A GEOMETRIC PARTIAL DIFFERENTIAL EQUATION

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Geometric Analysis

Interplay between DG and PDEs

If $f : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a solution of the Hessian one equation

$$f_{xx} f_{yy} - f_{xy}^2 = \varepsilon = \pm 1,$$

then

$$h = f_{xx} dx^2 + f_{yy} dy^2 + 2 f_{xy} dx dy$$

is the affine metric of an improper affine sphere in $\mathbb{R}^3$.

- Definite in the elliptic case ($\varepsilon = +1$).
- Indefinite in the non-elliptic case ($\varepsilon = -1$).

Global solutions

- $\varepsilon = +1 \Longrightarrow f(x, y) = \frac{1}{2}\left(x^2 + y^2\right)$, (Jörgens 1954).
- $\varepsilon = -1 \Longrightarrow f(x, y) = xy + g(x)$...

Introduction

- Affine spheres are the umbilical surfaces of the equiaffine theory in $\mathbb{R}^3$, (the study of invariants under the transformations which preserve the volume, $SL(3,\mathbb{R})$-invariants).
- Locally, they are the graphs of the solutions of some Monge-Ampère equations.
- The study of their PDEs, with geometric methods, was initiated by Calabi, Pogorelov and Cheng-Yau.
- The Monge-Ampère equation and its geometric applications, (Trudinger-Wang, 2008).
- Affine Bernstein Problems and Monge-Ampère equations, (Li-Jia-Simon-Xu, 2010).

Main Schedule

1. Definite improper affine spheres
2. Complex representation
3. Singularities I
4. Ribaucour transformations
5. Definite and indefinite Cauchy problem
6. Singularities II

Preliminaries

If $f : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a solution of the Hessian one equation

$$f_{xx} f_{yy} - f_{xy}^2 = 1,$$

then its graph $\psi = \{(x, y, f(x, y)) : (x, y) \in \Omega\}$ is an improper affine sphere in $\mathbb{R}^3$.
That is, ψ has constant affine normal

$$\xi = \frac{1}{2} \Delta_h \psi = (0, 0, 1),$$

where

$$h = \kappa^{\frac{-1}{4}} \sigma$$

is the affine metric, (the $SL(3, \mathbb{R})$-invariant metric obtained with the Gauss curvature κ and the second fundamental form σ of ψ).

Preliminaries

In this case, from the Hessian one equation, the affine metric

$$h = f_{xx}dx^2 + f_{yy}dy^2 + 2f_{xy}dxdy$$

and the affine conormal

$$N = (-f_x, -f_y, 1) \perp d\psi$$

satisfy

$$1 = \sqrt{det(h)} = det(\psi_x, \psi_y, \xi) = det(N_x, N_y, N).$$

Also, $h = -\langle dN, d\psi \rangle$ and $\langle N, \xi \rangle = 1$.

Preliminaries

Thus, for a conformal parameter z, we have $h = 2\rho|dz|^2$ with

$$\rho = \langle N, \psi_{z\bar{z}} \rangle = -\imath[\psi_z, \psi_{\bar{z}}, \xi] = -\imath[N_z, N_{\bar{z}}, N]$$

and $\xi = (0, 0, 1)$. Hence,

$$\psi_z = \imath N \times N_z, \qquad N_{\bar{z}} = -\imath \xi \times \psi_{\bar{z}}$$

and

$$\Phi = \frac{1}{2}\Big(N + \imath\xi \times \psi\Big) = \frac{1}{2}\Big(- f_x - \imath y, -f_y + \imath x, 1\Big)$$

is a holomorphic planar curve, such that $N = \Phi + \overline{\Phi}$.
In particular, ψ is an affine maximal surface $\equiv N_{z\bar{z}} = 0$ and

$$\psi_{z\bar{z}} = \imath N_{\bar{z}} \times N_z = \rho\xi.$$

Weierstrass-type Representation Formulas

Calabi (1988)

If ψ is an affine maximal surface (improper affine sphere), then

$$\psi = 2Re \int \imath(\Phi + \overline{\Phi}) \times \Phi_z dz,$$

with ϕ a holomorphic (planar) curve and $-\imath[\Phi + \overline{\Phi}, \Phi_z, \overline{\Phi_z}] > 0$.

Ferrer, Martínez, M (1996)

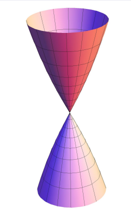
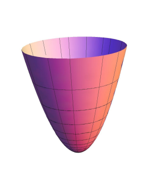
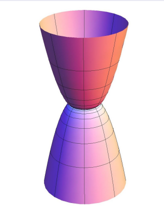
If ψ is an improper affine sphere in $\mathbb{R}^3 \equiv \mathbb{C} \times \mathbb{R}$, then

$$\psi = \left(G + \overline{F}, \frac{1}{2}|G|^2 - \frac{1}{2}|F|^2 + Re(GF) - 2Re \int FdG \right)$$

with F and G holomorphic functions, such that $N = (\overline{F} - G, 1)$ and $h = |dG|^2 - |dF|^2 > 0$.

Some applications

Rotational IAS: $G = z$, $F = \frac{a}{z}$, $|z|^2 > |a|$.

Isolated singularity ($a < 0$), complete ($a = 0$), cuspidal edge $a > 0$.

- An extension of a theorem by Jörgens and a maximum principle at infinity for IAS, (Ferrer, Martínez, M 99).

$$f(x, y) \approx \mathcal{E}(x, y) + a \log |z|^2.$$

- The space of IAS with fixed compact boundary, (FMM 00).

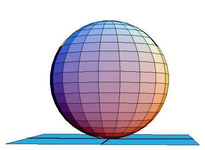
Some applications

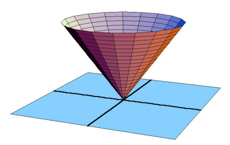
Flat surfaces in $\mathbb{H}^3$ have also a conformal representation, since

$$h = f_{xx}dx^2 + f_{yy}dy^2 + 2f_{xy}dxdy$$

is their second fundamental form, (Gálvez, Martínez, M 00).

Global classification





Horosphere and cone (in the half space model).

- Many authors begin to study a global theory with singularities.

Admissible singularities (from $\mathbb{H}^3$ to $\mathbb{R}^3$)

- Flat fronts in $\mathbb{H}^3$, with admissible singularities, (isolated singularities, cuspidal edges and swallowtails), (Kokubu, Umehara, Yamada 04).
- Improper affine maps, with admissible singularities, where

$$|dG| = |dF| \neq 0.$$

That is, $h = |dG|^2 - |dF|^2 \geq 0$, but

$$|d\Phi|^2 = 2(|dG|^2 + |dF|^2) > 0.$$

Isolated singularities

- The space of solutions to the Hessian one equation in the finitely punctured plane, (Gálvez, Martínez, Mira 05).
- Explicit construction for two singularities, with the annular Jacobi theta functions.
- Isolated singularities are in 1-1 correspondence with planar convex analytic Jordan curves, (see the Cauchy problem).
- Complete flat surfaces in $\mathbb{H}^3$ with two isolated singularities, (Corro, Martínez, M 10).

Ribaucour transformations (Martínez, M, Tenenblat 15)

Definition

Two improper affine maps $\psi, \widetilde{\psi} : \Sigma \longrightarrow \mathbb{R}^3$ are R-associated if there is a differentiable function $g : \Sigma \longrightarrow \mathbb{R}$ such that

1. $(\psi + gN)\times\xi = (\widetilde{\psi} + g\widetilde{N})\times\xi$.
2. $dGdF = d\widetilde{G}d\widetilde{F}$.

Theorem

Equivalently

$$(\widetilde{F}, \widetilde{G}) = \Big(F + \frac{1}{cR}, G + R\Big),$$

where $c \in \mathbb{R} - \{0\}$ and R is a holomorphic solution of the Riccati equation

$$dR + dG = cR^2 dF \qquad \Big(\iff d\Big(\frac{1}{cR}\Big) + dF = \frac{1}{cR^2} dG\Big).$$

Ribaucour transformations

Consequence

If ψ is helicoidal, then $FG = -a^2$ and

$$R = \frac{\exp(z)}{2ac} \frac{1 + b + (1 - b)k \exp(bz)}{1 + k \exp(bz)}$$

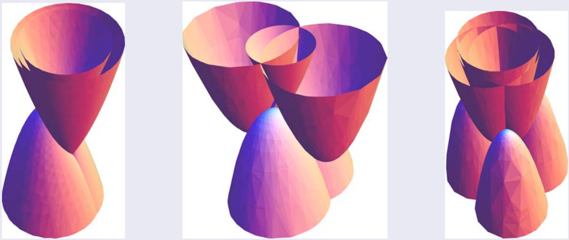
with $a, k \in \mathbb{C}$, $c \in \mathbb{R} - \{0\}$ and $b = \sqrt{1 + 4a^2c} \neq 0$.
In particular, if

$$b = \frac{n}{m} \in \mathbb{Q} - \{0, 1\}$$

is irreducible, then $\widetilde{\psi}$ is $2m\pi$-periodic in one variable and has $2n$ complete embedded ends of revolution type.

Ribaucour transformations

R-helicoidal examples



$(n, m) = (1, 3)$, $(2, 1)$ *and* $(3, 2)$.

The singular set is contained in a compact set.

Cauchy problem (M 14, Martínez, M 15)

A Björling-type problem

Find all (definite and indefinite) IAS containing a curve α in $\mathbb{R}^3$ with a prescribed affine conormal U along it.

1. Note that h definite implies

$$0 < h(\alpha'(s), \alpha'(s)) = -\langle \alpha'(s), U'(s) \rangle,$$

   with $\{\alpha, U\}$ analytic curves, (Aledo, Chaves, Gálvez 07).

2. In the indefinite case, $\langle \alpha', U' \rangle$ vanishes when α' is an asymptotic (also known as characteristic) direction.

Non-characteristic Cauchy problem

- First, we exclude asymptotic (characteristic) data.
- We consider

$$f_{xx} f_{yy} - f_{xy}^2 = \varepsilon = \pm 1$$

and the ε-complex numbers (Inoguchi, Toda 04)

$$\mathbb{C}_\varepsilon = \{ z = s + jt \ : \ s, t \in \mathbb{R}, \ j^2 = -\varepsilon, \ j1 = 1j \}.$$

Thus

$$\Phi = \frac{1}{2}\Big(N + j\xi \times \psi \Big) = \frac{1}{2}\Big(-f_x - jy, -f_y + jx, 1 \Big)$$

is a holomorphic curve and

$$\psi = 2Re \int j(\Phi + \overline{\Phi}) \times \Phi_z dz.$$

Non-characteristic Cauchy problem

Necessary conditions

If $\psi : \Sigma \longrightarrow \mathbb{R}^3$ is an IAS with $\xi = (0,0,1)$ and $\beta : I \longrightarrow \Sigma$ is a curve, then $\alpha = \psi \circ \beta$, $U = N \circ \beta$ and $\lambda = -\langle \alpha', U' \rangle$ satisfy

$$\begin{cases} 1 = \langle \xi, U \rangle, \\ 0 = \langle \alpha', U \rangle, \\ \lambda = \langle \alpha'', U \rangle. \end{cases}$$

Definition

A pair of (analytic) curves $\alpha, U : I \longrightarrow \mathbb{R}^3$ is a non-characteritic admissible pair if verify the above conditions with $\lambda : I \longrightarrow \mathbb{R}^+$.

Non-characteristic Cauchy problem

Geometric theorem

If $\{\alpha, U\}$ is a non-characteristic admissible pair, then there exits a unique IAS ψ containing $\alpha(I)$ with affine conormal U along α.

As $\lambda > 0$, from the inverse function theorem, in a domain around I, there is a conformal parameter $z = s + jt$ and a unique holomorphic extension of

$$\Phi(s) = \frac{1}{2}\Big(U(s) + j\xi \times \alpha(s)\Big),$$

which gives ψ.

Non-characteristic Cauchy problem

Analytic theorem

There exits a unique solution to the Cauchy problem

$$\begin{cases} f_{xx} f_{yy} - f_{xy}^2 = \varepsilon, & \\ f(x,0) = a(x), & a''(x) > 0, \\ f_y(x,0) = b(x). & \end{cases}$$

Apply the above theorem with

$$\alpha(s) = (s, 0, a(s)) \;\; \textit{and} \;\; U(s) = (-a'(s), -b(s), 1).$$

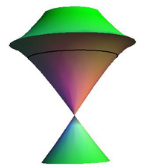
Consequences

1. If $[\alpha', \alpha'', \xi] \neq 0$, then α and λ determine
$$U = \frac{\alpha' \times (\alpha'' - \lambda\xi)}{[\alpha', \alpha'', \xi]} \qquad \textit{and} \qquad \psi.$$
2. In particular, any revolution IAS can be recovered with one their circles α and the affine metric along it. Moreover, α is geodesic when $\lambda = r^2$ and $\varepsilon = -1$.
3. In general, α is geodesic of some IAS if and only if
$$[\alpha', \alpha'', \xi] = -\varepsilon[U', U'', \xi],$$
with $\lambda = m \in R^+$.

Consequences

- We classify the IAS admitting a geodesic planar curve.

Non-ruled examples

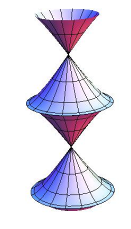
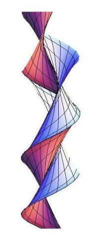


- Note that any symmetry of a non-characteristic admissible pair induces a symmetry of the IAS generated by it.

Consequences

- We obtain the IAS which are invariant under a one-parametric group of equiaffine transformations.

Helicoidal examples

Isolated singularities and cuspidal edges.

- Where are the swallowtails?

Prescribed singular curves

Theorem

If $[\alpha', \alpha'', \alpha''']^2 \neq -\varepsilon[\alpha', \alpha'', \xi]^4 \neq 0$, then there exists a unique improper affine map ψ with α as (cuspidal edge) singular curve.

Take $U = \frac{\alpha' \times \alpha''}{[\alpha', \alpha'', \xi]}$. Then, $\{\alpha, U\}$ gives ψ with

$$[\psi_s, \psi_t, \xi](s, 0) = [\alpha', U' \times U, \xi] = -\langle \alpha', U' \rangle = 0$$

and α is an admissible singular curve of ψ since

$$\frac{d}{dt}\Big|_{(s,0)} [\psi_s, \psi_t, \xi] = [\alpha', \alpha'', \xi]\Big(-\varepsilon - \frac{[\alpha', \alpha'', \alpha''']^2}{[\alpha', \alpha'', \xi]^4}\Big) \neq 0.$$

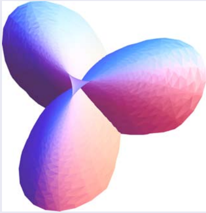
Note that $\varepsilon\psi_{ss} + \psi_{tt} || \xi$.

Prescribed singular curves

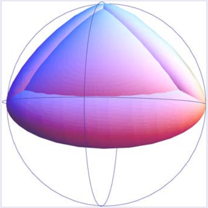
Theorem

If $[\alpha', \alpha'', \alpha''']^2 \neq -\varepsilon[\alpha', \alpha'', \xi]^4 \neq 0$ on $I - \{0\}$ and 0 is a zero of α', $\alpha' \times \alpha''$, $[\alpha', \alpha'', \xi]$ and $[\alpha', \alpha'', \alpha''']$ of order 1, 2, 2 and 3 respectively, then $\alpha(0)$ is a swallowtail of ψ.

Examples with three swallowtails



Improper affine map,



flat front (Martínez, M 14).

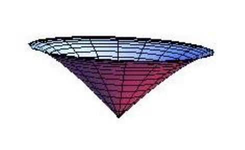
Prescribed isolated singularities

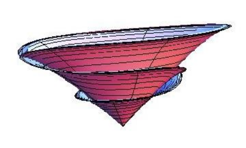
Theorem

If U is a periodic planar convex curve, then there exists a unique improper affine map with an isolated singularity at 0, where the affine conormal tends to U.

Here, $\Phi(s) = \frac{1}{2}\Big(U(s) + j\xi \times 0\Big) = \frac{1}{2}U(s)$.

Examples





Characteristic Cauchy problem

- Finally, we consider IAS generated by a characteristic admissible pair $\{\alpha, U\}$, that is, $\langle \alpha', U' \rangle$ vanishes when α' is an asymptotic direction.
- We use the Blaschke's representation for an indefinite IAS ψ with asymptotic parameters (u, v) and two planar curves $a(u)$ and $b(v)$ given by the harmonic maps

$$N = (a + b, 1) \qquad and \qquad \xi \times \psi = (b - a, 0).$$

- It is clear that an asymptotic curve $\psi(u, v_o)$ determines $a(u)$ and $N(u, v_o)$, but not $b(v)$.
- So, an admissible pair $\{\alpha, U\}$ generates many (indefinite) IAS, when $\langle \alpha', U' \rangle$ vanishes identically.

Characteristic Cauchy problem

- If $\langle \alpha'(s), U'(s) \rangle$ only vanishes at isolated points, then we can take the planar curves $\widetilde{a}(s)$ and $\widetilde{b}(s)$ with

$$U = (\widetilde{a} + \widetilde{b}, 1), \qquad \xi \times \alpha = (\widetilde{b} - \widetilde{a}, 1)$$

  and $2det(\widetilde{a}', \widetilde{b}') = det(U', \xi \times \alpha', \xi) = \langle \alpha', U' \rangle$.
- Thus, we can determine the curves $a(u)$, $b(v)$ and the IAS, up to a change of parameters $\widetilde{a}(s) = a(u(s))$, $\widetilde{b}(s) = b(v(s))$, when

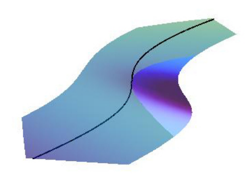
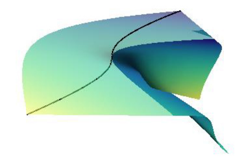
$$\langle \alpha', U' \rangle = 2det(\widetilde{a}', \widetilde{b}') = 2det(a', b')u'v'$$

  does not change sign.

Characteristic Cauchy problem

- The uniqueness fails when $\alpha(s) = \psi(u(s), v(s))$ is tangent to an asymptotic curve, that is, when $u'v'$ changes sign.

Examples

Theorem

Two solutions agree on a domain which contains $\alpha(I)$ except its characteristic points without sign.