{dh}{dG}} & \frac{i}{dh} d\left(G\sqrt{\frac{dh}{dG}}\right) \\ i\sqrt{\frac{dh}{dG}} & \frac{i}{dh} d\left(\sqrt{\frac{dh}{dG}}\right) \end{pmatrix}.$$

Thus, by considering the new parameter $\zeta = h$ where that is possible, one has

$$F = \frac{1}{2G_\zeta^{3/2}} \begin{pmatrix} 2G_\zeta^2 - G\,G_{\zeta\zeta} & 2G_\zeta\,(G - \zeta G_\zeta) + \zeta\,G\,G_{\zeta\zeta} \\ -G_{\zeta\zeta} & 2G_\zeta + \zeta\,G_{\zeta\zeta} \end{pmatrix},$$

that is, we recover the Small's formula for surfaces with constant mean curvature one (see [10], [23]).

3 Completeness of the immersions.

From Remark 3 it is clear that the Gauss curvature K_I of a BLW-surface $\psi : S \longrightarrow \mathbb{H}^3$ is non negative on S or non positive everywhere.

Moreover, if S is non compact then, from (16), (17), (19) and (20), its Gaussian curvature $K_I = K - 1$ can be calculated as

$$(23) \qquad K_I = \frac{-4\,\varepsilon\,|dh|^2}{(1+\varepsilon|h|^2)^4\,|\omega|^2 - (1-\varepsilon)^2\,|dh|^2}$$

and its mean curvature is given by

$$H = 1 + \frac{2\,(1-\varepsilon)\,|dh|^2}{(1+\varepsilon|h|^2)^4\,|\omega|^2 - (1-\varepsilon)^2\,|dh|^2}.$$

In particular, a point $p \in S$ is umbilical if and only if $dh(p) = 0$ or $\omega(p) = 0$.

We shall study separately the cases of complete immersions with non positive and non negative Gauss curvature.

3.1 Completeness with non negative Gauss curvature.

First, we give some examples of complete BLW-surfaces and then we classify them as the only non totally umbilical complete BLW-surfaces with non negative Gauss curvature.

Let us consider $S = \mathbb{C}$, $h(z) = z + c_0$ and $\omega(z) = c_1\,dz$ for some complex constants c_0 and c_1:

1. If we take $a = 0$ and $b = 1$ then, from Theorem 2, one obtains the flat immersion

   $$(24) \qquad \psi(u,v) = \begin{pmatrix} \frac{1+r^2}{2r}e^{-u} & \frac{1-r^2}{2r}e^{-iv} \\ \frac{1-r^2}{2r}e^{iv} & \frac{1+r^2}{2r}e^{u} \end{pmatrix},$$

   with $r = \sqrt{|c_1|}$, $u + iv = 2\sqrt{c_1}\,z$ and $\sigma = (|c_1|^2 - 1)|dz|^2$ positive definite when $|c_1| > 1$. Moreover, the induced metric $I = c_1\,dz^2 + (1 + |c_1|^2)|dz|^2 + \overline{c_1}\,d\bar{z}^2 \geq (|c_1| - 1)^2|dz|^2$ is complete.

2. If we put $a = 0$ and $b = -1$ then (24) is a complete flat immersion with $0 < |c_1| < 1$.

In any case, the immersion given by (24) may be regarded as the set of points at a fixed distance from the geodesic

$$\gamma(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix},$$

namely, a hyperbolic cylinder.

Theorem 3 *Let $\psi : S \longrightarrow \mathbb{H}^3$ be a complete BLW-surface with non negative Gauss curvature K_I. Then $\psi(S)$ is a totally umbilical round sphere, a horosphere or a hyperbolic cylinder.*

Proof: We can assume that S is simply connected, otherwise we would pass to the universal covering surface of S. Moreover, if S is compact then, from Remark 2, $\psi(S)$ is a totally umbilical round sphere.

First, let us suppose that ψ is a non flat immersion, that is, there exists a point on S such that $K_I > 0$ (in particular, from Remark 3, $a < 0$).

If $a + b = -1$ then from (3) and since $H^2 \geq K$, we have that $K_I \geq -4a/b^2 > 0$. Therefore, from Bonnet's theorem, S must be compact.

If $a + b = 1$, $\varepsilon < 0$ and from (19)

$$\frac{1}{2} I \leq \tau = \frac{(1 - \varepsilon)^2 |dh|^2}{(1 + \varepsilon|h|^2)^2} + (1 + \varepsilon|h|^2)^2 |\omega|^2$$

and τ must be a complete metric on S. Besides, using that σ is a positive definite metric and that the bounded holomorphic function h satisfies that $1 + \varepsilon|h|^2 < 1$ we obtain $\tau \leq 2(1 + \varepsilon|h|^2)^2 |\omega|^2 \leq 2|\omega|^2$. Therefore, $2|\omega|^2$ is a flat complete metric conformal to σ. Thus, S is conformally equivalent to the complex plane $\mathbb{C}$ and h must be constant, that is, $\psi(S)$ is a horosphere.

On the other hand, if ψ is a flat immersion then $a = 0 = \varepsilon$ and from (19)

$$\tau = |dh|^2 + |\omega|^2$$

is a complete metric. Since σ is a positive definite metric one has that if $b = 1$ (respectively, $b = -1$) then $\tau \leq 2|\omega|^2$ (resp. $\tau \leq 2|dh|^2$) and, hence, there exists a complete conformal flat metric on S. Thus, S is conformally equivalent to $\mathbb{C}$ and the function dh/ω (or ω/dh) is constant with modulus less than one when $b = 1$ (or $b = -1$).

Therefore, $\psi(S)$ is a horosphere or $\omega = c_1 dh$ for a complex constant $c_1 \neq 0$ with $|c_1| \neq 1$. Thus, in the second case $dh \neq 0$ everywhere because σ is non degenerate and we can take the new conformal parameter ζ such that $d\zeta = dh$. Then one obtains $h(\zeta) = \zeta + c_0$ and $\omega(\zeta) = c_1 d\zeta$ for a constant $c_0 \in \mathbb{C}$ and ψ lies on a hyperbolic cylinder. $\square$

3.2 Completeness with non positive Gauss curvature. A Plateau problem at infinity.

Now, we focus our attention on the study of the completeness of the immersions with non positive Gauss curvature. We remark that given a simply connected surface S and a non flat BLW-surface $\psi : S \longrightarrow \mathbb{H}^3$ with non positive Gauss curvature then S cannot be compact and $a > 0$. Therefore, the immersion lies in the set R_3 or R_4.

In the last case, the geometry of the surface is similar to the one of a surface with constant mean curvature $H = 1$. In fact, if (h, ω) are the Weierstrass data for a BLW-surface ψ_0 in

R_4 then $\varepsilon > 0$ and $(1 + \varepsilon |h|^2)|\omega|^2$ is a positive definite metric because so is σ. Moreover, we obtain a new associated immersion ψ_1 with constant mean curvature one and Weierstrass data $(\sqrt{\varepsilon} h, \omega)$. If ψ_0 is a complete immersion then, from (19) and (20), $(1 + \varepsilon |h|^2)|\omega|^2$ is also a complete metric, that is, ψ_1 is a complete immersion. Thus, the study of complete BLW-surfaces in R_4 can be reduced to the study of complete immersions with constant mean curvature one. Many things are known in this case and some very interesting results were proved in [2], [3] and [25].

However, the geometry of the surface is very different when the immersion lies in R_3. For instance,

Lemma 3 *Let $\psi : S \longrightarrow \mathbb{H}^3$ be a complete BLW-surface in R_3 with Weierstrass data (h, ω). Then, S is conformally equivalent to the unit disk $\mathbb{D}$ and h is a global diffeomorphism onto $\mathbb{D}_\varepsilon = \{z \in \mathbb{C} : |z|^2 < -1/\varepsilon\}$.*

Proof: Since the induced metric is complete and σ is positive definite, from (19) and (20)

$$\frac{1}{2} I \leq \frac{(1 - \varepsilon)^2 |dh|^2}{(1 + \varepsilon |h|^2)^2} + (1 + \varepsilon |h|^2)^2 |\omega|^2 \leq 2 \frac{(1 - \varepsilon)^2 |dh|^2}{(1 + \varepsilon |h|^2)^2}$$

and the metric $4 |dh|^2/(1 + \varepsilon |h|^2)^2$ must be complete. Therefore, $h : S \longrightarrow \mathbb{D}_\varepsilon$ is bijective and S is conformally equivalent to the unit disk. $\square$

From the above lemma, given a complete BLW-surface $\psi : S \longrightarrow \mathbb{H}^3$ with Weierstrass data (h, ω) we can consider, up to a change of parameter, $S = \mathbb{D}$ and $h(z) = z/\sqrt{-\varepsilon}$.

Moreover, from (20), σ is positive definite if and only if

$$(25) \qquad\qquad |\omega| < \frac{1 - \varepsilon}{\sqrt{-\varepsilon}} \frac{|dz|}{(1 - |z|^2)^2}, \qquad z \in \mathbb{D}.$$

On the other hand, if we consider the hyperbolic Gauss map of the immersion G and take $A = G$, $B = 1$ in (15) one gets from (21) and (22)

$$(26) \qquad g = \begin{pmatrix} iG\sqrt{\frac{dh}{dG}} & \frac{i}{dh} d\left(G\sqrt{\frac{dh}{dG}}\right) \\ i\sqrt{\frac{dh}{dG}} & \frac{i}{dh} d\left(\sqrt{\frac{dh}{dG}}\right) \end{pmatrix} = \begin{pmatrix} \dfrac{iG}{\sqrt[4]{-\varepsilon}\sqrt{G_z}} & i\sqrt[4]{-\varepsilon}\left(\dfrac{G}{\sqrt{G_z}}\right)_z \\ \dfrac{i}{\sqrt[4]{-\varepsilon}\sqrt{G_z}} & i\sqrt[4]{-\varepsilon}\left(\dfrac{1}{\sqrt{G_z}}\right)_z \end{pmatrix}.$$

Thus,

$$g^{-1} dg = \begin{pmatrix} 0 & \sqrt{-\varepsilon} \dfrac{3G_{zz}^2 - 2G_z G_{zzz}}{4G_z^2} \\ \dfrac{1}{\sqrt{-\varepsilon}} & 0 \end{pmatrix} dz,$$

and from (18) and (25), the Schwarzian derivative of G satisfies

$$\{G, z\} := \frac{d}{dz}\left(\frac{G_{zz}}{G_z}\right) - \frac{1}{2}\left(\frac{G_{zz}}{G_z}\right)^2 = \frac{2G_z G_{zzz} - 3G_{zz}^2}{2G_z^2} = \frac{-2}{\sqrt{-\varepsilon}}\frac{\omega}{dz}$$

and

$$|\{G, z\}| < \frac{2(1-\varepsilon)}{-\varepsilon}\frac{1}{(1-|z|^2)^2} \qquad z \in \mathbb{D}.$$

Consequently, G is a local diffeomorphism with bounded Schwarzian derivative. Otherwise, there would exist a point z_0 such that G is not locally one to one and we could write G_z in a neighbourhood of z_0 as

$$G_z = (z - z_0)^k \sum_{n=0}^{\infty} c_n(z - z_0)^n$$

with $c_0 \neq 0$ and an integer $k \neq 0$. Hence

$$\{G, z\} = -\frac{k(k+2)}{2}\frac{1}{(z-z_0)^2} - \frac{k\,c_1}{2\,c_0}\frac{1}{z - z_0} + \widetilde{h}(z)$$

where $\widetilde{h}$ is a holomorphic function in a neighbourhood of z_0. But, either $\{G, z\} = \infty$ at z_0 which contradicts the above inequality or $k = -2$ and G has a pole of order one at z_0, that is, G is locally one to one.

The above considerations are summarized in the following result:

Theorem 4 *Let S be a simply connected surface and $\psi : S \longrightarrow \mathbb{H}^3$ a complete BLW-surface in R_3. Then one has*

(i) *S can be identified with $\mathbb{D}$ and its Weierstrass data h can be taken as $h(z) = z/\sqrt{-\varepsilon}$,*

(ii) *the hyperbolic Gauss map $G : \mathbb{D} \longrightarrow \mathbb{C} \cup \{\infty\}$ is a local diffeomorphism satisfying*

$$(27) \qquad |\{G, z\}| < \frac{2(1-\varepsilon)}{-\varepsilon}\frac{1}{(1-|z|^2)^2}, \qquad z \in \mathbb{D},$$

(iii) *the immersion can be recover as*

$$(28) \quad \frac{1}{\sqrt{-\varepsilon}\,|G_z|(1-|z|^2)}\left(\begin{matrix} |G|^2 - \varepsilon \left|-G\mathcal{A} + G_z(1-|z|^2)\right|^2 & (1-\varepsilon|\mathcal{A}|^2)G + \varepsilon\,G_z(1-|z|^2)\overline{\mathcal{A}} \\ (1-\varepsilon|\mathcal{A}|^2)\overline{G} + \varepsilon\,\overline{G_z}(1-|z|^2)\mathcal{A} & 1 - \varepsilon|\mathcal{A}|^2 \end{matrix} \right)$$

and its Gauss map η as

$$(29) \quad \frac{1}{\sqrt{-\varepsilon}\,|G_z|(1-|z|^2)}\left(\begin{matrix} |G|^2 + \varepsilon \left|-G\mathcal{A} + G_z(1-|z|^2)\right|^2 & (1+\varepsilon|\mathcal{A}|^2)G - \varepsilon\,G_z(1-|z|^2)\overline{\mathcal{A}} \\ (1+\varepsilon|\mathcal{A}|^2)\overline{G} - \varepsilon\,\overline{G_z}(1-|z|^2)\mathcal{A} & 1 + \varepsilon|\mathcal{A}|^2 \end{matrix} \right)$$

with

$$\mathcal{A} = \frac{G_{zz}}{2G_z}(1 - |z|^2) - \bar{z}.$$

15

Conversely, let $G : \mathbb{D} \longrightarrow \mathbb{C} \cup \{\infty\}$ be a meromorphic map. Then, if G satisfies (27) one has that (28) is a BLW-surface in R_3 with hyperbolic Gauss map G and Weierstrass data $(z/\sqrt{-\varepsilon}, -\frac{1}{2}\sqrt{-\varepsilon}\{G, z\}\,dz)$. Moreover, if

$$(30) \qquad |\{G, z\}| \ \leq \ \frac{b_0}{(1 - |z|^2)^2} \qquad z \in \mathbb{D},$$

with $b_0 < \frac{2(1-\varepsilon)}{-\varepsilon}$, then the immersion is complete.

Proof: Let $\psi : S \longrightarrow \mathbb{H}^3$ be a complete BLW-surface in R_3 and S simply connected. Then, we can assume that (i) and (ii) are satisfied and, following Theorem 2, the immersion and its Gauss map can be recover as in (16) and (17) where g is given as in (22) for some meromorphic functions A and B.

Thus,

$$\psi = \frac{1}{1 - |z|^2} \left(\begin{array}{cc} |A|^2 - \varepsilon \mathcal{A}_1 \overline{\mathcal{A}_1} & A\overline{B} - \varepsilon \mathcal{A}_1 \overline{\mathcal{A}_2} \\ \overline{A}B - \varepsilon \overline{\mathcal{A}_1}\mathcal{A}_2 & |B|^2 - \varepsilon \mathcal{A}_2 \overline{\mathcal{A}_2} \end{array} \right),$$

where

$$\mathcal{A}_1 = A\bar{z} + A_z(1 - |z|^2), \quad \mathcal{A}_2 = B\bar{z} + B_z(1 - |z|^2).$$

Now, using that $G = A/B$, $g \in \mathbb{SL}(2, \mathbb{C})$, that is, $\sqrt{-\varepsilon}(AB_z - BA_z) = 1$ and

$$(31) \qquad G_z = \frac{-1}{\sqrt{-\varepsilon}\,B^2}, \qquad \frac{G_{zz}}{G_z} = -2\frac{B_z}{B},$$

one has

$$\mathcal{A}_2 = B\left(\bar{z} + \frac{B_z}{B}(1 - |z|^2)\right) = -B\mathcal{A},$$

$$\mathcal{A}_1 = G\mathcal{A}_2 - \frac{1}{\sqrt{-\varepsilon}\,B}(1 - |z|^2) = B(-G\mathcal{A} + G_z(1 - |z|^2)),$$

and bearing in mind that $\sqrt{-\varepsilon}|B|^2|G_z| = 1$ from (31), one obtains (28). Analogously, one can compute (29).

From Theorem 2, the converse part is a straightforward computation taking g as in (26). Moreover, if G satisfies (30), the induced metric I can be estimated as follows

$$I \ = \ \left| \frac{(1 - \varepsilon)\,dh}{1 + \varepsilon\,|h|^2} + (1 + \varepsilon\,|h|^2)\,\overline{\varpi} \right|^2 \geq \left(\frac{(1 - \varepsilon)\,|dh|}{1 + \varepsilon\,|h|^2} - (1 + \varepsilon\,|h|^2)\,|\omega| \right)^2 =$$

$$\left(\frac{1 - \varepsilon}{\sqrt{-\varepsilon}\,(1 - |z|^2)} - (1 - |z|^2)\frac{\sqrt{-\varepsilon}}{2}|\{G, z\}| \right)^2 |dz|^2 \ \geq \ \left(\frac{1 - \varepsilon}{\sqrt{-\varepsilon}} - \frac{\sqrt{-\varepsilon}}{2}b_0 \right)^2 \frac{|dz|^2}{(1 - |z|^2)^2}.$$

Since $b_0 < \frac{2(1-\varepsilon)}{-\varepsilon}$, then $\frac{1-\varepsilon}{\sqrt{-\varepsilon}} - \frac{\sqrt{-\varepsilon}}{2}b_0$ is a positive number and the induced metric is, up to a constant, greater than or equal to the Poincaré metric on the unit disk. Hence, the immersion is complete. $\qquad \square$

Remark 5 Let $\psi_0 : \mathbb{D} \longrightarrow \mathbb{H}^3$ be a BLW-surface in R_3 with Weierstrass data $h_0(z) = z/\sqrt{-\varepsilon}$ and hyperbolic Gauss map G_0. If we consider the new immersion $\psi_1 : \mathbb{D} \longrightarrow \mathbb{H}^3$ with $h_1(\zeta) = \zeta/\sqrt{-\varepsilon}$ and $G_1(\zeta) = G_0(\varphi(\zeta))$ where $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ is a conformal equivalence, then $\psi_0(\varphi(\zeta)) = \psi_1(\zeta)$.

Indeed, since φ is a biholomorphic map onto the unit disk

$$\varphi(\zeta) = e^{i\theta} \frac{\zeta + \zeta_0}{\overline{\zeta_0}\zeta + 1}, \qquad \zeta \in \mathbb{D},$$

for some $\zeta_0 \in \mathbb{D}$, $\theta \in \mathbb{R}$ and

$$\frac{\varphi_\zeta(\zeta)}{1 - |\varphi(\zeta)|^2} = e^{i\theta} \frac{1 + \zeta_0 \overline{\zeta}}{1 + \overline{\zeta_0}\zeta} \frac{1}{1 - |\zeta|^2}.$$

Thus, one has

$$G_{1\,\zeta}(\zeta)\,(1 - |\zeta|^2) = e^{i\theta} \frac{1 + \zeta_0 \overline{\zeta}}{1 + \overline{\zeta_0}\zeta} G_{0\,z}(\varphi(\zeta))\,(1 - |\varphi(\zeta)|^2)$$

$$\mathcal{A}_{\psi_1}(\zeta) = e^{i\theta} \frac{1 + \zeta_0 \overline{\zeta}}{1 + \overline{\zeta_0}\zeta} \mathcal{A}_{\psi_0}(\varphi(\zeta))$$

and the above equality is clear from (28). □

Now, for the representation of some surfaces and the study of its geometric behaviour, we will identify the hyperbolic space $\mathbb{H}^3 \subset \mathbb{L}^4$ with the upper half space of $\mathbb{R}^3$ given by $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$ by considering the map

$$(32) \qquad\qquad (x_0, x_1, x_2, x_3) \longrightarrow \frac{1}{x_0 + x_3}\,(x_1, x_2, 1).$$

Under this identification the hyperbolic metric is given by

$$(33) \qquad\qquad ds^2 = \frac{1}{x_3^2}\,(dx_1^2 + dx_2^2 + dx_3^2)$$

and its ideal boundary $\mathbb{S}_\infty^2$ is identified with the one point compactification of the plane $\Pi \equiv \{x_3 = 0\}$, that is, $\mathbb{S}_\infty^2 = \Pi \cup \{\infty\}$. Thus, the asymptotic boundary of a set $\Sigma \subset \mathbb{H}^3$ is

$$\partial_\infty(\Sigma) = \mathrm{cl}(\Sigma) \cap \mathbb{S}_\infty^2$$

where $\mathrm{cl}(\Sigma)$ is the closure of Σ in $\{x_3 \geq 0\} \cup \{\infty\}$, that is, in the one point compactification of $\mathbb{H}^3 \cup \Pi$.

Example 1 Let $\psi : \mathbb{D} \longrightarrow \mathbb{H}^3$ be a complete totally umbilical BLW-surface with $\varepsilon < 0$. From the comments at the beginning of Section §3 and from Lemma 3, we have that the Weierstrass data of the immersion are $(z/\sqrt{-\varepsilon}, 0)$, $z \in \mathbb{D}$. In that case $G(z) = rz$, $r > 0$ is a holomorphic solution of $\{G, z\} = 0$. Consequently, from Theorem 4 and (32) the immersion is isometric to the following one:

$$\psi_1(u, v) = \frac{r}{1 - \varepsilon(u^2 + v^2)} \{(1 - \varepsilon)u, (1 - \varepsilon)v, \sqrt{-\varepsilon}(1 - u^2 - v^2)\},$$

$z = u + iv \in \mathbb{D}$. It is clear that the asymptotic boundary of ψ_1 is a circle in $\Pi \cup \{\infty\}$ of radius r, see Figure 2. $\qquad \square$

Example 2 For $\varepsilon = -1/4$, let $\psi : \mathbb{D} \longrightarrow \mathbb{H}^3$ be the BLW-surface associated with the Weierstrass data $(2z, 3dz/(8(1 - z^2)))$, $z \in \mathbb{D}$. Since a solution of the equation $2\{G, z\} = -3/(1 - z^2)$ is the holomorphic map $G(z) = (1 - z)^2/4$, $z \in \mathbb{D}$, we conclude from Theorem 4 and (32) that the immersion ψ is regular.

Consider the new conformal parameter $x + iy = \varphi(z) = 2i/(1 - z)$ and take τ the isometry of $\mathbb{H}^3$ given by the Euclidean inversion with respect to the unit sphere centered at $(0, 0, 0)$. Then $\tau \circ \psi$ is parametrized as $\psi_2 \circ \varphi$, where

$$
\psi_2(x, y) \;\; = \;\; \{ \frac{1 - 4y + 8y^3 + 8x^2 - 5x^4 + y^2 \, (5 - 3x^2)}{D},
$$

$$
\frac{-2x \, (6y^2 + 2y^3 + 2x^2 + 3y \, (-1 + x^2))}{D}, \frac{8 \, (-1 + y) \, (y^2 + x^2)^{\frac{3}{2}}}{D} \},
$$

and $D := 1 - 12y^5 + 8y^6 + 11x^2 - x^4 + 5x^6 + y^3 \, (16 - 24x^2) + 3y^4 \, (3 + 7x^2) + 2 \, (y + 3yx^2)^2 - 4y \, (1 + 3x^4)$.

It is clear, see Figure 3, that the immersion $\tau \circ \psi$ is regular but its asymptotic boundary is a non regular Jordan curve. $\qquad \square$

Example 3 Let $\varepsilon = -1/(4n^2 + 4n)$, with n a positive integer and $\psi : \mathbb{D} \longrightarrow \mathbb{H}^3$ be the BLW-surface associated with the Weierstrass data $\left(z/\sqrt{-\varepsilon}, -\frac{n+1}{2n}\sqrt{-\varepsilon}dz/(1 - z^2)^2 \right)$, $z \in \mathbb{D}$. Since a solution of the equation $n\{G, z\} = -(n + 1)/(1 - z^2)^2$ is the meromorphic map

$$G(z) = \tan\left(\frac{(1 + 2n) \left(-\pi + 2i \, \log(\frac{1+z}{1-z}) \right)}{4} \right),$$

$z \in \mathbb{D}$, it is not difficult to see that the immersion is not embedded and has as asymptotic boundary the set $\{(0, x_2, 0) : x_2 \in \mathbb{R}\} \cup \{\infty\}$ in $\Pi \cup \{\infty\}$.

In fact, for $n = 1$ if we consider τ the isometry of $\mathbb{H}^3$ given by the Euclidean inversion with respect to the unit sphere centered at $(1, 1, 0)$, then the asymptotic boundary of $\tau \circ \psi$ is the

circle of radius $1/2$ centered at $1/2 + i$ on $\Pi \cup \{\infty\}$. Moreover, by taking the new conformal parameter $x + iy = \varphi(z) = \log\left(\frac{1+z}{1-z}\right)$, the immersion $\tau \circ \psi$ is parametrized as $\psi_3 \circ \varphi$, where

$$
\begin{aligned}
\psi_3(x,y) \;=\; \frac{1}{B}\{ & 4\,\cos(y) - 4\,\cos(3\,y) + \cos(5\,y) + 2\,(13 + 4\,\cos(2\,y))\,(\cosh(3\,x) - \sinh(3\,x)),\\
& 2\,(4\,\cos(y) - 4\,\cos(3\,y) + \cos(5\,y)) + (13 + 4\,\cos(2\,y))\,(2\,\cosh(3\,x) - \sinh(3\,x)),\\
& 12\,\sqrt{2}\,\cos(y)\},
\end{aligned}
$$

and $B = 8\,\cos(y) - 8\,\cos(3\,y) + 2\,\cos(5\,y) + 3\,(13 + 4\,\cos(2\,y))\,\cosh(3\,x) - 4\,\sin(y) + 4\,\sin(3\,y) - \sin(5\,y) - 2\,(13 + 4\,\cos(2\,y))\,\sinh(3\,x)$, see Figure 4. $\qquad\square$

Example 4 We fix $\varepsilon = -1/8$ and take the BLW-surface $\psi : \mathbb{D} \longrightarrow \mathbb{H}^3$ associated with the Weierstrass data $\left(2\sqrt{2}z, -5/(2\sqrt{2}(1 - z^2)^2)dz\right)$, $z \in \mathbb{D}$. Since a solution of the equation $\{G, z\} = 10/(1 - z^2)^2$ is the holomorphic map

$$
G(z) = \tan(\log(\frac{z+1}{z-1})),
$$

$z \in \mathbb{D}$, from Theorem 4 and (32) ψ is a regular immersion and its asymptotic boundary are two disjoint circles in $\Pi \cup \{\infty\}$.

By taking the new conformal parameter $x + iy = \varphi(z) = \log\left(\frac{z+1}{z-1}\right)$, the immersion ψ is parametrized as $\psi_4 \circ \varphi$, where

$$
\begin{aligned}
\psi_4(x,y) \;=\; \{ & \frac{(-13 + 5\,\cos(2\,y))\,\sin(2\,x)}{\cos(2\,x)\,(-13 + 5\,\cos(2\,y)) - 3\,(7 + \cos(2\,y))\,\cosh(2\,y) - 4\,\sin(2\,y)\,\sinh(2\,y)},\\
& \frac{4\,\cosh(2\,y)\,\sin(2\,y) + 3\,(7 + \cos(2\,y))\,\sinh(2\,y)}{\cos(2\,x)\,(13 - 5\,\cos(2\,y)) + 3\,(7 + \cos(2\,y))\,\cosh(2\,y) + 4\,\sin(2\,y)\,\sinh(2\,y)},\\
& \frac{16\,\sqrt{2}\,\cos(y)}{\cos(2\,x)\,(-13 + 5\,\cos(2\,y)) - 3\,(7 + \cos(2\,y))\,\cosh(2\,y) - 4\,\sin(2\,y)\,\sinh(2\,y)}\},
\end{aligned}
$$

It is clear that the immersion can be considered as a embedding of a cylinder with two disjoint circles in $\Pi \cup \{\infty\}$ as its asymptotic boundary, see Figure 5. $\qquad\square$

For the study of these surfaces at infinity we pose the following *Plateau problem*:

> *Given $\varepsilon_0 < 0$ and a Jordan curve Γ on $\mathbb{S}_\infty^2 \equiv \Pi \cup \{\infty\}$, find a complete BLW-surface $\psi : S \longrightarrow \mathbb{H}^3$ satisfying*
>
> $$
> 2(-\varepsilon_0)(H - 1) + (\varepsilon_0 - 1)K_I = 0
> $$
>
> *with Γ as its asymptotic boundary.*

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Since every preserving orientation isometry of $\mathbb{H}^3$ induces a Möbius transformation on $\mathbb{S}^2_\infty$ (see, for instance, [24]) then we will assume that, up to a Möbius transformation, the Jordan curve Γ lies on Π.

Proposition 1 *Let Γ be a Jordan curve on $\Pi \equiv \{x_3 = 0\} \subset \mathbb{R}^3$, $int(\Gamma)$ the bounded component of $\Pi \setminus \Gamma$ and $G : \mathbb{D} \longrightarrow \mathbb{I}_\Gamma$ a conformal equivalence where $\mathbb{I}_\Gamma = \{x_1 + ix_2 \in \mathbb{C} : (x_1, x_2, 0) \in int(\Gamma)\}$. Then, if G satisfies (27) for a fixed $\varepsilon < 0$, the immersion $\psi : \mathbb{D} \longrightarrow \mathbb{H}^3$ associated with G by Theorem 4 lies in the region R_3 and it is a solution to the Plateau problem for the Jordan curve Γ.*

Moreover,

1. *ψ has a continuous extension $\widetilde{\psi}$ to the closure $\overline{\mathbb{D}}$ of $\mathbb{D}$, such that $\widetilde{\psi}_{|\partial \overline{\mathbb{D}}} : \partial \overline{\mathbb{D}} \longrightarrow \Gamma$ is a homeomorphism. If Γ is a C^∞ smooth Jordan curve then ψ and its derivatives have continuous extensions to $\overline{\mathbb{D}}$,*

2. *if ψ is an embedding and Γ is a differentiable curve such that each tangent line to Γ does not pass through a fixed point $p \in int(\Gamma)$ then the bounded domain by $\widetilde{\psi}(\overline{\mathbb{D}}) \cup int(\Gamma)$, $\mathcal{O}$, is starshaped from p in $\mathbb{R}^3$.*

Proof: First, we observe that G has an extension to a homeomorphism $\widetilde{G}$ of the closed disk $\overline{\mathbb{D}}$ onto the closure of $\mathbb{I}_\Gamma$ [8, Theorem II.3.4]. Moreover, if Γ is differentiable the above extension is a diffeomorphism [8, Theorem II.3.5].

On the other hand, from (28) and (32), the immersion can be written as

$$(34) \qquad \psi(z) = \left(G + \varepsilon \, G_z (1 - |z|^2) \, \frac{\overline{\mathcal{A}}}{1 - \varepsilon |\mathcal{A}|^2}, \sqrt{-\varepsilon} \, \frac{|G_z|(1 - |z|^2)}{1 - \varepsilon |\mathcal{A}|^2} \right).$$

Thus, if we consider the extension $\widetilde{\psi}$ of ψ in $\mathbb{R}^3$ such that $\widetilde{\psi}(z) = (\widetilde{G}(z), 0)$ when $|z| = 1$, $\widetilde{\psi}$ is a continuous map on $\overline{\mathbb{D}}$ since

$$1 - \varepsilon |\mathcal{A}|^2 \geq 1, \qquad \frac{|\mathcal{A}|}{1 - \varepsilon |\mathcal{A}|^2} \text{ is bounded}$$

and $|G_z|(1 - |z|^2) \to 0$ when $|z| \to 1$ (see, [4, Lemma 14.2.8]). In particular, ψ is a solution to the Plateau problem for the Jordan curve Γ.

Moreover, if Γ is differentiable then $\widetilde{G}$ is also differentiable and $\widetilde{G}_z \neq 0$ for $|z| = 1$, so $\mathcal{A}$ is well defined on $\overline{\mathbb{D}}$ and $\widetilde{\psi}$ is a smooth extension of ψ in $\mathbb{R}^3$.

Now, let us assume that ψ is an embedding and that Γ is a differentiable curve such that each tangent line to Γ does not pass through a fixed point $p \in int(\Gamma)$. We consider,

$$B_t^- = \{q \in \mathbb{R}^3 : \| q - p \| \leq t\} \cap \{x_3 \geq 0\}, \qquad B_t^+ = \{q \in \mathbb{R}^3 : \| q - p \| \geq t\} \cap \{x_3 \geq 0\}$$

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where $\| \,.\, \|$ denotes the usual Euclidean norm and $H_t = B_t^- \cap B_t^+$ the closed Euclidean hemisphere with radius t and centered at p.

The Euclidean inversion I_t with respect to H_t is a hyperbolic reflection when I_t is restricted to $\mathbb{H}^3$ and so we can use the Alexandrov reflection principle. That is, let

$$\Sigma_t^- = \widetilde{\psi}(\overline{\mathbb{D}}) \cap B_t^-, \qquad \Sigma_t^+ = I_t\left(\widetilde{\psi}(\overline{\mathbb{D}}) \cap B_t^+\right).$$

Since $\widetilde{\psi}(\overline{\mathbb{D}})$ is compact there exists t_0 such that $\widetilde{\psi}(\overline{\mathbb{D}}) \subset B_{t_0}^-$. Therefore, in a standard way, if $\mathcal{O}$ was not starshaped from p then, when the radius decreases from t_0 to 0, there would exist a first time t_1 such that either $\Sigma_{t_1}^-$ and $\Sigma_{t_1}^+$ are tangent at a point in their boundaries or they meet at a interior point.

These situations cannot happen for a point of Γ because $\widetilde{\psi}$ is differentiable and all tangent lines to Γ do not pass through p. Then, by using that (3) is an elliptic equation we have that $\Sigma_{t_1}^-$ and $\Sigma_{t_1}^+$ would be the same surface which is a contradiction. $\qquad\square$

Remark 6 From Remark 5, the solution to the Plateau problem associated with a Jordan curve Γ given by the above proposition does not depend on the selected conformal equivalence G from $\mathbb{D}$ onto $\mathbb{I}_\Gamma$.

Moreover, if ψ is an embedding, the hyperbolic normal to the immersion points towards $\mathcal{O}$. Otherwise, the hyperbolic normal would point upwards at the (Euclidean) highest point p_0 of the immersion, but the vertical straight half lines are hyperbolic geodesics and so $G(p_0) = \infty$ which contradicts that $G(\mathbb{D}) \subset \mathbb{I}_\Gamma$. $\qquad\square$

First, we solve the Plateau problem at infinity when the Jordan curve is convex.

Theorem 5 *Let Γ be a convex Jordan curve on $\Pi \equiv \{x_3 = 0\} \subset \mathbb{R}^3$. Then for any $\varepsilon < 0$ there exists a unique embedded solution to the Plateau problem for Γ with hyperbolic normal pointing downwards at its (Euclidean) highest point.*

Moreover if $-1/2 < \varepsilon < 0$, then there are exactly two embedded solutions to the Plateau problem for Γ.

Proof: Let us consider the bounded component $int(\Gamma)$ of $\Pi \setminus \Gamma$ and a conformal equivalence G from $\mathbb{D}$ onto $\mathbb{I}_\Gamma = \{x_1 + ix_2 \in \mathbb{C} : (x_1, x_2, 0) \in int(\Gamma)\}$. Then, from [14] or [9, Corollary 3], we obtain $|\{G, z\}| \leq 2/(1 - |z|^2)^2 < 2(1 - \varepsilon)/(-\varepsilon(1 - |z|^2)^2)$, $\forall \varepsilon < 0$. Therefore, from Proposition 1, the immersion ψ associated with ε and G is a solution of the Plateau problem for Γ.

Now, we shall prove that ψ is an embedding. We observe that the induced metric I of the immersion ψ is conformal to its usual metric I_0 as an immersion into $\mathbb{R}^3$, in fact, from (33), $I_0 = \psi_3^2 I$ where $\psi_3 > 0$ denotes the third coordinate immersion. Hence, the Euclidean curvature K_0 of ψ can be calculated as

$$\psi_3^2 K_0 = K_I - \Delta^I(\log \psi_3),$$

here Δ^I denotes the Laplacian associated with the induced metric I.

Since the Weierstrass data of the immersion ψ are given by $h(z) = z/\sqrt{-\varepsilon}$ and $\omega = -1/2\sqrt{-\varepsilon}\{G, z\}dz$ we obtain from (19), (23), (34) and by a straightforward computation of the above Laplacian that

$$K_I = \frac{-16\varepsilon}{\varepsilon^2|\{G, z\}|^2(1 - |z|^2)^4 - 4(1 - \varepsilon)^2}$$

and

$$K_0 = \frac{-4\varepsilon\left(4(1 - |\mathcal{A}|^2)^2 - |\{G, z\}|^2(1 - |z|^2)^4\right)}{|G_z|^2(1 - |z|^2)^2\left(4(1 - \varepsilon)^2 - \varepsilon^2(1 - |z|^2)^4\right)}.$$

Thus, by using [9, Corollary 3], one has $4(1 - |\mathcal{A}|^2)^2 - |\{G, z\}|^2(1 - |z|^2)^4 \geq 0$ and ψ is an immersion with non negative Euclidean curvature and planar boundary, that is, ψ is an embedding (see, for instance, [7] or [27]).

From Remark 6, the hyperbolic normal of the immersion ψ points downwards at its (Euclidean) highest point. Let us assume $\chi : S \longrightarrow \mathbb{H}^3 \subset \mathbb{R}^3$ is another solution to the Plateau problem for a fixed $\varepsilon_0 < 0$ and the convex Jordan curve Γ, with hyperbolic normal pointing downwards at its highest point, or equivalently, pointing towards the enclosed Euclidean domain by $\mathrm{cl}(\chi(S)) \cup int(\Gamma)$.

Without loss of generality we may assume the origin belongs to $int(\Gamma)$. Let C_0 and C_2 be two circles on Π centered at the origin and bounding a closed annulus A in Π containing Γ in its interior. Choose C_0 and C_2 so that the totally umbilical BLW-surfaces (spherical caps, with hyperbolic normal pointing downwards), S_0 and S_2 associated with ε_0 and asymptotic boundary C_0 and C_2, respectively, satisfy S_0 is below $\chi(S)$ and $\chi(S)$ is below S_2 (see Example 1).

Then, we consider Γ_t, $0 \leq t \leq 2$, a foliation of the annulus A such that Γ_t is convex for all $t \in [0, 2]$, $\Gamma_0 = C_0$, $\Gamma_1 = \Gamma$ and $\Gamma_2 = C_2$. Let G_t be the conformal equivalence from $\mathbb{D}$ onto $\{x_1 + ix_2 \in \mathbb{C} : (x_1, x_2, 0) \in int(\Gamma_t)\}$ such that $G_t(0) = 0$ and $(G_t)_z(0)$ is a positive real number. Since Γ_t is also convex we have $|\{G_t, z\}| \leq 2/(1 - |z|^2)^2$ (see [14]) and we can consider, from Proposition 1, the BLW-surface $\psi_t : \mathbb{D} \longrightarrow \mathbb{H}^3 \subset \mathbb{R}^3$ associated with ε_0 and hyperbolic Gauss map G_t. In particular, $\psi_0(\mathbb{D}) = S_0$, $\psi_1(\mathbb{D}) = \psi(\mathbb{D})$ and $\psi_2(\mathbb{D}) = S_2$. Moreover, ψ_t is an embedding as it was proved for ψ.

We want to show that $\chi(S) = \psi_1(\mathbb{D})$. First, we observe that if $t_n \in [0, 2]$ and $\{t_n\}$ is a sequence converging to t_0 then $\{\widetilde{G_{t_n}}\}$ converges uniformly to $\widetilde{G_{t_0}}$ (see [8, Theorem II.5.2]), where $\widetilde{G_t}$ is the extension of G_t to $\overline{\mathbb{D}}$.

Let us prove that if $0 \leq t < 1$ then $\psi_t(\mathbb{D})$ is below $\chi(S)$. If $J = \{t \in [0, 1[: \psi_t(\mathbb{D}) \cap \chi(S) \neq \emptyset\}$ is non empty then we see that J has a minimum; indeed we observe that given a sequence $\{t_n\} \subset J$ converging to the infimum t_0 of J, there would exist $p_n \in \mathbb{D}$ such that $\psi_{t_n}(p_n) \in \mathrm{cl}(\chi(S))$ and a subsequence $\{p_m\}$ would converge to a point $p_0 \in \overline{\mathbb{D}}$ and, since $\{\psi_{t_m}\}$ converges uniformly to ψ_{t_0} then $\psi_{t_0}(p_0) \in \mathrm{cl}(\chi(S))$. But, $p_0 \notin \partial\mathbb{D}$ because of $\Gamma_{t_0} \cap \Gamma_1 = \emptyset$, that is, $t_0 \in J$.

Assume $J \neq \emptyset$ then, using the uniform convergence for any $t \in [0, t_0[$, one has that ψ_t and ψ_{t_0} are below $\chi(S)$. Thus, since $\psi_{t_0}(\mathbb{D})$ and $\chi(S)$ are tangent at some point with the same hyperbolic normal and $\psi_{t_0}(\mathbb{D})$ is below $\chi(S)$ we have that $\psi_{t_0}(\mathbb{D})$ and $\chi(S)$ agree, but this is not possible because they have different asymptotic boundary.

Consequently, $J = \emptyset$ and using another time the uniform convergence one has $\psi_1(\mathbb{D})$ is below $\chi(S)$. Analogously, it can be showed that $\chi(S)$ is below $\psi_1(\mathbb{D})$ reasoning for the interval $[1, 2]$ and it yields $\psi_1(\mathbb{D}) = \chi(S)$.

Let us now consider $\varepsilon_0 \in]-1/2, 0[$, we want to prove that there exists another embedded solution to the Plateau problem for ε_0 and for the convex Jordan curve Γ. Let Γ_t, $0 \leq t \leq 1$, be a foliation of the topological annulus bounded by the circle C_0 and Γ in Π such that Γ_t is smooth for any $t \in [0, 1[$, $\Gamma_0 = C_0$ and $\Gamma_1 = \Gamma$.

Let $f : \mathbb{R}^3 \setminus \{(0,0,0)\} \longrightarrow \mathbb{R}^3 \setminus \{(0,0,0)\}$ be the map given by

$$(35) \qquad f(x_1, x_2, x_3) = \left(\frac{x_1}{x_1^2 + x_2^2 + x_3^2}, \frac{-x_2}{x_1^2 + x_2^2 + x_3^2}, \frac{x_3}{x_1^2 + x_2^2 + x_3^2} \right),$$

f is a preserving orientation hyperbolic isometry when it is restricted to $\mathbb{H}^3$.

Bearing in mind that the origin belongs to $int(\Gamma_t)$ for any $t \in [0, 1]$, we take the Jordan curves $\varphi_t = f(\Gamma_t) \subset \Pi$ (non necessarily convex). And we also call G_t to the conformal equivalence from $\mathbb{D}$ onto $\{x_1 + ix_2 : (x_1, x_2, 0) \in int(\varphi_t)\}$ such that $G_t(0) = 0$ and $(G_t)_z(0)$ is a positive real number. Then, from [13], one has $|\{G_t, z\}| \leq 6/(1 - |z|^2)^2$ and, using that $-1/2 < \varepsilon_0$ and Proposition 1, the immersion $\psi_t : \mathbb{D} \longrightarrow \mathbb{H}^3$ associated with ε_0 is well defined and ψ_t is a solution to the Plateau problem for the Jordan curve φ_t, or equivalently, $f \circ \psi_t$ is a solution to the Plateau problem for Γ_t (since $f \circ f$ is the identity). Moreover, if ψ_t is an embedding with hyperbolic normal pointing downwards at its highest point then $f \circ \psi_t$ is an embedding with hyperbolic normal pointing upwards at its highest point because the origin is in $int(\varphi_t)$.

We want to prove that ψ_1 is an embedding. If we define the interval $L = \{t \in [0, 1] : \psi_s$ is an embedding for all $s \in [0, t]\}$, we know that $0 \in L$ and only need to prove that L is open and closed in $[0, 1]$.

We claim L must be open. Otherwise, since L is an interval, $L = [0, t_0]$ with $t_0 \neq 1$ and there would exist a sequence $\{t_n\} \subset]t_0, 1[$ converging to t_0 such that ψ_{t_n} is not an embedding. Hence, there would exist $p_n, q_n \in \mathbb{D}$, $p_n \neq q_n$, satisfying $\psi_{t_n}(p_n) = \psi_{t_n}(q_n)$. But a subsequence $\{p_m\}$ of $\{p_n\}$ must converge to a point $p_0 \in \overline{\mathbb{D}}$ and a subsequence $\{q_r\}$ of $\{q_m\}$ converges to a point $q_0 \in \overline{\mathbb{D}}$.

Reasoning as above, $\{\widetilde{\psi_{t_r}}\}$ converges uniformly to $\widetilde{\psi_{t_0}}$, where $\widetilde{\psi_t}$ denotes the differentiable extension of ψ_t to $\overline{\mathbb{D}}$. In that way, $\widetilde{\psi_{t_0}}(p_0) = \widetilde{\psi_{t_0}}(q_0)$, but $\widetilde{\psi_{t_0}}$ is an embedding, that is, $p_0 = q_0$.

From the uniform convergence, there exists a neighbourhood U of $p_0 = q_0$ in $\overline{\mathbb{D}}$ and r_0 such that $\widetilde{\psi_{t_r}}(U)$ is a graph on the tangent plane to the immersion $\widetilde{\psi_{t_0}}$ at $p_0 = q_0$ for all $r \geq r_0$, which contradicts that $\widetilde{\psi_{t_r}}(p_r) = \widetilde{\psi_{t_r}}(q_r)$ with $p_r \neq q_r$ for r big enough such that $p_r, q_r \in U$.

L must be closed. We observe that the origin is a point in $int(\varphi_t)$ for any $t \in [0,1]$ and the tangent line l to φ_t at a point p does not pass through the origin, otherwise, we would obtain that the tangent line to $\Gamma_t = f(\varphi_t)$ at $f(p)$ is $f(l) = l$ which is a contradiction because Γ_t is convex and the origin is in $int(\Gamma_t)$.

Consequently, if the interval L is not closed then $L = [0, t_0[$ and one has, from Proposition 1, that the bounded Euclidean domain $\mathcal{O}_t$ enclosed by $\widetilde{\psi_t}(\overline{\mathbb{D}}) \cup int(\Gamma_t)$ is starshaped from the origin for all $t \in L$. Using again the uniform convergence of $\widetilde{\psi_t}$ for $t \leq t_0$, the domain $\mathcal{O}_{t_0}$ enclosed by $\widetilde{\psi_{t_0}}(\overline{\mathbb{D}}) \cup int(\Gamma_{t_0})$ is also starshaped from the origin and then ψ_{t_0} must be an embedding. Hence, $L = [0,1]$ and ψ_1 is an embedding.

The uniqueness of $f \circ \psi_1$ as the only embedding with asymptotic boundary Γ and hyperbolic normal pointing upwards at its highest point can be proven in a similar way to the above case. $\qquad\square$

Remark 7 Notice that the Euclidean homotheties with center at a point of Π are preserving orientation hyperbolic isometries when they are restricted to $\mathbb{H}^3 \subset \mathbb{R}^3$ (see [24]) and then the uniqueness part of the above theorem could be proven in an easier way changing the above foliation and considering the foliation given by the image of $\psi(\mathbb{D})$ using the homotheties with center at a point in $int(\Gamma)$.

Though this easier proof can be applied when $int(\Gamma)$ is starshaped, it does not work for a general Jordan curve as it will be needed in Theorem 6. $\qquad\square$

Finally, we shall consider the case of a general Jordan curve Γ.

Theorem 6 *Let Γ be a Jordan curve on $\Pi \equiv \{x_3 = 0\} \subset \mathbb{R}^3$. Then for any $\varepsilon \in]-1/2, 0[$ there exist at least two solutions to the Plateau problem for Γ. Moreover, if $-1/4 < \varepsilon < 0$ then there only exist two embedded solutions to this Plateau problem.*

Proof: Without loss of generality we can assume the origin belongs to $int(\Gamma)$, thus reasoning as in Theorem 5, if we take $\varphi = f(\Gamma)$, where f is given by (35), and

$$G_1 : \mathbb{D} \longrightarrow \{x_1 + ix_2 : (x_1, x_2, 0) \in int(\Gamma)\} \qquad G_2 : \mathbb{D} \longrightarrow \{x_1 + ix_2 : (x_1, x_2, 0) \in int(\varphi)\}$$

are two conformal equivalences then, from [13], $|\{G_i, z\}| \leq 6/(1 - |z|^2)^2$, for $i = 1, 2$ and, using $\varepsilon \in]-1/2, 0[$ and Proposition 1, we can consider the immersions $\psi_1, \psi_2 : \mathbb{D} \longrightarrow \mathbb{H}^3 \subset \mathbb{R}^3$ with associated hyperbolic Gauss map G_1 and G_2, respectively. Therefore, ψ_1 and $f \circ \psi_2$ are two solutions to the Plateau problem for Γ.

On the other hand, if $-1/4 < \varepsilon < 0$ then ψ_1 and ψ_2 must be embedded. In fact, we shall prove that ψ_1 is a graph on the plane Π and analogously could be done for ψ_2.

From (29) and (32), the hyperbolic normal η of ψ_1 is given by

$$\eta = \left(-2\varepsilon\, (G_1)_z (1 - |z|^2)\, \frac{\overline{\mathcal{A}}}{(1 - \varepsilon|\mathcal{A}|^2)^2}, -\sqrt{-\varepsilon}\, |(G_1)_z|(1 - |z|^2)\frac{1 + \varepsilon|\mathcal{A}|^2}{(1 - \varepsilon|\mathcal{A}|^2)^2} \right).$$

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And, since from (33), the induced metrics on ψ_1 from $\mathbb{R}^3$ and $\mathbb{H}^3$ are conformal then the standard Euclidean normal N to ψ_1 is

$$N = \left(2\sqrt{-\varepsilon}\, \frac{(G_1)_z}{|(G_1)_z|}\, \frac{\overline{\mathcal{A}}}{1 - \varepsilon|\mathcal{A}|^2}, -\frac{1 + \varepsilon|\mathcal{A}|^2}{1 - \varepsilon|\mathcal{A}|^2} \right).$$

On the other hand, $|\mathcal{A}| \leq 2$ because of $|\{G_1, z\}| \leq 6/(1 - |z|^2)^2$ (see [16, Folgerung 2.3]) and the third coordinate of N is negative. Consequently, the projection $p : \psi_1(\mathbb{D}) \longrightarrow int(\Gamma)$ given by $p(x_1, x_2, x_3) = (x_1, x_2)$ is a local diffeomorphism and it is proper, therefore, p is a global diffeomorphism and $\psi_1(\mathbb{D})$ is a graph on the plane Π.

By considering a foliation Γ_t, $0 \leq t \leq 2$, of an annulus A centered at a point in $int(\Gamma)$ such that Γ_0 is a small enough circle in $int(\Gamma)$, $\Gamma_1 = \Gamma$ and Γ_2 is a big enough circle containing Γ in its interior, one can prove the uniqueness part in a similar way to Theorem 5, bearing in mind that any immersion ψ_t associated with Γ_t is an embedding. $\qquad \square$

Remark 8 A solution to the Plateau problem for a non regular Jordan curve can be found in Example 2. Moreover, from Example 3, we can see that the condition of embeddedness in Theorems 5 and 6 is essential. $\qquad \square$

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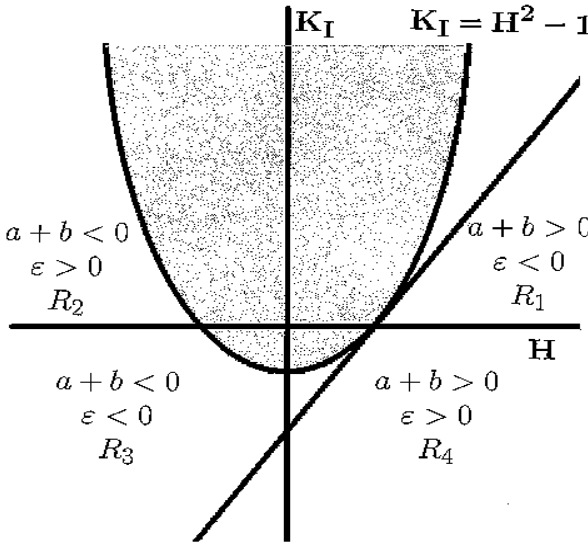email: jagalvez@goliat.ugr.es, amartine@goliat.ugr.es, milan@goliat.ugr.es
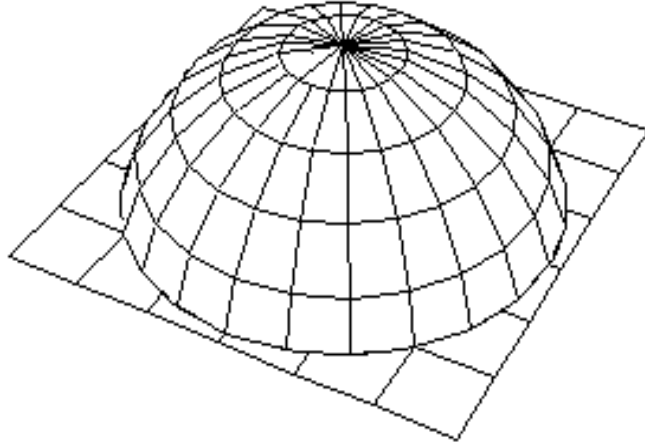
Figure 1.

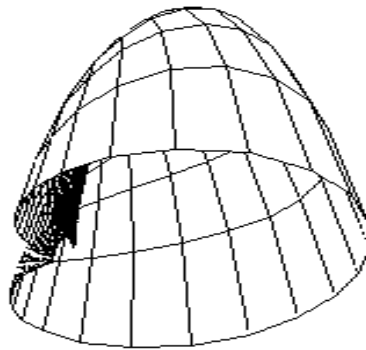Figure 2: Totally umbilical BLW-surfaces.

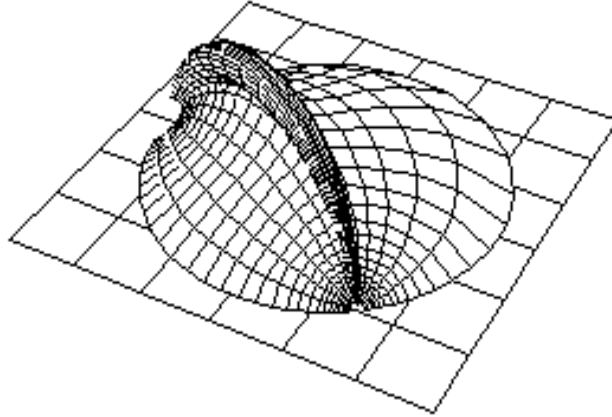Figure 3: BLW-surface with non regular asymptotic boundary.

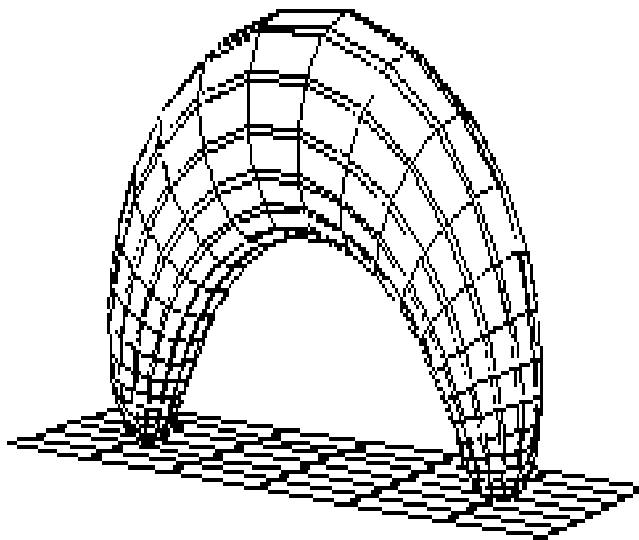Figure 4: Non embedded BLW-surface with embedded asymptotic boundary.

Figure 5: BLW-surface with a non connected embedded asymptotic boundary.