An Extension of the Affine Bernstein Problem

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Dedicated to Professor Keti Tenenblat on her 65^{th} *birthday*

Abstract. In this work we study a class of affine maximal surfaces with singularities that we called *affine maximal maps*. Our main goal is the classification of the complete affine maximal maps with one or two embedded regular ends. In particular, we obtain a new characterization of the elliptic paraboloid and a large family of global examples.

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1. Introduction

In the classical work [B], Blaschke showed that the Euler-Lagrange equation of the equiaffine area functional is of fourth-order and nonlinear. To be more precise, the affine invariant area functional is given by

$$\mathcal{A}(\phi) = \int \left(det\left(\nabla^2\phi\right)\right)^{1/4} dxdy = \int K_e^{1/4} d\sigma, \tag{1.1}$$

where K_e is the Euclidean Gauss curvature of the graph of $\phi(x,y)$ and $d\sigma$ its volume element, and its Euler-Lagrange equation becomes

$$L[\phi] := \phi_{yy}\rho_{xx} - 2\phi_{xy}\rho_{xy} + \phi_{xx}\rho_{yy} = 0, \qquad \rho = \left(det\left(\nabla^2\phi\right)\right)^{-3/4}, \tag{1.2}$$

$\nabla^2\phi > 0$ being the positive definite Hessian matrix of ϕ.

Blaschke also showed that Equation (1.2) is equivalent to the vanishing of the affine mean curvature, which along with the fact that, for locally strongly convex surfaces, the second variation formula is always negative (see [5]) led to the notion of *affine maximal surfaces*.

Equation (1.2) has been widely studied from a global point of view, see the recent book [17]. For instance, Trudinger and Wang [22, 23] proved that

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the entire convex solution of (1.2) are quadratic polynomials, answering in this way the so called *Affine Bernstein Problem* conjectured by Chern [8] in 1978. Another celebrated global result is the one characterizing the elliptic paraboloid as the only affine complete affine maximal surface (see [16, 20, 24])

Motivated by the lack of global examples, as becomes clear in view of the results above, the study of singularities of Equation (1.2) has lately received many contributions [1, 2, 3, 4, 11, 13, 19] which has revealed an interesting global theory for this class of surfaces.

The aim of this work is to study in a global way affine maximal surfaces with some *admissible* singularities. We call such surfaces *affine maximal maps*.

The paper is organized as follows. In Section 2 we revise some fundamental facts about affine surfaces and introduce the concept of affine maximal map. We pay special attention to the notion of *completeness* of an affine maximal map and define the *ends* of such maps.

Section 3 is devoted to study when the regular ends of a complete affine maximal map are embedded.

Finally, in Section 4 we classify the complete affine maximal maps with only one or two embedded regular ends. To be more precise, we see that the elliptic paraboloid is the only complete affine maximal map with a unique embedded regular end. Regarding to complete affine maximal maps with two embedded regular ends, we show that they are contained in a family of canonical examples that we describe in detail.

2. Affine Maximal Maps

In order to motivate the notion of affine maximal maps, we will revise some basic facts about affine surfaces. We refer the reader to [18, 21] for more detailed discussions about this topic.

Let $\psi : \Sigma \longrightarrow \mathbb{R}^3$ be a locally strongly convex immersion of a surface Σ, oriented so that its second fundamental form, σ_e, is positive definite everywhere. Denote by K_e and dA_e its Gaussian curvature and the element of Euclidean area, respectively. Then, the positive density $dA := K_e^{\frac{1}{4}} dA_e$, the positive quadratic form $g := K_e^{-\frac{1}{4}} \sigma_e$ and the transversal vector field $\xi := \frac{1}{2}\Delta_g \psi$, where Δ_g is the Laplace-Beltrami operator associated to g, are the most elementary unimodular affine invariants of the immersion and they are called the *equiaffine area element*, the *Berwald-Blaschke metric*, and the *Blaschke normal* or *affine normal*, respectively.

On Σ we have a canonical Riemann surface structure such that g is Hermitian. Moreover, the equiaffine normalization of ψ is given by the affine normal ξ and the *affine co-normal* vector field $N := K_e^{-1/4} N_e$, where N_e is the unit normal vector field to the immersion. With this normalization

$$\begin{aligned} 1 &= \langle N, \xi\rangle, \qquad \langle N, d\psi\rangle = 0, \\ g &= -\langle d\psi, dN\rangle = \det[N, *dN, dN], && (2.1)\\ d\psi &= -N\wedge *dN, && (2.2)\\ \xi &= \frac{*dN\wedge dN}{\det[N,*dN,dN]}, \end{aligned}$$

where $\langle .,.\rangle$ is the usual inner product in $\mathbb{R}^3$, $\wedge$ denotes the cross product in $\mathbb{R}^3$ and $*$ is the standard conjugation operator acting on 1-forms.

The Euler Lagrange equation (1.2) for the affine invariant area functional (1.1) is equivalent to the system of PDE's

$$\Delta_g N = 0.$$

So, when Σ is simply-connected, $\frac{1}{2}N$ is locally the real part of a holomorphic curve $\Phi : \Sigma \longrightarrow \mathbb{C}^3$ determined by ψ up to a real translation which satisfies

$$\begin{aligned} N &= \Phi + \overline{\Phi}, && (2.3)\\ g &= -2\,\mathrm{i}\det\left[\Phi+\overline{\Phi}, d\Phi, \overline{d\Phi}\ \right], && (2.4)\\ 2\,d\Phi &= dN + \mathrm{i}*dN. && (2.5) \end{aligned}$$

Conversely, (2.2) allows us to recover ψ from its affine co-normal and the conformal class of g, ([15]), as

$$\psi = -\int N\wedge *dN, \qquad (2.6)$$

which, along with (2.4) and (2.5), says that ψ is uniquely determined, up to a real translation, by a holomorphic curve Φ satisfying that $-2\ \mathrm{i}\det\left[\Phi+\overline{\Phi}, d\Phi, \overline{d\Phi}\ \right]$ is a Riemannian metric (see also [6]). To be precise,

$$\psi = 2\Re\int \mathrm{i}\ \left(\Phi+\overline{\Phi}\right)\wedge d\Phi = -\,\mathrm{i}\ \left(\ \Phi\wedge\overline{\Phi} - \int \Phi\wedge d\Phi + \int \overline{\Phi}\wedge \overline{d\Phi}\ \right). \qquad (2.7)$$

Remark 1. *Calabi* [7] *calls torsion-free affine maximal surfaces to those obtained from a holomorphic curve* $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ *satisfying that*

$$c_1\Phi_1 + c_2\Phi_2 + c_3\Phi_3 = c_4$$

for $c_j \in \mathbb{C},\ j = 1,2,3,4$. *Furthermore if* $c_j,\ j = 1,2,3$, *are real constants, then such surfaces belong to the special class of improper affine spheres and, up to an eqquiaffine tranformation, one can assume that* Φ_3 *is constant.*

From (2.6) and (2.7) we have a recipe that lets us obtain affine maximal immersions from either harmonic vector fields or holomorphic curves. But when the expressions in (2.1) and (2.4) fail to be Riemannian metrics, it appears a natural set of singularities of ψ that motivates the study of affine maximal surfaces with some admissible singularities.

Definition 2. *Let* Σ *be a Riemann surface. We say that a map* $\psi : \Sigma \longrightarrow \mathbb{R}^3$ *is an* affine maximal map *if there exists a harmonic vector field* $N : \Sigma \longrightarrow \mathbb{R}^3$ *such that* $\det[N, *dN, dN]$ *does not vanish identically and* ψ *is given as in*

(2.6). We shall say that N is the affine co-normal vector field *of the affine maximal map.*

Remark 3. *Our definition extends the notion of improper affine map introduced in* [19]*. In fact, when the affine co-normal vector field $N = (N_1, N_2, N_3)$ is such that $N(p)$ is contained in a plane π for all $p \in \Sigma$, up to an equiaffine transformation we can assume that N_3 is constant and ψ becomes an improper affine map (see Remark 1). Besides, in this case, it is proved in* [9], [?] *and* [19] *that there exist holomorphic functions Φ_1 and Φ_2 globally defined on Σ (even although Σ is not simply-connected) such that*

$$N = (\Phi_1 + \overline{\Phi}_1, \Phi_2 + \overline{\Phi}_2, 1).$$

The singular set $\mathcal{S}_\psi$ of an affine maximal map ψ is the set of points where the quadratic form

$$g = \det[N, *dN, dN] \tag{2.8}$$

vanishes. It is clear that $\Sigma \setminus \mathcal{S}_\psi$ is dense in Σ.

Although the affine metric is not well defined on $\mathcal{S}_\psi$, we can define completeness of affine maximal maps in a similar way as Kokubu, Umehara and Yamada have done for other kind of surfaces with singularities (see, for instance, [14, 25]).

Definition 4. *Let $\psi : \Sigma \longrightarrow \mathbb{R}^3$ be an affine maximal map. We say that ψ is complete if there exist compacts sets K, K', $K \subset K' \subset \Sigma$, and a symmetric 2-tensor T with compact support in Σ such that if $\mu : \Sigma \longrightarrow [0, 1]$ is a differentiable function satisfying*

$$\mu(p) = \begin{cases} 0 & \text{if } p \in K \\ 1 & \text{if } p \in \Sigma \setminus K' \end{cases}$$

then

$$\widetilde{g} = (1 - \mu)T + \mu|g|$$

is a complete metric on Σ.

Using a Theorem by Hubber (see [12][Th 13]), one can prove

Theorem 5. *Let $\psi : \Sigma \longrightarrow \mathbb{R}^3$ be a complete affine maximal map with affine co-normal vector field N. Then Σ is conformally equivalent to the complement of a finite point set $\{p_1, \ldots, p_n\}$ in a compact Riemann surface $\overline{\Sigma}$.*

The points $\{p_1, \ldots, p_n\}$ will be called ends of ψ. We will say that an end p is regular if $dN + \mathrm{i} * dN$ extends meromorphically to p; otherwise the end will be called irregular.

Remark 6. *The ends of a complete improper affine map (see* [19]*) are always regular.*

A regular end is particularly interesting because around it we can assume that it comes as in (2.6) from a harmonic map $N : \mathbb{D}^\star \longrightarrow \mathbb{R}^3$ well-defined on the unit punctured disk $\mathbb{D}^\star$ and such that $dN + \mathrm{i} * dN$ extends meromorphically to the origin. Under this assumption N admits a series development (see Section 3) which will allow us to determine when the end is embedded.

3. Embedded Ends of Affine Maximal Maps

Let $\psi : \mathbb{D}^\star \longrightarrow \mathbb{R}^3$ be an affine maximal regular complete end with affine co-normal vector field N. Let us take (u, v) conformal parameters in $\mathbb{D}^\star$ for the affine metric g. Note that it must be $[N, N_u, N_v] > 0$ in $\mathbb{D}^\star$.

Under our assumptions, there exist meromorphic functions F, G, H : $\mathcal{D} \longrightarrow \mathbb{C}$ (which are holomorphic in $\mathcal{D}^\star$) and real constants a, b and c such that the affine conormal $N = (N_1, N_2, N_3)$ of the affine maximal map ψ can be written as

$$\begin{aligned} N_1(z) &= \Re(F_1(z)) + b\log|z|, \\ N_2(z) &= \Re(F_2(z)) + c\log|z|, \\ N_3(z) &= \Re(F_3(z)) + a\log|z|, \end{aligned} \tag{3.1}$$

where $z = u + \mathrm{i}\,v$. Observe that, up to an equiaffine transformation, we can assume that $b = c = 0$.

Let us take polar coordinates (R, t) in $\mathcal{D}$ such that $u = R\cos t$ and $v = R\sin t$. Since F, G and H are meromorphic functions in $\mathcal{D}$, near the origin we can write

$$N = (\sum_{m\geq p} R^m A_{1m}(t), \sum_{m\geq q} R^m A_{2m}(t), \sum_{m\geq k} R^m A_{3m}(t) + a\log(R)), \tag{3.2}$$

where p, q, k are integers and

$$A_{jm}(t) = a_{jm}\cos(mt) + b_{jm}\sin(mt), \qquad j = 1, 2, 3,$$

$$\begin{aligned} &A_{1p}(t) \neq 0, \qquad A_{2q}(t) \neq 0, \\ &A_{3k}(t) \neq 0 \qquad \textit{or} \qquad a \neq 0, \end{aligned}$$

being possible that the third coordinate is simply $a\log(R)$ or $\sum_{m\geq k} R^m A_{3m}(t)$ $(a = 0)$.

The case were $p, q, k \geq 0$ was studied in [AMM1]. Under those conditions the type of singularity at the origin is not an end. Hence, we will assume that at least one the integers p, q, k is negative. Actually, we are interested in studying the *embedded* ends.

We must bear in mind the following considerations regarding the integers p, q, k.

Remark 7. *First, observe that we can assume that the three integers p, q, k are not equal. In fact, if that was the case, we would be able by means of a suitable reparametrization of the parameter t and an equiaffine transformation to rewrite N so that no more than two of the integers p, q, k were equal. Even more, when two of the integers p, q, k are equal, let us say for instance p and q, we can assume that the vectors (a_{1p}, b_{1p}) and (a_{2q}, b_{2q}) are not proportional because in such a case, again up to an equiaffine transformation,* *we can rewrite N so that $p < q$. Furthermore, being $p = q$ (and (a_{1p}, b_{1p}) and (a_{2q}, b_{2q}) not proportional), we can take $A_{1p} = \cos(pt)$ and $A_{2q} = \sin(qt)$.*

Reasoning in a similar way, we can also assume that no more than one of the integers p, q, k is zero. Furthermore, if for instance $p = 0$ we can take $A_{1p}(t) = 1$.

Finally, up to a reparametrization of the parameter t and an equiaffine transformation, we can assume in general that one of the coefficients $A_{1p}(t), A_{2q}(t)$ or $A_{3k}(t)$, let us say, for instance, the first one, can be simplified to $A_{1p}(t) = \cos(pt)$.

In order to determine how the integers p, q, k must be for the affine maximal map has a well-defined end at the origin, we will use that the determinant

$$[N, N_u, N_v] = \frac{1}{R}[N, N_R, N_t] \tag{3.3}$$

cannot change signs. Note that the sign of this determinant is determined by the coefficient of the lowest power of R. In addition, we will use the following technical result

Lemma 8. *Let $\phi_1, \ldots, \phi_n, \lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n$ be integers such that:*

- *$\phi_i \neq 0$, for $i = 1, \ldots, n$.*
- *At least one of the integers $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n$ is not zero.*

Then there exist real numbers t_1, t_2 such that the function

$$f(t) = \sum_{i=1}^{n} (\lambda_i \cos(\phi_i t) + \mu_i \sin(\phi_i t))$$

verifies that $f(t_1) < 0$ and $f(t_2) > 0$.

Proof. The function

$$F(t) = -\sum_{i=1}^{n} \left(\frac{1}{\phi_i^2} \lambda_i \cos(\phi_i t) + \frac{1}{\phi_i^2} \mu_i \sin(\phi_i t) \right)$$

is a periodic function such that $F''(t) = f(t)$. Observe that if $f(t)$ did not change signs, then $F(t)$ would be a concave (or convex) function, which is impossible because it is periodic. This proves the claim of the lemma. □

We will study separately the cases $a = 0$ and $a \neq 0$.

Proposition 9. *Let $\psi : \mathbb{D}^\star \longrightarrow \mathbb{R}^3$ be an affine maximal regular complete end with affine co-normal vector field N given by (3.2). If $a = 0$, then either*

i) *two of the integers p, q, k are equal and different from zero, and the other one is zero.*

or

ii) *the three integers p, q, k are pairwise different and, assuming without loss of generality that $p < q < k$, one of the following situations happens:*
   - *$p = q + k$ and $q = \mu k$ for a certain integer μ, $\mu \geq 2$.*
   - *$q = p + k$ and either $p = \mu k$ or $k = \mu p$ for a certain integer μ, $\mu \leq -2$.*

Moreover, the end is embedded only when two of the integers p, q, k are equal to -1 and the other one is zero.

Proof. From Remark 7, we can assume that the three integers p, q, k are not all equal and no more than one of them is zero.

It is a straightforward computation to check that the coefficient of the lowest power of R, $R^{-1+p+q+k}$, is given by

$$C(t) = (p-k)A_{1p}(t)A_{3k}(t)A'_{2q}(t)+A_{2q}(t)\left((k-q)A_{3k}(t)A'_{1p}(t) + (q-p)A_{1p}(t)A'_{3k}(t)\right) \tag{3.4}$$

with

$$\begin{aligned} A_{1p} &= a_{1p}\cos(pt) + b_{1p}\sin(pt), \\ A_{2q} &= a_{2q}\cos(qt) + b_{2q}\sin(qt), \\ A_{3k} &= a_{3k}\cos(kt) + b_{3k}\sin(kt). \end{aligned} \tag{3.5}$$

Let us start studying the case when two of the integers p, q, k are equal; let us say, for instance, $p = q$. As we have already pointed out in Remark 7, it must be $p \neq 0$, we can assume that the vectors (a_{1p}, b_{1p}) and (a_{2q}, b_{2q}) are not proportional and

$$A_{1p} = \cos(pt), \qquad A_{2q} = \sin(pt).$$

Then the coefficient (3.4) becomes

$$C(t) = p(p-k)\left(a_{3k}\cos(kt) + b_{3k}\sin(kt)\right). \tag{3.6}$$

When A_{3k} is not a constant (i.e. $k \neq 0$), this coefficient changes signs depending on the angle t and so this case is not possible. Consequently the only possibility is that $p = q < 0$ and $k = 0$, and *i)* is proved.

In order to analyze when, being $p = q < 0$ and $k = 0$, the end is embedded, we will recover the affine maximal map ψ from N. Given a local parameter (u, v) on $\mathcal{D}^\star$ around the origin, we have that $\psi_u = N \times N_v$ and $\psi_v = -N \times N_u$. Equivalently, if we take polar coordinates (R, t) such that $u = R\cos t$ and $v = R\sin t$, then $\psi_R = (1/R)N \times N_t$ and $\psi_t = -RN \times N_R$.

Hence, from (3.2) and bearing in mind that $a = 0$, $p = q$ and $k = 0$, we get

$$\begin{aligned} \psi_R &= \left(-pR^{p-1}\cos(pt) + o(p), -pR^{p-1}\sin(pt) + o(p), pR^{2p-1} + o(2p)\right) \\ \psi_t &= \left(pR^p\sin(pt) + o(p+1), -pR^p\cos(pt) + o(p+1), o(2p+1)\right) \end{aligned}$$

where $o(l)$, $l \in \mathbb{Z}$, stands for a function depending on R and t which can be written as $o(l) = R^l f(R,t)$ for a certain function $f(R,t)$ bounded in a neighborhood of the origin. Thus, we obtain by integrating the expressions above

$$\psi = \left(-R^p\cos(pt) + o(p+1), -R^p\sin(pt) + o(p+1), R^{2p}/2 + o(2p+1)\right). \tag{3.7}$$

From this expression becomes clear that ψ is an embedding around the origin only when $p = -1$.

Now, let us focus on the case when the integers p, q, k are pairwise different. We can assume without loss of generality that $p < q < k$ and $A_{1p} = \cos(pt)$.

We can write

$$\begin{aligned} C(t) &= \lambda_1 k(p-q)\cos(\alpha t) + \lambda_2 q(k-p)\cos(\beta t) + \lambda_3 p(k-q)\cos(\gamma t) \\ &\quad + \lambda_4 k(p-q)\sin(\alpha t) + \lambda_5 q(k-p)\sin(\beta t) + \lambda_6 p(k-q)\sin(\gamma t), \end{aligned}$$

where

$$\alpha = k-q-p, \qquad \beta = k+p-q, \qquad \gamma = k-p+q,$$

and

$$\begin{aligned} \lambda_1 &= (1/2)(a_{2q}b_{3k} - b_{2q}a_{3k}), \\ \lambda_2 &= (1/2)(b_{2q}a_{3k} - a_{2q}b_{3k}), \\ \lambda_3 &= (1/2)(b_{2q}a_{3k} + a_{2q}b_{3k}), \\ \lambda_4 &= -(1/2)(a_{2q}a_{3k} + b_{2q}b_{3k}), \\ \lambda_5 &= (1/2)(a_{2q}a_{3k} + b_{2q}b_{3k}), \\ \lambda_6 &= (1/2)(b_{2q}b_{3k} - a_{2q}a_{3k}). \end{aligned}$$

Observe that at least one of the integers λ_i must be not zero. Otherwise it is easy to see that either A_{2q} or A_{3k} would be zero, which is a contradiction.

If α, β and γ are not zero, from Lemma 8 we deduce that $C(t)$ changes signs, which is not possible.

Hence we must study this particular case when one of the integers α, β, γ is zero.

First, observe that if more than one of the integers α, β, γ were zero, then the integers p, q, k would not be pairwise different, which is not possible.

Since $p < q < k$, $\alpha = k - q - p$ cannot be zero. In fact, since $p < 0$, if $\alpha = 0$ then $q = k - p > k$ which is a contradiction. So, only β or γ (nor both of them simultaneously) can be zero.

Let us analyze the case when $\gamma = 0$, that is, $p = q + k$. Observe that, since $p < q < k$, it must be $k < 0$.

We can rewrite $C(t)$ as

$$C(t) = k^2 C_q(t) - q^2 C_k(t),$$

where

$$\begin{aligned} C_q(t) &= \nu_1 \cos(qt)^2 + \nu_4 \sin(qt)^2 + (\nu_2 + \nu_3)\cos(qt)\sin(qt) \\ C_k(t) &= \nu_4 \cos(kt)^2 + \nu_1 \sin(kt)^2 + (\nu_2 + \nu_3)\cos(kt)\sin(kt) \end{aligned}$$

with

$$\nu_1 = a_{2q}b_{3k}, \quad \nu_2 = a_{2q}a_{3k}, \quad \nu_3 = b_{2q}b_{3k}, \quad \nu_4 = a_{3k}b_{2q}.$$

We can relate $C_q(t)$ and $C_k(t)$ with the quadrics defined by the matrices

$$C_q = \begin{pmatrix} \nu_1 & \frac{\nu_2+\nu_3}{2} \\ \frac{\nu_2+\nu_3}{2} & \nu_4 \end{pmatrix}, \qquad C_k = \begin{pmatrix} \nu_4 & \frac{\nu_2+\nu_3}{2} \\ \frac{\nu_2+\nu_3}{2} & \nu_1 \end{pmatrix}$$

respectively. Both matrices have the same trace and determinant, and so the same eigenvalues κ_1 and κ_2. These eigenvalues are the minimum and maximum values, respectively, which attain the quadrics C_q and C_k on the set of unitary vectors. Moreover, since

$$\det(C_q) = \det(C_k) = -\frac{(\nu_2 - \nu_3)^2}{4} \leq 0$$

it follows that $\kappa_1 \leq 0 \leq \kappa_2$. Let t_1, t_2 be real numbers such that

$$C_k(t_1) = \kappa_1 \leq C_q(t_1),$$
$$C_k(t_2) = \kappa_2 \geq C_q(t_2).$$

Then, bearing in mind that $q < k < 0$, we have

$$C(t_1) \geq (k^2 - q^2)\kappa_1 \geq 0,$$
$$C(t_2) \leq (k^2 - q^2)\kappa_2 \leq 0$$

and so $C(t)$ changes signs whenever $\det(C_q) \neq 0$.

On the other hand, if $\det(C_q) = 0$, we can put $a_{2q} = \lambda b_{3k}$ and $b_{2q} = \lambda a_{3k}$ for a certain $\lambda \in \mathbb{R}$, $\lambda \neq 0$ because $A_{2q} \neq 0$. Hence

$$C_q(t) = \lambda(b_{3k}\cos(qt) + a_{3k}\sin(qt))^2$$
$$C_k(t) = \lambda(a_{3k}\cos(kt) + b_{3k}\sin(kt))^2$$

and

$$C(t) = \lambda\left(k^2(b_{3k}\cos(qt) + a_{3k}\sin(qt))^2 - q^2(a_{3k}\cos(kt) + b_{3k}\sin(kt))^2\right).$$

If there exist $t_q, t_k \in \mathbb{R}$ such that $C_q(t_q) = 0$, $C_q(t_k) \neq 0$, $C_k(t_q) \neq 0$ and $C_k(t_k) = 0$, then $C(t_q)$ and $C(t_k)$ have different sign, and so the affine maximal map have not a well-defined end under that assumption. Otherwise, it must exist $t_0 \in \mathbb{R}$ such that $C_q(t_0) = 0 = C_k(t_0)$ and, since a_{3k} and b_{3k} cannot be both zero, we have

$$0 = \cos(qt_0)\cos(kt_0) - \sin(qt_0)\sin(kt_0) = \cos((q+k)t_0). \tag{3.8}$$

Let us assume, without loss of generality, that $a_{3k} \neq 0$.

Observe that if $\cos(qt_0) = 0$, it follows from (3.8) that $\sin(kt_0) = 0$. But then $C_k(t_0) \neq 0$, which is a contradiction. Hence we can put

$$\frac{b_{3k}}{a_{3k}} = \tan(-qt_0) = \cot(-kt_0).$$

Then, after a straightforward computation we can write

$$\begin{aligned} A_{1p} &= \cos((k+q)t), \\ A_{2q} &= \lambda a_{3k}\sec(qt_0)\sin(q(t-t_0)), \\ A_{3k} &= -a_{3k}\csc(kt_0)\sin(k(t-t_0)). \end{aligned} \tag{3.9}$$

Now, by means of the reparametrization $s = t - t_0$ (we will continue calling t to the new parameter) and a suitable equiaffine transformation, we can rewrite the coefficients in (3.9) as

$$\begin{aligned} A_{1p} &= \cos((k+q)t), \\ A_{2q} &= \sin(qt), \\ A_{3k} &= \sin(kt), \end{aligned} \tag{3.10}$$

and so $C(t)$ becomes

$$C(t) = -q^2 \sin(kt)^2 + k^2 \sin(qt)^2.$$

Finally, observe that $C(t)$ changes signs except in the case when $q = \mu k$ for a certain integer μ, $\mu \geq 2$, where we are using that $q < k < 0$.

In order to see that under these assumptions the affine maximal map is not embedded, we obtain from (3.10) and reasoning as in *i)* that $\psi_R = (\psi_R^1, \psi_R^2, \psi_R^3)$ and $\psi_t = (\psi_t^1, \psi_t^2, \psi_t^3)$ are given by

$$\begin{aligned}
\psi_R^1 &= R^{(\mu+1)k-1}\left(k\sin(\mu kt)\cos(kt) - \mu k\sin(kt)\cos(\mu kt)\right) \\
&\quad + o((\mu+1)k) \\
\psi_R^2 &= R^{(\mu+2)k-1}\left((\mu+1)k\sin(kt)\cos((\mu+1)kt) - k\sin((\mu+1)kt)\cos(kt)\right) \\
&\quad + o((\mu+2)k) \\
\psi_R^3 &= R^{(2\mu+1)k-1}\left(\mu k\sin((\mu+1)kt)\cos(\mu kt) - (\mu+1)k\sin(\mu kt)\cos((\mu+1)kt)\right) \\
&\quad + o((2\mu+1)k)
\end{aligned}$$

$$\begin{aligned}
\psi_t^1 &= -R^{(\mu+1)k}\left(k\sin(\mu kt)\sin(kt) - \mu k\sin(kt)\sin(\mu kt)\right) \\
&\quad + o((\mu+1)k+1) \\
\psi_t^2 &= -R^{(\mu+2)k}\left((\mu+1)k\sin(kt)\sin((\mu+1)kt) - k\sin((\mu+1)kt)\sin(kt)\right) \\
&\quad + o((\mu+2)k+1) \\
\psi_t^3 &= -R^{(2\mu+1)k}\left(\mu k\sin((\mu+1)kt)\sin(\mu kt) - (\mu+1)k\sin(\mu kt)\sin((\mu+1)kt)\right) \\
&\quad + o((2\mu+1)k+1)
\end{aligned}$$

and so

$$\begin{aligned}
\psi^1 &= \frac{R^{(\mu+1)k}}{(\mu+1)k}\left(k\sin(\mu kt)\cos(kt) - \mu k\sin(kt)\cos(\mu kt)\right) \\
&\quad + o((\mu+1)k+1) \\
\psi^2 &= \frac{R^{(\mu+2)k}}{(\mu+2)k}\left((\mu+1)k\sin(kt)\cos((\mu+1)kt) - k\sin((\mu+1)kt)\cos(kt)\right) \\
&\quad + o((\mu+2)k+1) \\
\psi^3 &= \frac{R^{(2\mu+1)k}}{(2\mu+1)k}\left(\mu k\sin((\mu+1)kt)\cos(\mu kt) - (\mu+1)k\sin(\mu kt)\cos((\mu+1)kt)\right) \\
&\quad + o((2\mu+1)k+1)
\end{aligned}$$

Now, it is not difficult to check that ψ is not embedded whichever the values of the integers $k < 0$ and $\mu \geq 2$.

The case $\beta = 0$ is quite similar, although we must make some considerations. First, since now $q = p + k$ and $p < 0$, it could be $q = 0$ and $k = -p$. In this case we can take (see Remark 7)

$$\begin{aligned}
A_{1p} &= \cos(pt) \\
A_{2q} &= 1 \\
A_{3k} &= a_{3k}\cos(pt) + b_{3k}\sin(pt)
\end{aligned}$$

and $C(t)$ becomes

$$C(t) = -p^2(b_{3k}\cos(2pt) + a_{3k}\sin(2pt))$$

which changes signs by Lemma 8. On the other hand, as long as $\beta = 0$ and $q \neq 0$ it must be $k > 0$. In this case we can reasoning as in the case $\gamma = 0$ to conclude that either $p = \mu k$ or $k = \mu p$ for a certain integer μ, $\mu \leq -2$.

None of these examples are embedded, which can be reasoning as above. □

Regarding the case $a \neq 0$, we have the following:

Proposition 10. *Let $\psi : \mathbb{D}^\star \longrightarrow \mathbb{R}^3$ be an affine maximal regular complete end with affine co-normal vector field N given by (3.2). If $a \neq 0$, then one of the following situations happens:*

i) *The third coordinate is $a\log(R)$ or $k \geq 0$, and $p = q \neq 0$. Under these assumptions the end is embedded if, and only if, $p = q = -1$.*

ii) *$k \leq -1$ and either*
   - iia) *$p = k$ (resp. $q = k$) and $q = 0$ (resp. $p = 0$) or*
   - iib) *the three integers p, q, k are pairwise different and, assuming without loss of generality that $p < q$, either*
     - *$p < q < k$, $p = q + k$ and $q = \mu k$ for a certain $\mu \in \mathbb{Z}$, or*
     - *$p < k < q$, $p = q + k$ and $k = \mu q$ for a certain $\mu \in \mathbb{Z}$, or*
     - *$p < k < q$, $k = p + q$ and either $p = \mu q$ or $q = \mu p$ for a certain $\mu \in \mathbb{Z}$, or*
     - *$k < p < q$, $k = p + q$ and $p = \mu q$ for a certain $\mu \in \mathbb{Z}$, or*
     - *$k < p < q$, $p = q + k$ and either $q = \mu k$ or $k = \mu q$ for a certain $\mu \in \mathbb{Z}$.*

   Under these assumptions, the end is embedded only when $p = k = -1$ (resp. $q = k = -1$) and $q = 0$ (resp. $p = 0$).

Proof. Let us start studying jointly the case when the third coordinate is $a\log(R)$ and the case $k \geq 0$. If we take $A_{1p} = \cos(pt)$ and $A_{2q} = a_{2q}\cos(qt) + b_{2q}\sin(qt)$, then in both cases the sign of $[N, N_R, N_t]$ near the origin depends on the coefficient of $R^{-1+p+q}\log(R)$ which is given by

$$C(t) = -apq(b_{2q}\cos((p-q)t) + a_{2q}\sin((p-q)t).$$

Hence, if $p \neq q$, $p \neq 0$ and $q \neq 0$, it follows from Lemma 8 that $C(t)$ changes signs. On the other hand, if one of the integers p, q is equal to zero (note that from Remark 7 we can assume that only one of them is zero), let us say $p = 0$, then we can take

$$A_{1p} = 1, \qquad A_{2q} = \cos(qt).$$

Since it must be $q < 0$, now the sign of $[N, N_R, N_t]$ near the origin depends on the coefficient of R^{-1+q} which is given by

$$C(t) = aq\sin(qt).$$

Again, $C(t)$ changes signs depending of the parameter t and *i)* is proved.

If $k \leq -1$, the sign of $[N, N_R, N_t]$ near the origin depends on the coefficient of $R^{-1+p+q+k}$. Thus we can follow the sketch of proof of Proposition 9 to prove *ii)*.

The reasoning regarding the embedding of the end is totally analogous to the one in Proposition 9. □

As an immediate consequence of Propositions 9 and 10 we have

Theorem 11. *Let p be an embedded regular end of a complete affine maximal map with affine co-normal vector field N. Then p is a pole of order 2 of $dN + i * dN$ and there exists $j \in \{1, 2, 3\}$ such that p is a pole of order less than 2 of $dN_j + i * dN_j$.*

4. Main results and Examples

Let $\psi : \Sigma \longrightarrow \mathbb{R}^3$ be a complete affine maximal map defined from its harmonic affine co-normal vector field $N = (N_1, N_2, N_3) : \Sigma \longrightarrow \mathbb{R}^3$. As we have seen in Section 2, from Theorem 5, the Riemann surface Σ is conformally equivalent to

$$\overline{\Sigma} - \{p_1, ..., p_n\}$$

for a compact Riemann surface $\overline{\Sigma}$, $\{p_1, ..., p_n\}$ being the ends of ψ that we will assume to be regular. In particular, the 1-form $dN + i * dN$ extends meromorphically to the ends.

Next, we obtain the following extension of the affine Bernstein problem:

Theorem 12. *The elliptic paraboloid is the only complete affine maximal map with a unique embedded regular end.*

Proof. Let $\psi : \Sigma \longrightarrow \mathbb{R}^3$ be a complete affine maximal map with a unique embedded regular end p.

Since p is the only pole of $dN + i * dN$ in the compact Riemann surface $\overline{\Sigma}$, from the Residue Theorem we get that

$$Res_p(dN + i * dN) = 0$$

and so, from Theorem 11, there exists $j \in \{1, 2, 3\}$ such that N_j is harmonic on the compact Riemann surface $\overline{\Sigma}$. Then, by the maximum principle, N_j is constant and $\psi : \Sigma \longrightarrow \mathbb{R}^3$ is an improper affine map.

In this case, see Remark 3, there are holomorphic maps

$$\Phi_1, \Phi_2 : \overline{\Sigma} \longrightarrow \mathbb{C} \cup \{\infty\} = \mathbb{S}^2$$

such that

$$N = (\Phi_1 + \overline{\Phi}_1, \Phi_2 + \overline{\Phi}_2, 1)$$

and $deg(\Phi_1) = 1 = deg(\Phi_2)$, since p is their unique simple pole. Thus, Φ_1 and Φ_2 are two biholomorphisms from $\overline{\Sigma}$ into $\mathbb{S}^2$ and Σ is conformally equivalent to $\mathbb{S}^2 - \{p\} = \mathbb{C}$. So we can take without loss of generality

$$\Phi_1(z) = z, \;\; \Phi_2(z) = -iz, \;\; \mathbb{N}_3(z) = 1,$$

with $z \in \mathbb{C}$, which are the data of an elliptic paraboloid. □

Remark 13. *We must point out that for whichever Riemann surface* Σ *and whichever holomorphic function* $F : \Sigma \longrightarrow \mathbb{C}$*,*

$$\Phi_1 = F, \quad \Phi_2 = -iF, \quad \Phi_3 = 1,$$

are the data of an elliptic paraboloid.

In contrast to this uniqueness for the case of one end, we are able to construct and characterize a wide family of complete affine maximal map with two embedded regular ends.

4.1. Canonical Examples

Let us take $\Sigma = \mathbb{C} \setminus \{0\}$ and N as in (3.1), F_1, F_2, F_3 being holomorphic functions on $\mathbb{C} \setminus \{0\}$ and meromorphic on $\mathbb{C} \cup \infty = \mathbb{S}^2$. Observe that, up to an equiaffine transformation, we can assume that $b = c = 0$.

We will call *canonical examples (of genus 0)* to the affine maximal maps such that

$$0 \leq deg(F_j) \leq 2, \quad deg(F_1) + deg(F_2) + deg(F_3) = 4, \; j = 1, 2, 3.$$

Remark 14. *In* [19] *it is proved that a complete improper affine map with exactly two embedded ends is affinely equivalent to a canonical example with* $a = 0$ *and* $deg(F_3) = 0$*.*

Moreover when $deg(F_j) = 0$ *for any* $j \in \{1, 2\}$*,* ψ *also becomes an improper affine map. In this case, from Remark* 3 *we know that* $a = 0$*.*

Of course, besides the improper affine maps described in the remark above, our family of canonical examples contains many others complete affine maximal maps with exactly two embedded ends. In essence, we can classify them into the following three types:

1. 220-type: If $deg(F_1) = 2 = deg(F_2)$, F_3 is constant and $a \in \mathbb{R}$ then, from Section 3, the affine co-normal is given by

$$\begin{aligned} N_1 &= \frac{\cos(t)}{R} + c_1 + d_1 R\cos(t) + e_1 R\sin(t), \\ N_2 &= \frac{\sin(t)}{R} + c_2 + d_2 R\cos(t) + e_2 R\sin(t), \\ N_3 &= c_3 + a\log(R) \end{aligned} \tag{4.1}$$

   and straightforward computations show that the corresponding affine maximal map is single-valued if the real coefficients in (4.1) verify

$$ac_1 = 0 = ac_2 = -d_2 + e_1.$$

   Therefore either $a = 0$ and ψ is an improper affine map (see Remark 14) or $a \neq 0$ and we have a family of new examples of affine maximal maps (see Figure 1).
2. 211/121-type: In the same way, if $deg(F_1) = 2$, $deg(F_2) = 1 = deg(F_3)$, (resp. $deg(F_2) = 2$, $deg(F_1) = 1 = deg(F_3)$) and $a \in \mathbb{R}$, we also have a family of new examples of affine maximal maps (see Figure 2). Note that the two types of affine maximal maps described in this family are affinely equivalent.

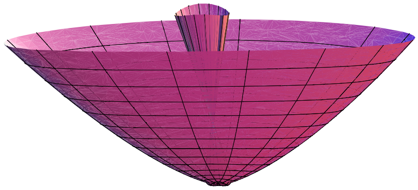

Figure 1. 220-affine maximal map.

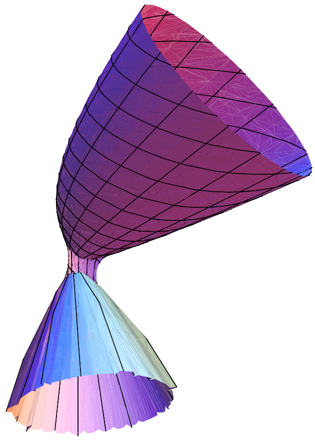

Figure 2. 211-affine maximal map.

3. 112-type: For $deg(F_1) = 1 = deg(F_2)$, $deg(F_3) = 2$ and $a \in \mathbb{R} \setminus \{0\}$, we have another family of new examples of complete affine maximal maps (see Figure 3). Observe that if $a = 0$, these surfaces are affinely equivalent to the ones of 211-type.

Similarly, we could describe the canonical examples with arbitrary genus. In fact, for any compact Riemann surface $\overline{\Sigma}$ and two different points p_1, p_2 on $\overline{\Sigma}$, if we take local coordinates z_j vanishing at p_j, $j = 1, 2$, then there exist on $\overline{\Sigma}$ real-valued functions u_1, v_1, u_2, v_2, L such that

1. u_j, v_j are harmonic on $\overline{\Sigma} - \{p_j\}$, $j = 1, 2$,

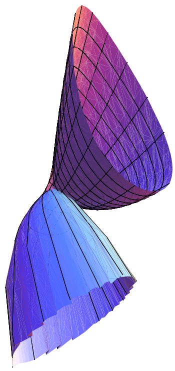

FIGURE 3. 112-affine maximal map.

2. $u_j - \Re z_j^{-1}$ and $v_j - \Im z_j^{-1}$ are harmonic in a neighborhood of p_j, $j = 1, 2$,
3. L is harmonic on $\overline{\Sigma} - \{p_1, p_2\}$,
4. $L - \log|z_1|$ is harmonic in a neighborhood of p_1 and
5. $L + \log|z_2|$ is harmonic in a neighborhood of p_2.

Again, besides the improper affine maps, the family of canonical examples contains the above three types. In particular, the 220-type is given by the following affine co-normal

$$\begin{aligned} N_1 &= u_1 + c_1 + d_1 u_2 + e_1 v_2, \\ N_2 &= v_1 + c_2 + d_2 u_2 + e_2 v_2, \\ N_3 &= c_3 + aL \end{aligned} \tag{4.2}$$

and in general

$$N = Au_1 + Bv_1 + C + Du_2 + Ev_2 + (0, 0, a)L,$$

for some $A, B, C, D, E \in \mathbb{R}^3$ and $a \in \mathbb{R}$ which must be chosen satisfying

$$\int_\gamma N \wedge *dN = 0, \tag{4.3}$$

for any loop γ in Σ. The important fact is that any complete affine maximal maps with two embedded regular ends can be described in this way:

Theorem 15. *A complete affine maximal map $\psi : \Sigma \longrightarrow R^3$ with exactly two embedded regular ends is a canonical example.*

Proof. Let p_1, p_2 the ends of ψ. Up to an affine transformation, we can assume

$$Res_{p_1}(dN_j + i * dN_j) = 0, \qquad j = 1, 2.$$

Then, from the Residue Theorem we get

$$0 = Res_{p_1}(dN_j + i * dN_j) + Res_{p_2}(dN_j + i * dN_j) = Res_{p_2}(dN_j + i * dN_j)$$

and so, from Theorem 11, there exist constant vectors $A, B, D, E \in \mathbb{R}^3$ such that

$$N - Au_1 - Bv_1 - Du_2 - Ev_2 - (0, 0, a)L$$

is harmonic on the compact Riemann surface $\overline{\Sigma}$. Thus, we conclude that it is a constant vector $C \in \mathbb{R}^3$ and the theorem is proved. □

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