Applications of the Affine Cauchy Problem.

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Abstract

We present the resolution of the problem of existence and uniqueness of affine maximal surfaces containing a regular analytic curve and with a given affine normal along it, see [1]. As applications we give results about symmetries, characterize when a curve in $\mathbb{R}^3$ can be a geodesic of a such surface and study helicoidal affine maximal surfaces, that is, surfaces invariant under a one-parametric group of equiaffine transformations. We obtain new examples with an analytic curve in its singular set, which have been studied in [2].

1. Affine Maximal Surfaces

The equiaffine area functional

$$\int dA = \int |K_e|^{\frac{1}{4}} dA_e,$$

with K_e the euclidean Gauss curvature and dA_e the element of euclidean area, has attracted to many geometers since the beginning of the last century.

Well-known Facts:

- Blaschke (1923): the associated fourth order Euler-Lagrange equation is equivalent to the vanishing of the affine mean curvature.
- Calabi (1982): locally strongly convex surfaces have always a negative second variation (affine maximal surfaces).

1.1. Recent developments

- Affine Weierstrass formulas that have provided an important tool in their study, (Calabi, Li, 1990).
- Entire solutions of the fourth order affine maximal surface equation

$$\phi_{yy}\omega_{xx} - 2\phi_{xy}\omega_{xy} + \phi_{xx}\omega_{yy} = 0, \quad \omega = \left(det\left(\nabla^2\phi\right)\right)^{-3/4}, \tag{1..1}$$

  are always quadratic polynomials (Trudinger-Wang, 2000).
- Every Affine complete affine maximal surface is an elliptic paraboloid, (Li-Jia, 2001, Trudinger-Wang, 2002).
- The Affine Plateau Problem (Trudinger-Wang, 2005).
- Their extension to different nonlinear fourth order equations (Li-Jia, 2003, Trudinger-Wang, 2002).
- The validity of the results in affine maximal surfaces with some natural singularities that may arise (Aledo, Chaves, Gálvez, Martínez, Milán, Mira, 2005-2008).

1.2. Basic Notations

Let $\psi : \Sigma \rightarrow \mathbb{R}^3$ be a l.s.c immersion, with second fundamental form σ_e definite positive,

$$g = K_e^{-\frac{1}{4}}\sigma_e, \quad \text{Berwald-Blaschke metric}$$
$$\xi = \frac{1}{2}\Delta_g\psi, \quad \text{affine normal}$$

with Δ_g:= Laplace-Beltrami operator associated to g.

The affine conormal field $N := K_e^{-1/4}N_e$, satisfies

$$\langle N, \xi\rangle = 1, \qquad \langle N, d\psi(v)\rangle = 0, \quad v \in T_p\Sigma, \tag{1..2}$$

and the Euler-Lagrange equation:= $\Delta_g N = 0$.

1.3. Weierstrass-type Representation Formulas

In the simply-connected case ψ can be recovered from N and the conformal class of the Blaschke metric:

Lelieuvre formula

$$\psi = 2\,\mathrm{Re}\int \imath\, N \times N_z dz$$

Calabi's Representation ψ determine a holomorphic curve $\Phi : \Omega \subset \Sigma \rightarrow \mathbb{C}^3$ s.t.

$$N = \Phi + \overline{\Phi}, \qquad g = -\imath Det\left[\Phi + \overline{\Phi}, \Phi_z, \overline{\Phi_z}\,\right] dzd\overline{z}. \tag{1..3}$$

ψ is determined, up to real translation, by a holomorphic curve Φ satisfying $-\imath Det\left[\Phi + \overline{\Phi}, \Phi_z, \overline{\Phi_z}\,\right] > 0$. To be precise,

$$\psi = 2\,\mathrm{Re}\int \imath\,\left(\Phi + \overline{\Phi}\right) \times \Phi_z dz = -\imath\left(\Phi \times \overline{\Phi} - \int \Phi \times d\Phi + \int \overline{\Phi} \times \overline{d\Phi}\,\right).$$

2. The Affine Björling Problem

Let $\beta : I \rightarrow \Sigma$ be a regular analytic curve. $\alpha = \psi \circ \beta$, $Y = \xi \circ \beta$ and $U = N \circ \beta$, then, along the curve α

$$\left.\begin{array}{l} 0 = \langle \alpha'(s), U(s)\rangle, \\ 1 = \langle Y(s), U(s)\rangle, \\ 0 = \langle Y'(s), U(s)\rangle, \\ 0 < \lambda(s) = -\langle \alpha'(s), U'(s)\rangle = \langle \alpha''(s), U(s)\rangle, \end{array}\right\} \tag{2..1}$$

where by prime we indicate derivation respect to s, for all $s \in I$.

Definition Given $Y, U, \alpha : I \longrightarrow \mathbb{R}^3$ regular analytic curves. $\{Y, U\}$ is an *analytic equiaffine normalization* of α if there is an analytic positive function $\lambda : I \rightarrow \mathbb{R}^+$ such that all the equations in (2..1) hold on I.

Theorem Let $\{Y, U\}$ be an analytic equiaffine normalization of α, then there exists a unique affine maximal surface ψ containing $\alpha(I)$, with conormal field and Blaschke normal along α, U and Y respectively.

(ψ:= a.m.s. along α generated by $\{Y, U\}$).

Outline of the Proof

- By the Inverse Function Theorem $\exists z : s + \imath t$, $s \in I$
- Identity Principle: $N_z = \frac{1}{2}(U_z + \imath Y \times \alpha_z)$, $\quad z \in \Omega$
- Via Calabi's representation.

$$\psi = \alpha(s_0) + 2\,\mathrm{Re}\int_{s_0}^{z} \imath(\Phi + \overline{\Phi}) \times \Phi_\zeta d\zeta, \tag{2..2}$$

  where, $\Phi(z) = \frac{1}{2}\left(U + \imath\int_{s_0}^{z} Y \times \alpha_\zeta d\zeta\right)$, $\quad z \in \Omega, \quad s_0 \in I$, on a complex domain Ω containing I.

Corollary Let $\alpha, Y : I \rightarrow \mathbb{R}^3$ be two regular analytic curves

$$Det[\,Y', \alpha', Y\,]Det[\,Y', \alpha', \alpha''\,] > 0, \qquad \text{on} \quad I. \tag{2..3}$$

$\Rightarrow \exists_1\, \psi$ containing $\alpha(I)$ with Y as Blaschke normal along α.

Proof $\exists_1\, U$ and λ,

$$U = \frac{Y' \times \alpha'}{Det[\,Y', \alpha', Y\,]}, \qquad 0 < \lambda = \frac{Det[\,Y', \alpha', \alpha''\,]}{Det[\,Y', \alpha', Y\,]}$$

s.t. $\{Y, U\}$ is an a.e.n. of α. The result follows from above Theorem, taking in Calabi's representation,

$$\Phi(z) = \frac{Y_z \times \alpha_z}{2Det[\,Y_z, \alpha_z, Y\,]} + \frac{\mathrm{i}}{2}\int_{s_0}^{z} Y \times \alpha_\zeta d\zeta, \qquad z \in \Omega, \quad s_0 \in I.$$

Corollary $\alpha, Y : I \rightarrow \mathbb{R}^3$ regular analytic curves

$$[\,Y, \alpha', \alpha''\,] \neq 0, \qquad Y' \times \alpha' = 0, \qquad \text{on} \quad I. \tag{2..4}$$

Given $\lambda : I \rightarrow \mathbb{R}^+$, $\exists_1\, \psi$ containing $\alpha(I)$, such that its Blaschke normal along α is Y and $g(\alpha', \alpha') = \lambda$.

3. Applications

3.1. The Cauchy Problem

If $\psi : \Omega \longrightarrow \mathbb{R}^3$ is the graph of a l.s.c. function $\phi(x, y)$, $(x, y) \in \Omega$. The Euler-Lagrange equation for the affine area functional is

$$\phi_{yy}\omega_{xx} - 2\phi_{xy}\omega_{xy} + \phi_{xx}\omega_{yy} = 0, \qquad \omega = \left(det\left(\nabla^2\phi\right)\right)^{-3/4}.$$

In this situation

$$\begin{aligned} g_\phi &= \sqrt[3]{\omega}\left(\phi_{xx}\,dx^2 + 2\phi_{xy}\,dx\,dy + \phi_{yy}\,dy^2\right), \\ N &= \sqrt[3]{\omega}\,(-\phi_x, -\phi_y, 1), \\ \xi &= \left(\varphi_y, -\varphi_x, \frac{1}{\sqrt[3]{\omega}} - \phi_y\varphi_x + \phi_x\varphi_y\right), \end{aligned} \tag{3..1}$$

where $\varphi_x = \frac{1}{3}(\phi_{xy}\omega_x - \phi_{xx}\omega_y)$ and $\varphi_y = \frac{1}{3}(\phi_{yy}\omega_x - \phi_{xy}\omega_y)$.

The Cauchy Problem

$$\begin{cases} \phi_{yy}\omega_{xx} - 2\phi_{xy}\omega_{xy} + \phi_{xx}\omega_{yy} = 0, \qquad \omega = \left(det\left(\nabla^2\phi\right)\right)^{-3/4} \\ \phi(x, 0) = a(x), \\ \phi_y(x, 0) = b(x), \\ \phi_{yy}(x, 0) = c(x), \\ \phi_{yyy}(x, 0) = d(x), \\ c(x)a''(x) - b'(x)^2 > 0 \end{cases}$$

where a, b, c, d are analytic functions on I, ϕ is defined on Ω containing $I \times \{0\}$, **has solution**

$$(x, y, \phi(x, y)) = (s_0, 0, a(s_0)) + 2\,\mathrm{Re}\int_{s_0}^{z = s + \mathrm{i}\,t} \left(\Phi + \overline{\Phi}\right) \times \Phi_\zeta\, d\zeta,$$

with

$$\begin{aligned} \Phi(z) &= \frac{1}{2}\left(U(z) + \imath\int_{s_0}^{z} Y(\zeta) \times A(\zeta)\, d\zeta\right), \\ U(s) &= \left(c(s)a''(s) - b'(s)^2\right)^{-1/4}(-a'(s), -b(s), 1), \\ A(s) &= (1, 0, a'(s)), \\ Y(s) &= \frac{1}{4}\left(c(s)a''(s) - b'(s)^2\right)^{-7/4}\left(b'(da'' + 3cb'') - 2b'^2c' - c(c'a'' + ca'''),\right. \\ &\quad b'(3c'a'' + ca''') - 2b'^2b'' - a''(da'' + cb''), \\ &\quad +4b'^4 - 2b'^2(a'c' + 4ca'' + bb'') - a''((-4c^2 + bd)a'' + bcb'') \\ &\quad \left. - ca'(c'a'' + ca''') + b'\left(a'(da'' + 3cb'') + b(3c'a'' + ca''')\right)\right). \end{aligned}$$

3.2. Symmetry and Geodesics

Consider $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, the equiaffine transformation given by $T(v) = Av + b$ and $\{Y, U\}$ an analytic equiaffine normalization of $\alpha : I \rightarrow \mathbb{R}^3$. We say T is a symmetry of $\{Y, U\}$ if $\exists\, \Gamma : I \rightarrow I$ analytic diffeomorphism such that $\alpha \circ \Gamma = T \circ \alpha$, $\quad Y \circ \Gamma = AY$, $\quad U \circ \Gamma = (A^t)^{-1}U$.

Theorem. (Generalized symmetry principle). Any symmetry of an analytic equiaffine normalization induces a global symmetry of the affine maximal surface generated by the equiaffine normalization.

If $\beta : I \rightarrow \Sigma$ is a regular curve s.t., $\alpha = \psi \circ \beta$, $Y = \xi \circ \beta$ and $U = N \circ \beta$ are analytic $\Rightarrow$ α is a pre-geodesic for the Blaschke metric if and only if

$$[\alpha', \alpha'', Y] + [U, U', U''] = 0 \qquad \text{on} \quad I. \tag{3..2}$$

Then a regular analytic curve $\alpha : I \rightarrow \mathbb{R}^3$ is the geodesic of some affine maximal surface if and only if there exists an affine equiaffine normalization $\{Y, U\}$ of α satisfying (3..2) and $\langle\alpha'', U\rangle = c$ for a positive constant c.

Thus, we can obtain that every planar analytic curve whose curvature does not vanish at any point is pre-geodesic of an affine maximal surface which has the plane containing the curve as a symmetry plane. Also, every analytic helix, $(k/\tau$ constant), is pre-geodesic of an affine maximal surface.

4. Helicoidal affine maximal surfaces

Consider $T_s(v) = A_s v + b_s$ a one-parametric subgroup of equiaffine transformations. From our existence Theorem and generalized symmetry Principle, an affine maximal surface invariant under T_s, $s \in \mathbb{R}$, is locally given by the surface generated by the following $\{T_s\}$-symmetric a.e.n $\{Y, U\}$, along the orbit $\alpha_p(s) = T_s(p)$ of a fixed point $p = (p_1, p_2, p_3)$,

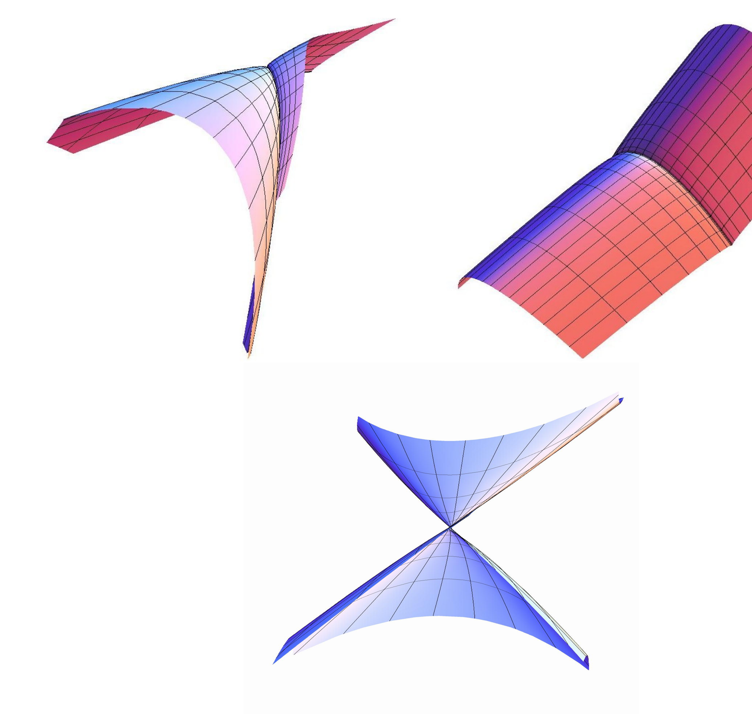
$$Y(s) = A_s Y_p, \qquad U(s) = (A_s^t)^{-1}U_p$$

and $Y_p, U_p \in \mathbb{R}^3$ satisfy the necessary conditions for (2..1) holds. In particular, the Berwald-Blaschke metric must be constant along α_p.

We apply our representation to classify the affine maximal surfaces invariant under these groups.

4.1. Some $G_{1,a}$-invariant affine maximal surfaces

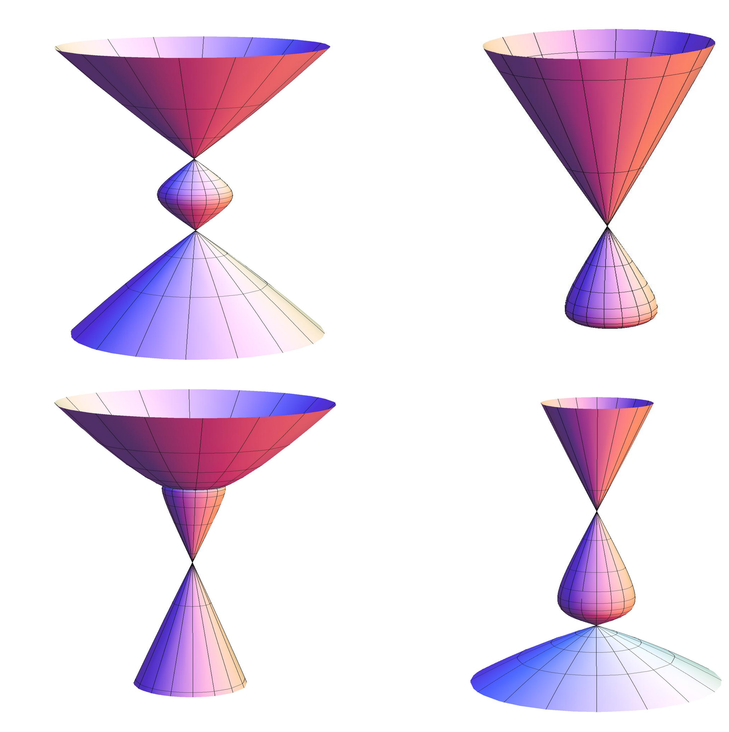
For this one-parametric group the orbit of a point p is given by $\alpha_p(s) = \left(p_1 + p_2as + p_3a\frac{s^2}{2} + a\frac{s^3}{6}, p_2 + p_3s + \frac{s^2}{2}, p_3 + s\right)$.



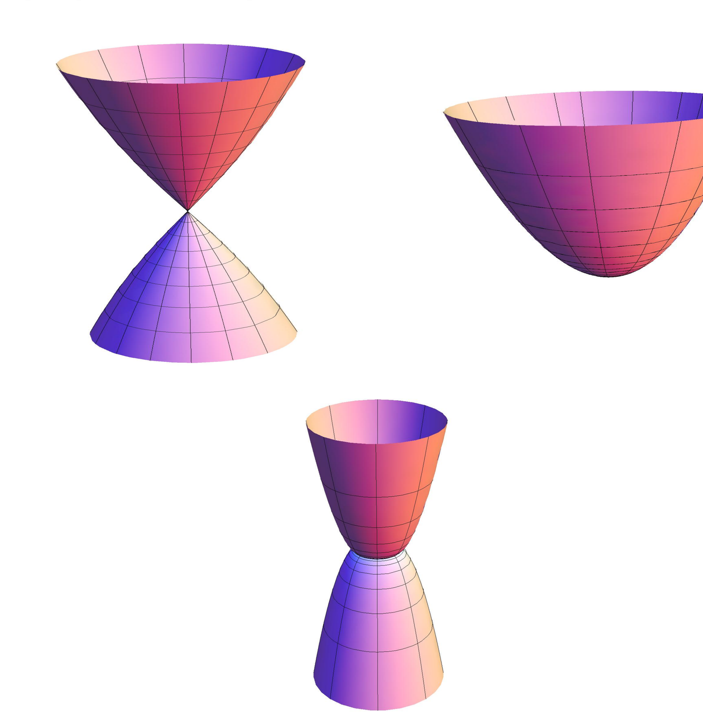
4.2. Some $G_{2,a}$-invariant affine maximal surfaces

In this case $\alpha_p(s) = (p_1\cos(s) + p_2\sin(s), -p_1\sin(s) + p_2\cos(s), p_3 + as)$.

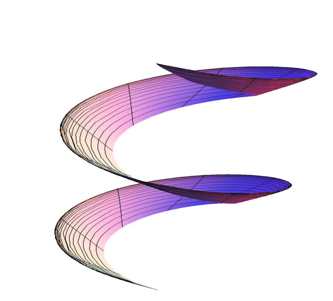
Rotational affine maximal surfaces:



Rotational improper affine spheres:



Non rotational $G_{2,a}$-invariant affine maximal surface:



5. Affine maximal maps

Some Helicoidal affine maximal surfaces:

- Are glued by analytic curves where the affine metric is degenerated but the affine normal and the affine conormal are well defined.
- Can be represented as in (2..2), where Φ is a well-defined holomorphic regular curve on the Riemann surface Σ.

Definition If a map $\psi : \Sigma \longrightarrow \mathbb{R}^3$ admits a representation as in (2..2) for a certain holomorphic curve Φ which satisfies that $[\Phi + \overline{\Phi}, \Phi_z, \overline{\Phi_z}]|dz|^2$ does not vanish identically, we say that ψ is an *affine maximal map*

Theorem $\alpha : I \longrightarrow \mathbb{R}^3$ a regular analytic curve with non-vanishing curvature. Then, for any non-vanishing regular analytic function $h : I \longrightarrow \mathbb{R}$ there exists a unique affine maximal map ψ_h containing $\alpha(I)$ in its set of singularities.

References

[1] *J. A. Aledo, A. Martínez and F. Milán*, The Affine Cauchy Problem. J. Math. Anal. Appl. 351 (2009) 70-83

[2] *J. A. Aledo, A. Martínez and F. Milán*, Affine maximal maps. Preprint.

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