On affine maximal surfaces. The affine Cauchy Problem.

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Joint work with J. A. Aledo and A. Martínez

About the Problem and its Motivation

PDE's Theory

- Cauchy Problem

Surfaces Theory

- Björling Problem

Classical Björling Problem

- Asks for the existence of minimal surfaces in $\mathbb{R}^3$ containing a given curve with a prescribed unit normal along it.
- Was proposed by Björling in 1844, solved by Schwarz in 1890 (using holomorphic data) and used to prove interesting geometric properties of minimal surfaces in $\mathbb{R}^3$.
- Has been extended and global applications of it have been developed to other geometric theories:
  1. maximal surfaces in $\mathbb{L}^3$
  2. flat surfaces in $\mathbb{H}^3$
  3. improper affine spheres

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Goal: Its extension to the affine case

A Björling-type problem

Existence and uniqueness of affine maximal surfaces containing a curve in $\mathbb{R}^3$ with a given affine normal along it.

Main Schedule

Affine maximal surfaces

The equiaffine area functional

$$\int dA = \int |K_e|^{\frac{1}{4}} dA_e,$$

K_e the euclidean Gauss curvature and dA_e the element of euclidean area, has attracted to many geometers since the beginning of the last century.

Well-known Facts:

- Blaschke (1923): a fourth order Euler-Lagrange equation equivalent to the vanishing of the affine mean curvature (affine minimal surfaces)
- Calabi (1982): l.s.c.s have always a negative second variation (affine maximal surfaces)

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Recent developments

- Affine Weierstrass formulas that have provided an important tool in their study, (Calabi, Li, 1990).
- Entire solutions of the fourth order affine maximal surface equation

$$\phi_{yy}\omega_{xx} - 2\phi_{xy}\omega_{xy} + \phi_{xx}\omega_{yy} = 0, \quad \omega = \left(det\left(\nabla^2\phi\right)\right)^{-3/4}, \quad (1)$$

  are always quadratic polynomials (Trudinger-Wang, 2000)
- Every Affine complete affine maximal surface must be an elliptic paraboloid, (Li-Jia, 2001, Trudinger-Wang, 2002).
- There is a formulation of the Affine Plateau Problem as a geometric variational problem for the equiaffine area functional for which the existence and regularity of maximizers have been proved (Trudinger-Wang, 2005)

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- Their extension to different nonlinear fourth order equations (Li-Jia, 2003,Trudinger-Wang, 2002)
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Recent developments

IA spheres $\Leftrightarrow$ constant affine normal $\Leftrightarrow$ $Det(\nabla^2\Phi) = 1$

About singularities

- Classification of global graphs with a finite number of isolated singularities.
- Improper affine maps.
- Isolated singularities are in 1-1 correspondence with planar convex curves.

The Björling problem, an interesting tool

A previous study of the corresponding Björling problem has been very useful to understand and motivate the study of singularities. (A curve in $\mathbb{R}^3$ determine a unique IA-map containing the curve in its singular set, Aledo-Chaves-Gálvez, 2007).

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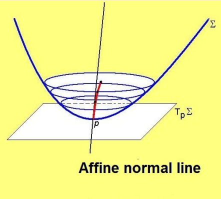
Basic Notations

$\psi : \Sigma \to \mathbb{R}^3$ l.s.c immersion, σ_e definite positive.

$$\begin{aligned} g &= K_e^{-\frac{1}{4}} \sigma_e, \quad \text{Berwald-Blaschke metric} \\ dA &= K_e^{\frac{1}{4}} dA_e, \quad \text{equiaffine area} \\ \xi &= \frac{1}{2} \Delta_g \psi, \quad \text{affine normal} \end{aligned}$$



$\Delta_g :=$ Laplace-Beltrami operator associated to g.

The affine conormal field $N := K_e^{-1/4} N_e$, satisfies

$$\langle N, \xi \rangle = 1, \qquad \langle N, d\psi(v) \rangle = 0, \quad v \in T_p\Sigma, \tag{2}$$

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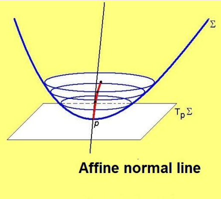

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Weierstrass-type Representation Formulas

Euler-Lagrange equation:$= \Delta_g N = 0$.

Lelieuvre formula

$$\psi = 2\,\mathrm{Re} \int \imath\, N \times N_z dz$$

Calabi's Representation

ψ determine a holomorphic curve $\Phi : \Omega \subset \Sigma \to \mathbb{C}^3$ s.t.

$$N = \Phi + \overline{\Phi}, \qquad g = -\imath Det \left[\Phi + \overline{\Phi}, \Phi_z, \overline{\Phi_z}\,\right] dz d\bar{z}. \qquad (3)$$

ψ is determined, up to real translation, by a holomorphic curve Φ satisfying $-\imath Det \left[\Phi + \overline{\Phi}, \Phi_z, \overline{\Phi_z}\,\right] > 0$. To be precise,

$$\psi = -\imath \left(\Phi \times \overline{\Phi} - \int \Phi \times d\Phi + \int \overline{\Phi} \times \overline{d\Phi}\right)$$

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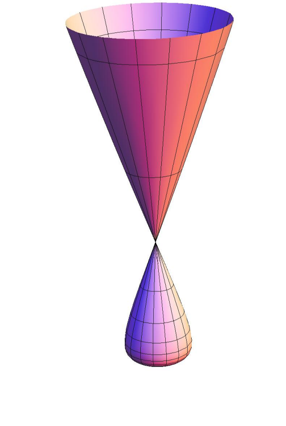
Some Examples

$N = (u, v, Log\sqrt{u^2 + v^2} + 1)$,
$g = Log\sqrt{u^2 + v^2}(du^2 + dv^2)$



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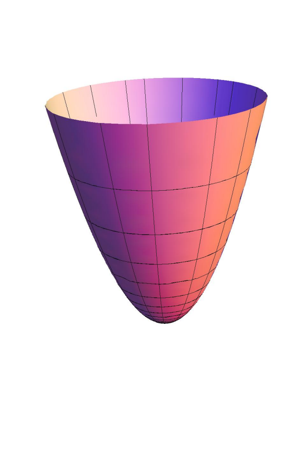

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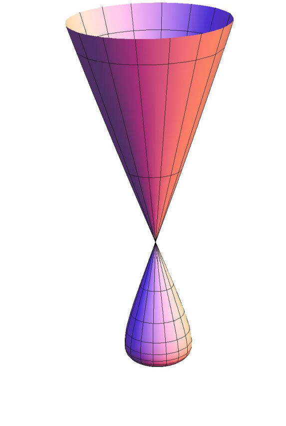

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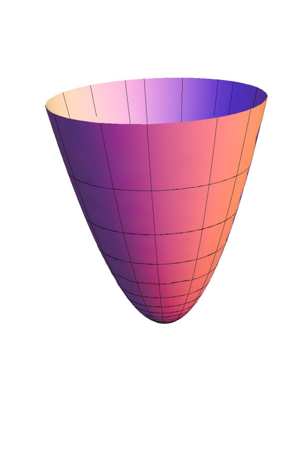

Solving the Problem

$\psi : \Sigma \to \mathbb{R}^3$, ξ, N.
$\beta : I \to \Sigma$ regular curve. $\alpha = \psi \circ \beta$, $Y = \xi \circ \beta$ and $U = N \circ \beta$, then, along the curve α

$$\left.\begin{array}{l} 0 = \langle \alpha'(s), U(s) \rangle, \\ 1 = \langle Y(s), U(s) \rangle, \\ 0 = \langle Y'(s), U(s) \rangle, \\ 0 < \lambda(s) = -\langle \alpha'(s), U'(s) \rangle = \langle \alpha''(s), U(s) \rangle, \end{array}\right\} \qquad (4)$$

where by prime we indicate derivation respect to s, for all $s \in I$.

Definition

Given $Y, U, \alpha : I \longrightarrow \mathbb{R}^3$ regular analytic curves.
$\{Y, U\}$ is an *analytic equiaffine normalization* of α if there is an analytic positive function $\lambda : I \to \mathbb{R}^+$ such that all the equations in (4) hold on I.

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Results

Main Theorem

$\{Y, U\}$ a.e.n. of $\alpha \Rightarrow \exists_1\ \psi$ containing $\alpha(I)$ with conormal field and Blaschke normal along α, U and Y respectively.
$\psi :=$ a.m.s. along α generated by $\{Y, U\}$

Outline of the Proof

- By the Inverse Function Theorem $\exists\ z = s + \imath t,\ s \in I$
- Identity Principle: $N_z = \frac{1}{2}\left(U_z + \imath Y \times \alpha_z\right), \qquad z \in \Omega$
- $$\psi = \alpha(s_0) + 2\,\mathrm{Re}\int_{s_0}^{z} \imath(\Phi + \overline{\Phi}) \times \Phi_\zeta d\zeta, \qquad (5)$$

  where,
  $\Phi(z) = \frac{1}{2}\left(U(z) + \imath\int_{s_0}^{z} Y \times \alpha_\zeta d\zeta\right), \qquad z \in \Omega, \quad s_0 \in I$, on a complex domain Ω containing I.

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Some consequences

(4) Corollary

$\alpha, Y : I \to \mathbb{R}^3$ be two regular analytic curves

$$Det[\ Y', \alpha', Y\ ]Det[\ Y', \alpha', \alpha''\ ] > 0, \quad \text{on} \quad I. \tag{6}$$

$\Rightarrow \exists_1\ \psi$ containing $\alpha(I)$ with Y as Blaschke normal along α.

$\exists_1\ U$ and λ,

$$U = \frac{Y' \times \alpha'}{Det[\ Y', \alpha', Y\ ]}, \qquad 0 < \lambda = \frac{Det[\ Y', \alpha', \alpha''\ ]}{Det[\ Y', \alpha', Y\ ]}$$

s.t. $\{Y, U\}$ is an a.e.n. of α. The result follows from above Theorem, taking in Calabi's representation,

$$\Phi(z) = \frac{Y_z \times \alpha_z}{2Det[\ Y_z, \alpha_z, Y\ ]} + \frac{\mathrm{i}}{2}\int_{s_0}^{z} Y \times \alpha_\zeta d\zeta, \qquad z \in \Omega, \quad s_0 \in I,$$

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Corollary

$\alpha, Y : I \to \mathbb{R}^3$ regular analytic curves

$$Det[\ Y, \alpha', \alpha''\ ] \neq 0, \quad Y' \times \alpha' = 0, \quad \text{on} \quad I. \tag{7}$$

Given $\lambda : I \to \mathbb{R}^+$, $\exists_1\ \psi$ containing $\alpha(I)$, such that its Blaschke normal along α is Y and $g(\alpha', \alpha') = \lambda$.
ψ can be written via Calabi's representation by taking

$$\Phi(z) = \frac{(-\alpha_{zz} + \lambda Y) \times \alpha_z}{2Det[\ \alpha_z, \alpha_{zz}, Y\ ]} + \frac{\imath}{2} \int_{s_0}^{z} Y \times \alpha_\zeta d\zeta, \quad z \in \Omega, \quad s_0 \in I,$$

As before, U is determined uniquely from Y and λ as

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Some consequences

Corollary

$\psi : \Sigma \to \mathbb{R}^3$, connected a.m.s. and $\beta : I \to \Sigma$ a regular curve s.t. $\alpha = \psi \circ \beta$ is analytic and $Y = \xi \circ \beta$ is constant. If (7) holds, $\Rightarrow$ ψ is an IA sphere.

The proof follows from the existence of IA spheres containing a given analytic curve under the above condition, and the uniqueness in our Theorem.

Remark

If $Y' \times \alpha' = 0$, $Det[\, Y, \alpha', \alpha''\,] = 0$ and there is an affine maximal surface ψ containing $\alpha(I)$ whose Blaschke normal along α is Y, then there exist infinitely many affine maximal surfaces containing $\alpha(I)$, with Y as Blaschke normal along α (Actually, if $\{Y, U\}$ a.e.n. $\Leftrightarrow$ $\{Y, U + \mu Y \times \alpha'\}$ a.e.n.

Some consequences

Corollary

$\psi : \Sigma \to \mathbb{R}^3$, connected a.m.s. and $\beta : I \to \Sigma$ a regular curve s.t. $\alpha = \psi \circ \beta$ is analytic and $Y = \xi \circ \beta$ is constant. If (7) holds, $\Rightarrow \psi$ is an IA sphere.

The proof follows from the existence of IA spheres containing a given analytic curve under the above condition, and the uniqueness in our Theorem.

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If $Y' \times \alpha' = 0$, $Det[\ Y, \alpha', \alpha''\ ] = 0$ and there is an affine maximal surface ψ containing $\alpha(I)$ whose Blaschke normal along α is Y, then there exist infinitely many affine maximal surfaces containing $\alpha(I)$, with Y as Blaschke normal along α (Actually, if $\{Y, U\}$ a.e.n. $\Leftrightarrow \{Y, U + \mu Y \times \alpha'\}$ a.e.n.

The Cauchy Problem

If $\psi : \Omega \longrightarrow \mathbb{R}^3$ is the graph of a l.s.c. function $\phi(x,y)$, $(x,y) \in \Omega$. The Euler-Lagrange equation for the affine area functional is

$$\phi_{yy}\omega_{xx} - 2\phi_{xy}\omega_{xy} + \phi_{xx}\omega_{yy} = 0, \qquad \omega = \left(det\left(\nabla^2\phi\right)\right)^{-3/4}.$$

In this situation

$$\begin{aligned} g_\phi &= \sqrt[3]{\omega}\left(\phi_{xx}\,dx^2 + 2\phi_{xy}\,dx\,dy + \phi_{yy}\,dy^2\right), \\ N &= \sqrt[3]{\omega}\left(-\phi_x, -\phi_y, 1\right), \\ \xi &= \left(\varphi_y, -\varphi_x, \frac{1}{\sqrt[3]{\omega}} - \phi_y\varphi_x + \phi_x\varphi_y\right), \end{aligned} \tag{8}$$

where

$$\varphi_x = \frac{1}{3}\left(\phi_{xy}\omega_x - \phi_{xx}\omega_y\right), \qquad \varphi_y = \frac{1}{3}\left(\phi_{yy}\omega_x - \phi_{xy}\omega_y\right).$$

The Cauchy Problem

A Initial Value Problem

$$\begin{cases} \phi_{yy}\omega_{xx} - 2\phi_{xy}\omega_{xy} + \phi_{xx}\omega_{yy} = 0, \qquad \omega = \left(det\left(\nabla^2\phi\right)\right)^{-3/4} \\ \\ \phi(x,0) = a(x), \\ \phi_y(x,0) = b(x), \\ \phi_{yy}(x,0) = c(x), \\ \phi_{yyy}(x,0) = d(x), \\ c(x)a''(x) - b'(x)^2 > 0 \end{cases}$$

where a, b, c, d are analytic functions on I, ϕ is defined on Ω containing $I \times \{0\}$. We are assuming that $c(x)a''(x) - b'(x)^2 > 0$ because the convexity. In particular, up to change of orientation, we can take $a''(x) > 0$ on I.

Its solution

$\exists_1\ \phi(x,y)$ solution to the above C.P. such that

$$(x,y,\phi(x,y)) = (s_0, 0, a(s_0)) + 2\,\mathrm{Re}\int_{s_0}^{z=s+\mathrm{i}\,t} \left(\Phi + \overline{\Phi}\right) \times \Phi_\zeta \, d\zeta,$$

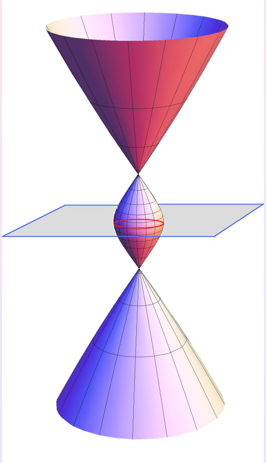
$$\Phi(z) = \frac{1}{2}\left(U(z) + \imath \int_{s_0}^{z} Y(\zeta) \times A(\zeta)\, d\zeta\right),$$

$$\begin{aligned}
U(s) &= \left(c(s)a''(s) - b'(s)^2\right)^{-1/4} (-a'(s), -b(s), 1),\\
A(s) &= (1, 0, a'(s)),\\
Y(s) &= \frac{1}{4}\left(c(s)a''(s) - b'(s)^2\right)^{-7/4} \left(b'(da'' + 3cb'') - 2b'^2c' - \right.\\
&\quad c(c'a'' + ca'''), b'(3c'a'' + ca''') - 2b'^2b'' - a''(da'' + cb''),\\
&\quad 4b'^4 - 2b'^2(a'c' + 4ca'' + bb'') - a''((-4c^2 + bd)a'' + bcb'')\\
&\quad \left. - ca'(c'a'' + ca''') + b'\left(a'(da'' + 3cb'') + b(3c'a'' + ca''')\right)\right).
\end{aligned}$$

Symmetry

$T : \mathbb{R}^3 \to \mathbb{R}^3, \; T(v) = Av + b, \;\; v \in \mathbb{R}^3.$
$\{Y, U\}$ a.e.n. of $\alpha : I \to \mathbb{R}^3$.
Say T is a symmetry of the a.e.n. if $\exists$
$\Gamma : I \to I$ analytic diffeomorphism s.t.

$$\alpha \circ \Gamma = T \circ \alpha, \quad Y \circ \Gamma = AY, \quad U \circ \Gamma = (A^t)^{-1} U.$$



Generalized symmetry principle

Any symmetry of an analytic equiaffine normalization induces a global symmetry of the affine maximal surface generated by the equiaffine normalization.

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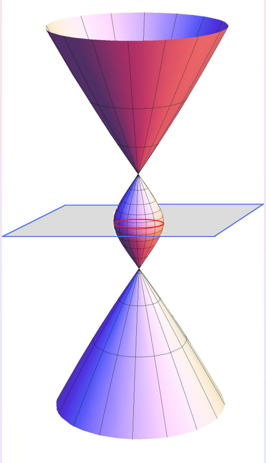

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Geodesics

pre-geodesics

$\psi : \Sigma \rightarrow \mathbb{R}^3$, ξ, N.
If $\beta : I \rightarrow \Sigma$ is a regular curve s.t., $\alpha = \psi \circ \beta$, $Y = \xi \circ \beta$ and $U = N \circ \beta$ are analytic $\Rightarrow$ α is a pre-geodesic for the Blaschke metric if and only if

$$Det[\alpha', \alpha'', Y] + Det[U, U', U''] = 0 \qquad \text{on} \quad I. \tag{9}$$

geodesics

$\alpha : I \rightarrow \mathbb{R}^3$ be a regular analytic curve. $\Rightarrow$ α is pre-geodesic (geodesic) of some affine maximal surface $\Leftrightarrow$ there exists an affine equiaffine normalization $\{Y, U\}$ of α satisfying (9) ((9) and $\langle \alpha'', U \rangle = constant > 0$.)

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Planar geodesics or pre-geodesics

Theorem

Every planar analytic l.s.c. curve is pre-geodesic of an affine maximal surface which has the plane containing the curve as a symmetry plane.

Proof

Choose $\{Y, U\}$ a.e.n. of α s.t. Y and U are also contained in the plane of α (Takes $Y = U = n$ for example).
(9) is fulfilled trivially and α is a pre-geodesic (geodesic if $\langle \alpha'', U \rangle$ is a positive constant) of the generated a.m.s.

The above fact in not true for IA spheres. The curve $\alpha(s) = (\cos(s), \sin(s), 0)$ cannot be the geodesic of an improper affine sphere (Aledo-Chaves-Gálvez, 2007).

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Non planar geodesics or pre-geodesics

$\alpha(s)$ analytic whose curvature $k(s)$ and torsion $\tau(s)$ do not vanish at any point. If we take $Y(s)$ as the unit normal vector field $n(s)$ of $\alpha(s)$,

$$Det[Y', \alpha', Y] = -\tau \neq 0 \quad \text{and} \quad Det[Y', \alpha', \alpha''] = -k\tau \neq 0$$

and so (6) is satisfied. Then there exists a unique affine maximal surface containing the curve $\alpha(s)$ such that its Blaschke normal along α is Y, and the affine conormal is

$$U = \frac{Y' \times \alpha'}{Det[Y', \alpha', Y]} = n.$$

It is easy to check that (9) is satisfied if k/τ is constant, that is, if α is a helix.
In particular

Every analytic helix is pre-geodesic of an affine maximal surface.

Helicoidal affine maximal surfaces

We identify the group $\mathcal{A}$ of equiaffine transformation of $\mathbb{R}^3$ with a subgroup of matrices of $\mathbb{SL}(4,\mathbb{R})$ in the following way:
$T(v) = Av + b$ will be identified to the matrix
$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \in \mathbb{SL}(4,\mathbb{R})$. Under this identification,

$$\mathcal{A} = \left\{ \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \ : \ A \in \mathbb{SL}(3,\mathbb{R}), \ b \in \mathbb{R}^3 \right\}$$

and its Lie algebra $\mathfrak{a}$ is given by

$$\mathfrak{a} = \left\{ \begin{pmatrix} C & d \\ 0 & 0 \end{pmatrix} \ : \ \text{Trace } C = 0, \ d \in \mathbb{R}^3 \right\}.$$

Helicoidal affine maximal surfaces

Since the one-parameter groups of equiaffine transformations are obtained as $\exp(sG)$, $s \in \mathbb{R}$, $G \in \mathfrak{a}$, the Jordan matrix decomposition Theory gives Up to a conjugation in $\mathcal{A}$, seven one-parametric groups of equiaffine transformations:

$$G_1 = \begin{pmatrix} 1 & as & \frac{as^2}{2} & \frac{as^3}{6} \\ 0 & 1 & s & \frac{s^2}{2} \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad G_2 = \begin{pmatrix} \cos(s) & \sin(s) & 0 & 0 \\ -\sin(s) & \cos(s) & 0 & 0 \\ 0 & 0 & 1 & as \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Helicoidal affine maximal surfaces

$$G_3 = \begin{pmatrix} e^s & 0 & 0 & 0 \\ 0 & e^{-s} & 0 & 0 \\ 0 & 0 & 1 & as \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad G_4 = \begin{pmatrix} 1 & as & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$G_5 = \begin{pmatrix} e^{as} & e^{as}s & 0 & 0 \\ 0 & e^{as} & 0 & 0 \\ 0 & 0 & e^{-2as} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad G_6 = \begin{pmatrix} e^{as} & 0 & 0 & 0 \\ 0 & e^{s} & 0 & 0 \\ 0 & 0 & e^{-as-s} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$G_7 = \begin{pmatrix} e^{as}\cos(s) & e^{as}\sin(s) & 0 & 0 \\ -e^{as}\sin(s) & e^{as}\cos(s) & 0 & 0 \\ 0 & 0 & e^{-2as} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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$T_s(v) = A_s v + b_s$ one-parametric subgroup eq. trans.
From our existence Theorem and generalized symmetry Principle, an a.m.s. invariant under T_s, $s \in \mathbb{R}$, is locally given as the surface generated by the following $\{T_s\}$-symmetric a.e.n $\{Y, U\}$, along the orbit $\alpha_p(s) = T_s(p)$ of a fixed point p,

$$Y(s) = A_s Y_p, \qquad U(s) = (A_s^t)^{-1} U_p$$

and $Y_p, U_p \in \mathbb{R}^3$ satisfy the necessary conditions for (4) holds.

The Berwald-Blaschke metric must be constant along α_p.

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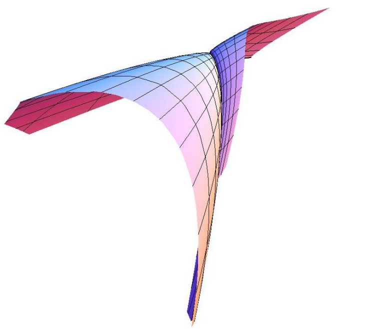
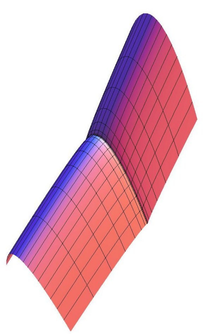
Helicoidal affine maximal surfaces

Helicoidal affine maximal surface $G_{1,a}$-invariant



$G_{1,0}$-invariant improper affine sphere



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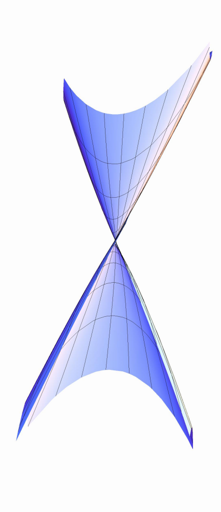

Rotational IA maximal surfaces

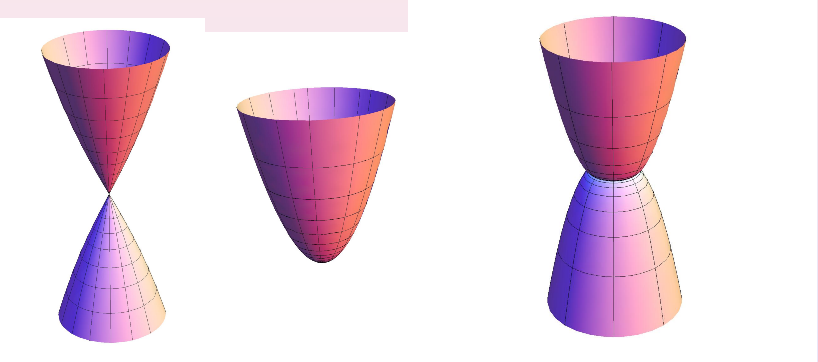

Figura: Rotational improper affine spheres.

Helicoidal affine maximal surfaces

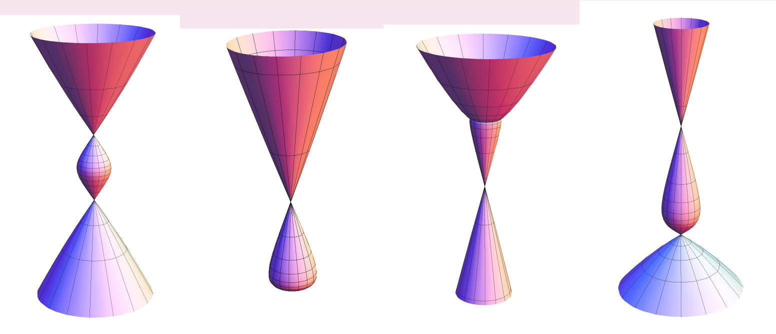

Figura: Rotational affine maximal surfaces.

Helicoidal affine maximal surfaces

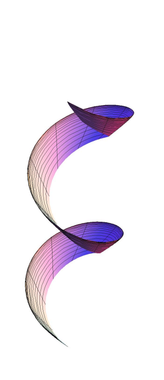

Figura: Non rotational G_2-invariant affine maximal surfaces

A class of affine maximal surfaces with singularities

Some Helicoidal affine maximal surfaces:

- Are glued by analytic curves where the affine metric is degenerated but the affine normal and the affine conormal are well defined.
- Can be represented as in (5), where Φ is a well-defined holomorphic regular curve on the Riemann surface Σ.

Definition

If a map $\psi : \Sigma \longrightarrow \mathbb{R}^3$ admits a representation as in (5) for a certain holomorphic curve Φ which satisfies that $[\Phi + \overline{\Phi}, \Phi_z, \overline{\Phi_z}]|dz|^2$ does not vanish identically, we say that ψ is an *affine maximal map* (Aledo-Martínez-M,2009).

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Theorem

$\alpha : I \longrightarrow \mathbb{R}^3$ a regular analytic l.s.c curve $\Rightarrow$ there exists an affine maximal map ψ containing $\alpha(I)$ in its singular set and determined up to an analytic function h.

From (4) we find that

1. When α has non vanishing torsion

$$U(s) = h(s)\alpha'(s) \times \alpha''(s), \qquad Y(s) = \frac{U''(s) \times U'(s)}{Det[U'', U', U](s)}$$

2. When $\alpha(s) = (f(s), g(s), 0)$ is a planar curve

$$U = (0,0,1), \qquad Y = \left(\frac{f'h' - f''h}{g'f'' - g''f'}, \frac{g'h' - g''h}{g'f'' - g''f'}, 1 \right),$$

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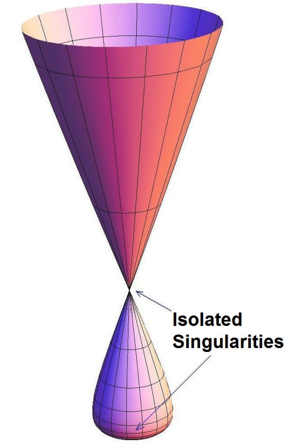
About Isolated Singularities

In some singular points the normalization $\{N, \xi\}$ is not well defined. It is the case of isolated singularities: graphs of solutions of (1) on a puncture disk. Two possibilities arise:

1. With the affine conformal structure of a puncture disk (the tangent plane is well defined at the puncture)
2. With the affine conformal structure of an annulus (the tangent plane is not well defined at the puncture)

Rotational a.m.s. with isolated singularities



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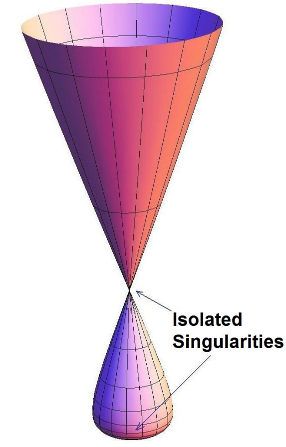

About Isolated Singularities

Theorem

Let ϕ be a solution of (1) on a punctured disk. If its graph is affine conformal to a punctured disk and ϕ has a non removable singularity at the origin $\Rightarrow$ ϕ is asymptotic to the rotational solution.

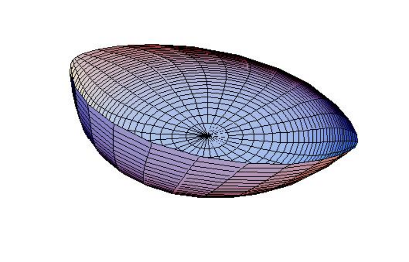

Figura: Non-rotational example with $N = (u, v, -\log(u^2 + v^2) + u^2 - v^2)$

Examples with the Underlying conformal Structure of an Annulus

$\gamma : \mathbb{R} \longrightarrow \mathbb{R}^3$, $\gamma(u) = (\gamma_1(u), \gamma_2(u), 1)$, 2π-periodic analytic parametrization of a strictly convex Jordan curve and $\lambda : \mathbb{R} \longrightarrow \mathbb{R}$, 2π-periodic analytic function. On
$\Delta^r = \{z = u + \imath v \mid -r < v < r\}$

$$F^{\gamma\lambda}(z) = \gamma(z) - \imath \int_0^z \lambda(w)\gamma(w)dw, \qquad z \in \Delta^r.$$

Theorem

$N = \mathrm{Re}\, F^{\gamma\lambda} \Rightarrow \exists$ a solution ϕ of (1) on a punctured disk s.t.

1. N is the affine conormal vector field of the graph of ϕ.
2. ϕ extends continuously at the origin and its graph has the affine conformal structure of an annulus.
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Let $N = \mathrm{Re}\, G^{\gamma\alpha}$. Then there exists a solution ϕ of (1) on a punctured disk s.t. ϕ satisfies the properties in above Theorem.

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