A class of Weingarten surfaces in the hyperbolic 3-space.

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Abstract

We study a large class of Weingarten surfaces which includes the constant mean curvature one surfaces and flat surfaces in the hyperbolic 3-space. We show that these surfaces can be parametrized by holomorphic data like minimal surfaces in the Euclidean 3-space and we use it to study their completeness. We also establish some existence and uniqueness theorems by studying the Plateau problem at infinity: when is a given curve on the ideal boundary the asymptotic boundary of a complete surface in our family? and, how many embedded solutions are there?

1. BLW-Surfaces.

Let S be a surface, an immersion $\psi : S \longrightarrow \mathbb{H}^3$, with Gauss map η, is called a linear Weingarten immersion of Bryant type, (in short, BLW-surface), if the mean curvature H and the Gauss curvature K_I satisfy

$$2\,a\,(H-1) + b\,K_I = 0, \tag{1}$$

for some $a, b \in \mathbb{R}$, $a+b \neq 0$.

Remark 1..1 This family includes the Bryant surfaces ($H = 1$) and the flat surfaces.
The case $a + b = 0$ is studied in [1]. □

2. The hyperbolic 3-space.

In the Lorentz-Minkowski model, $\mathbb{L}^4$, the hyperbolic 3-space and the positive null cone are given by

$$\mathbb{H}^3 = \left\{(x_0,x_1,x_2,x_3) \in \mathbb{R}^4 :\ -x_0^2 + x_1^2 + x_2^2 + x_3^2 = -1,\ x_0 > 0\right\}$$
$$\mathbb{N}^3_+ = \left\{(x_0,x_1,x_2,x_3) \in \mathbb{R}^4 :\ -x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0,\ x_0 > 0\right\}$$

and $\mathbb{N}^3_+/\mathbb{R}^+$ is the ideal boundary $\mathbb{S}^2_\infty$ of $\mathbb{H}^3$.
If we regard $\mathbb{L}^4$ as the space of 2×2 Hermitian matrices, Herm(2), by identifying

$$(\mathrm{x}_0,\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3) \longleftrightarrow \begin{pmatrix} x_0 - x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 + x_3 \end{pmatrix},$$

then $\mathbb{H}^3 = \{m \in Herm(2)\,|\,det(m) = 1\}$ and the action of $\mathbb{SL}(2,\mathbb{C})$ defined by

$$g \cdot m = gmg^*, \quad g^* = {}^t\bar{g},$$

preserves the inner product and leaves $\mathbb{H}^3$ invariant. Moreover the map

$$w\,{}^t\overline{w} \longrightarrow [(w_1,w_2)] \in \mathbb{CP}^1, \quad {}^t\overline{w} = (w_1,w_2) \in \mathbb{C}^2,$$

let us identify $\mathbb{S}^2_\infty$ to $\mathbb{CP}^1$ and the action of $\mathbb{SL}(2,\mathbb{C})$ on $\mathbb{S}^2_\infty$ is the action on $\mathbb{CP}^1$ by Möbius transformations.

3. Conformal representation.

Lema 3..1 *Let $\psi : S \longrightarrow \mathbb{H}^3$ be a BLW-surface, with Gauss map η. Then, we can consider that $|a+b| = 1$,*

$$2\,a\,(H-1) + b\,(K-1) = 0 \tag{2}$$

and $\sigma = a\,I + b\,II$ is a positive definite metric, where K is the Gauss-Kronecker curvature, $I = \langle d\psi, d\psi\rangle$ and $II = \langle d\psi, -d\eta\rangle$, the first and second fundamental form of the immersion, respectively.

From now on, we shall regard S as a Riemann surface with the conformal structure induced by $\sigma = a\,I + b\,II$.

Theorem 3..2 *Let $\psi : S \longrightarrow \mathbb{H}^3$ be a BLW-surface, with Gauss map η. Then $\psi + \eta$ is a conformal map with respect to the metric $\sigma = aI + bII$ and*

$$\Delta^\sigma(\psi + \eta) = \frac{2}{a+b}\{(H-1)\psi + (K-H)\eta\}, \tag{3}$$

where Δ^σ denotes the Laplacian of σ. In particular, its hyperbolic Gauss map $G := [\psi + \eta]$ is conformal.
Moreover, the immersion lies in a horosphere or $\langle d(\psi+\eta), d(\psi+\eta)\rangle$ is a pseudometric of constant curvature $\frac{a}{a+b}$.

Remark 3..3 From the above Theorem, the two 2-forms $Q_I = \langle \psi_z, \psi_z\rangle dz^2$ and $Q_{II} = \langle \psi_z, -\eta_z\rangle dz^2$ are holomorphic on S, for any conformal parameter z for σ.
If S is a topological sphere then, Q_I and Q_{II} vanish identically and $\psi(S)$ is a totally umbilical round sphere. □

Theorem 3..4 *(Conformal representation)*
i) Let S be a non compact, simply connected surface and $\psi : S \longrightarrow \mathbb{H}^3$ a BLW-surface. Then, there exist a meromorphic curve $g : S \longrightarrow \mathbb{SL}(2,\mathbb{C})$ and a pair (h,ω) consisting of a meromorphic function h and a holomorphic 1-form ω on S, such that the immersion and its Gauss map can be recovered as

$$\psi = g\Omega g^* \quad \text{and} \quad \eta = g\widetilde{\Omega}g^*, \tag{4}$$

where

$$\Omega = \begin{pmatrix} \frac{1+\varepsilon^2|h|^2}{1+\varepsilon|h|^2} & -\varepsilon\bar{h} \\ -\varepsilon h & 1+\varepsilon|h|^2 \end{pmatrix} \quad \text{and} \quad \widetilde{\Omega} = \begin{pmatrix} \frac{1-\varepsilon^2|h|^2}{1+\varepsilon|h|^2} & \varepsilon\bar{h} \\ \varepsilon h & -1-\varepsilon|h|^2 \end{pmatrix}, \tag{5}$$

with $\varepsilon = \frac{a}{a+b}$ and $1 + \varepsilon|h|^2 > 0$. Moreover, the curve g satisfies

$$g^{-1}dg = \begin{pmatrix} 0 & \omega \\ dh & 0 \end{pmatrix}. \tag{6}$$

The induced metric and $\sigma = a\,I + b\,II$ are given, respectively, by

$$I = (1-\varepsilon)\omega dh + \left(\frac{(1-\varepsilon)^2|dh|^2}{(1+\varepsilon|h|^2)^2} + (1+\varepsilon|h|^2)^2|\omega|^2\right) + (1-\varepsilon)\bar{\omega}d\bar{h} \tag{7}$$

and

$$\sigma = (a+b)\left((1+\varepsilon|h|^2)^2|\omega|^2 - \frac{(1-\varepsilon)^2|dh|^2}{(1+\varepsilon|h|^2)^2}\right). \tag{8}$$

ii) Conversely, let S be a Riemann surface, $g : S \longrightarrow \mathbb{SL}(2,\mathbb{C})$ a meromorphic curve and (h,ω) a pair as above satisfying (6) and such that (8) is a positive definite metric. Then $\psi = g\Omega g^* : S \longrightarrow \mathbb{H}^3$, ($\Omega$ as in (5)), is a BLW-surface satisfying (2) with induced metric and σ given by (7) and (8).

Remark 3..5 *Following the same notation as in [2] and [4], the pair (h,ω) given by the above theorem will be called the Weierstrass data. BLW-surfaces with the same Weierstrass data are congruent.*

Remark 3..6 *If $\psi : S \longrightarrow \mathbb{H}^3$ is an immersion with $H = 1$ then $\varepsilon = 1$,*

$$\Omega = \begin{pmatrix} 0 & -i \\ -i & ih \end{pmatrix}\begin{pmatrix} 0 & i \\ i & -i\bar{h} \end{pmatrix}$$

and $\psi = g\Omega g^* = FF^*$, where

$$F = g\begin{pmatrix} 0 & -i \\ -i & ih \end{pmatrix} \in \mathbb{SL}(2,\mathbb{C}),$$

that is, the conformal representation becomes the Bryant's representation (see [2], [9]).
Moreover, if the immersion does not lie in a horosphere and we denote by G its hyperbolic Gauss map then one gets

$$g = \begin{pmatrix} iG\sqrt{\frac{dh}{dG}} & \frac{i}{dh}d\left(G\sqrt{\frac{dh}{dG}}\right) \\ i\sqrt{\frac{dh}{dG}} & \frac{i}{dh}d\left(\sqrt{\frac{dh}{dG}}\right) \end{pmatrix}.$$

and we recover the Small's formula for surfaces with constant mean curvature one (see [6], [8]).

Remark 3..7 *If S is non compact then, from (4), (5), (7) and (8), its Gaussian curvature $K_I = K-1$ can be calculated as*

$$K_I = \frac{-4\,\varepsilon\,|dh|^2}{(1+\varepsilon|h|^2)^4\,|\omega|^2 - (1-\varepsilon)^2\,|dh|^2} \tag{9}$$

and its mean curvature is given by

$$H = 1 + \frac{2\,(1-\varepsilon)\,|dh|^2}{(1+\varepsilon|h|^2)^4\,|\omega|^2 - (1-\varepsilon)^2\,|dh|^2}.$$

In particular, a point $p \in S$ is umbilical if and only if $dh(p) = 0$ or $\omega(p) = 0$.

4. Completeness of the immersions.

4.1. Completeness with non negative Gauss curvature.

Theorem 4..1 *Let $\psi : S \longrightarrow \mathbb{H}^3$ be a complete BLW-surface with non negative Gauss curvature K_I. Then $\psi(S)$ is a totally umbilical round sphere, a horosphere or a hyperbolic cylinder.*

4.2. Completeness with negative Gauss curvature and $\varepsilon > 0$.

If (h,ω) are the Weierstrass data for a BLW-surface ψ_0 with $K_I < 0$ and $\varepsilon > 0$, we obtain a new associated immersion ψ_1 with constant mean curvature one and Weierstrass data $(\sqrt{\varepsilon}h, \omega)$.
Thus, the study of complete BLW-surfaces with $K_I < 0$ and $\varepsilon > 0$ can be reduced to the study of complete immersions with constant mean curvature one. Many things are known in this case and some very interesting results were proved in [2], [3] and [9].

4.3. Completeness with negative Gauss curvature and $\varepsilon < 0$.

The geometry of the surface is very different when the immersion lies in the case R_3, that is, $K_I < 0$ and $\varepsilon < 0$. For instance

Lema 4..2 *Let $\psi : S \longrightarrow \mathbb{H}^3$ be a complete BLW-surface in R_3 with Weierstrass data (h,ω). Then, S is conformally equivalent to the unit disk $\mathbb{D}$ and h is a global diffeomorphism onto $\mathbb{D}_\varepsilon = \{z \in \mathbb{C} :\ |z|^2 < -1/\varepsilon\}$.*

From the above lemma, given a complete BLW-surface $\psi : S \longrightarrow \mathbb{H}^3$ with Weierstrass data (h,ω) we can consider, up to a change of parameter, $S = \mathbb{D}$ and $h(z) = z/\sqrt{-\varepsilon}$.
Moreover, from (8), σ is positive definite if and only if

$$|\omega| < \frac{1-\varepsilon}{\sqrt{-\varepsilon}}\,\frac{|dz|}{(1-|z|^2)^2}, \qquad z \in \mathbb{D}. \tag{10}$$

The Schwarzian derivative of G satisfies

$$\{G,z\} := \frac{d}{dz}\left(\frac{G_{zz}}{G_z}\right) - \frac{1}{2}\left(\frac{G_{zz}}{G_z}\right)^2 = \frac{2G_zG_{zzz} - 3G_{zz}^2}{2G_z^2} = \frac{-2}{\sqrt{-\varepsilon}}\,\frac{\omega}{dz}$$

and

$$|\{G,z\}| < \frac{2(1-\varepsilon)}{-\varepsilon}\frac{1}{(1-|z|^2)^2}, \qquad z \in \mathbb{D}, \tag{11}$$

Consequently, G is a local diffeomorphism with bounded Schwarzian derivative.

Theorem 4..3 *Conversely, let $G : \mathbb{D} \longrightarrow \mathbb{C} \cup \{\infty\}$ be a meromorphic map. Then, if G satisfies (11) one has a BLW-surface in R_3 with hyperbolic Gauss map G and Weierstrass data $(z/\sqrt{-\varepsilon}, -\frac{1}{2}\sqrt{-\varepsilon}\,\{G,z\}\,dz)$. Moreover, if*

$$|\{G,z\}| \leq \frac{b_0}{(1-|z|^2)^2} \qquad z \in \mathbb{D}, \tag{12}$$

with $b_0 < \frac{2(1-\varepsilon)}{-\varepsilon}$, then the immersion is complete.

5. A Plateau problem at infinity.

From the above Theorem there are a lot of complete BLW-surfaces in the case R_3, that is, $K_I < 0$ and $\varepsilon < 0$. Moreover, they give geometric meaning to classical families of complex functions which have been studied in connection with the Schwarzian derivative (see [7]). We use this relation in order to study the following Plateau problem at infinity:

Given $\varepsilon_0 < 0$ and a Jordan curve Γ on $\mathbb{S}^2_\infty \equiv \Pi \cup \{\infty\}$, find a complete BLW-surface $\psi : S \longrightarrow \mathbb{H}^3$ satisfying

$$2(-\varepsilon_0)(H-1) + (\varepsilon_0 - 1)K_I = 0$$

with Γ as its asymptotic boundary.

Here we identify the hyperbolic space $\mathbb{H}^3 \subset \mathbb{L}^4$ with the upper half space of $\mathbb{R}^3$. The ideal boundary $\mathbb{S}^2_\infty$ is identified with the one point compactification of the plane $\Pi \equiv \{x_3 = 0\}$. We will assume that, up to a Möbius transformation, the Jordan curve Γ lies on Π.

Theorem 5..1 *(Convex case) Let Γ be a convex Jordan curve on $\Pi \equiv \{x_3 = 0\} \subset \mathbb{R}^3$. Then for any $\varepsilon < 0$ there exists a unique embedded solution to the Plateau problem for Γ with hyperbolic normal pointing downwards at its (Euclidean) highest point.*
Moreover if $-1/2 < \varepsilon < 0$, then there are exactly two embedded solutions to the Plateau problem for Γ.

Theorem 5..2 *(General case) Let Γ be a Jordan curve on $\Pi \equiv \{x_3 = 0\} \subset \mathbb{R}^3$. Then for any $\varepsilon \in]-1/2, 0[$ there exist at least two solutions to the Plateau problem for Γ. Moreover, if $-1/4 < \varepsilon < 0$ then there only exist two embedded solutions to this Plateau problem.*

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