The Indefinite Affine Cauchy Problem [1]

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Abstract

We solve the geometric Cauchy problem for the class of affine maximal surfaces, with indefinite affine metric, in the affine space $\mathbb{R}^3$, that is, we find all the surfaces of this class which contain a given regular curve of $\mathbb{R}^3$ with prescribed affine normal and affine conormal along it. We prove the problem is well-posed when the initial data are non-characteristic and show that uniqueness of the solution can fail at characteristic (asymptotic) directions. As application we obtain some results about geodesics and symmetries of indefinite affine maximal surfaces.

1 Introduction

The affine surfaces theory provides important geometric models for some well known partial differential equations (PDEs), see [13, 22]. Thus, the improper affine spheres are locally the graphs of the solutions of the classical Monge-Ampére equation

$$f_{xx}f_{yy} - f_{xy}^2 = \pm 1. \tag{1.1}$$

These umbilical affine surfaces, (with constant affine normal), are a particular case of affine maximal surfaces, with harmonic affine conormal with respect to their affine metric, see [12, 19]. Equivalently, their affine mean curvature vanishes and they satisfy the nonlinear fourth order PDE

$$f_{yy}\omega_{xx} - 2f_{xy}\omega_{xy} + f_{xx}\omega_{yy} = 0, \qquad \omega = \left|f_{xx}f_{yy} - f_{xy}^2\right|^{-3/4}. \tag{1.2}$$

Of course, the situation changes completely if the Hessian $f_{xx}f_{yy} - f_{xy}^2$ is positive or negative, since the nature of the above PDE changes from elliptic to hyperbolic and the associated affine metric changes from definite to indefinite. For instance, in the first case, the solution of the affine Bernstein problem gives that the only global definite affine

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maximal surface is the elliptic paraboloid, see [12, 23, 24]. However, in the second case, we have many ruled affine maximal surfaces, with complete flat affine metric, see [14]. Also, in [17] we found complete non flat examples using a conformal representation for indefinite improper affine spheres.

In fact, since the affine conormal of an affine maximal surface is harmonic, one has different holomorphic and split-holomorphic representations with many analytic and geometric applications, see [1, 3, 5, 15, 17, 18]. In particular, one can solve the associated geometric Cauchy problem in order to construct interesting examples and study their properties.

In general, the geometric Cauchy problem for a class of surfaces immersed in a 3-manifold M is to find all surfaces of this class which contain a given curve in M and with the tangent plane distribution prescribed along this curve.

This is a generalization of the classical *Björling* problem for the class of minimal surfaces in $\mathbb{R}^3$, see [8, 9, 20], which has been extended to different families of surfaces, such as constant mean curvature surfaces in $\mathbb{R}^3$ and $\mathbb{H}^3$ in [6, 10], maximal surfaces in $\mathbb{L}^3$ in [4, 7] or flat surfaces in $\mathbb{S}^3$ and $\mathbb{H}^3$ in [2, 11].

Here, we consider the geometric Cauchy problem for the class of indefinite affine maximal surfaces in $\mathbb{R}^3$. Again, if we compare with the definite case in [3], we have some important differences, since their affine conormal is a Lorentzian harmonic map and its coordinates satisfy the 2-dimensional wave equation. Hence, they are not necessarily analytic and the uniqueness can fail at characteristic (asymptotic) directions.

So, we organize the work as follows. In Section 2 we briefly review the theory of indefinite affine maximal surfaces and remind the Blaschke's representation for this class of surfaces. In Section 3 we remember the Blaschke's representation for the family of ruled affine maximal surfaces and remark some interesting facts about the geometric Cauchy problem for this subclass.

Section 4 is devoted to solve the geometric Cauchy problem for the class of indefinite affine maximal surfaces when the initial data are non-characteristic. In Section 5 we extend our study to the characteristic case and show that uniqueness of solution can fail at characteristic directions. Finally, in Section 6 we obtain some consequences about geodesics and symmetries.

2 Blaschke's representation

Consider $\psi : \Sigma \longrightarrow \mathbb{R}^3$ an indefinite affine maximal surface, that is, an immersion with Lorentzian affine metric h and vanishing affine mean curvature, $H = 0$.

Then, see [12], up to an equiaffine transformation, ψ can be locally seen as the graph of a solution $f(x, y)$ of (1.2).

In this case, the affine conormal of ψ is given by

$$N = \omega^{1/3} (\psi_x \times \psi_y) = \omega^{1/3} (-f_x, -f_y, 1), \tag{2.1}$$

where $\times$ denote the cross product in $\mathbb{R}^3$ and the affine normal can be written as

$$\xi = \varphi_y \, \psi_x - \varphi_x \, \psi_y + \omega^{-1/3}(0, 0, 1) \tag{2.2}$$

with

$$\varphi_x = \frac{1}{3}\left(f_{xy}\omega_x - f_{xx}\omega_y\right), \qquad \varphi_y = \frac{1}{3}\left(f_{yy}\omega_x - f_{xy}\omega_y\right).$$

Note that $\varphi_{xy} = \varphi_{yx}$ is equivalent to (1.2) and $H = 0$.

In general, see [19], if $\langle , \rangle$ is the standard inner product, then the affine metric

$$h = -\langle dN, d\psi \rangle = \omega^{1/3}\left(f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2\right), \tag{2.3}$$

the affine conormal N and the affine normal ξ are determined by the conditions

$$\langle N, \xi \rangle = 1, \qquad \langle N, d\psi \rangle = 0 = \langle N, d\xi \rangle \tag{2.4}$$

and

$$\sqrt{|det(h)|} = [\psi_x, \psi_y, \xi] = -[N_x, N_y, N], \tag{2.5}$$

that is, the volume element of h coincides with the determinant $[.,.,\xi]$.

Moreover, from the above expressions, one can obtain

$$\Delta_h N = -2HN, \qquad \Delta_h \psi = 2\xi,$$

where Δ_h is the Laplace-Beltrami operator associated to h.

Actually, see [5, 19], if we take asymptotic parameters (u, v) for h, then from (2.3), (2.4) and (2.5) we have

$$h = 2\rho \; du \; dv, \qquad \rho = \langle N, \psi_{uv} \rangle = [\psi_u, \psi_v, \xi] = -[N, N_u, N_v] \neq 0 \tag{2.6}$$

and

$$\xi = \frac{1}{\rho} N_v \times N_u, \qquad N = \frac{1}{\rho}\psi_u \times \psi_v.$$

Also, we get

$$\psi_u = N \times N_u, \qquad \psi_v = N_v \times N, \qquad N_u = \psi_u \times \xi, \qquad N_v = \xi \times \psi_v \tag{2.7}$$

and

$$\psi_{uv} = \rho\xi, \qquad N_{uv} = -\rho H N.$$

Hence, if $H = 0$, then there exist two regular curves $a(u)$ and $b(v)$ in $\mathbb{R}^3$ such that

$$N(u, v) = a(u) + b(v), \qquad \xi(u, v) = \frac{-1}{\rho(u, v)} a'(u) \times b'(v), \tag{2.8}$$

$$2da = dN - \xi \times d\psi, \qquad 2db = dN + \xi \times d\psi \tag{2.9}$$

and

$$\rho(u, v) = -[a(u) + b(v), a'(u), b'(v)] \neq 0. \tag{2.10}$$

Thus, from (2.7) and (2.8), Blaschke recovered any indefinite affine maximal surface with the Lelieuvre formula

$$\psi = \int (a + b) \times (da - db), \tag{2.11}$$

for any curves $a(u)$ and $b(v)$ satisfying (2.10).

Conversely, from (2.10) and (2.11), one has that the map

$$\widetilde{N} = \frac{1}{\rho}\psi_u \times \psi_v = a + b$$

is harmonic with respect to the metric

$$-\langle d\widetilde{N}, d\psi\rangle = 2\rho\ du\ dv,$$

with $\rho = \langle \widetilde{N}, \psi_{uv}\rangle$. Thus, from (2.4) and (2.6), ψ is an indefinite affine maximal surface with affine conormal $\widetilde{N}$ and affine normal

$$\widetilde{\xi} = \frac{1}{\rho}\psi_{uv} = \frac{-1}{\rho}a' \times b' = \frac{-1}{\rho}\widetilde{N}_u \times \widetilde{N}_v.$$

Remark 1. From (2.8), the affine conormal $\eta(u) = N(u, v_0)$ along the asymptotic curve $\beta(u) = \psi(u, v_0)$, determines the curve $a(u)$, but not the curve $b(v)$. So, in the characteristic case,

$$0 = h\left(\beta'(u), \beta'(u)\right) = -\langle \eta'(u), \beta'(u)\rangle,$$

there exist many affine maximal surfaces containing the curve β, with a prescribed affine conormal η along β.

3 Ruled examples

As application of the above representation, we remember, from [5, 14], that locally the family of ruled affine maximal surfaces is affinely equivalent to the family $M(\delta, \gamma)$, with immersion

$$\psi(u, v) = (u, v\ f_1(u) + g_1(u), v\ f_2(u) + g_2(u)) \tag{3.1}$$

and flat indefinite affine metric

$$h = 2\ du\ dv.$$

The functions f_j and g_j, $j = 1, 2$, satisfy

$$f_j'' = -\delta(u) f_j, \qquad g_j'' = \gamma(u) f_j, \qquad f_1 f_2' - f_1' f_2 = 1, \tag{3.2}$$

for some regular functions $\delta,\ \gamma : I \longrightarrow \mathbb{R}$.

In this family, the affine normal is

$$\xi = \psi_{uv} = \left(0, f_1', f_2'\right) \tag{3.3}$$

and the affine conormal

$$N = \psi_u \times \psi_v = \left(-v + g_1' f_2 - g_2' f_1,\ -f_2,\ f_1\right) = a(u) + b(v) \tag{3.4}$$

is given by the regular curves

$$a(u) = \left(g_1' f_2 - g_2' f_1,\ -f_2,\ f_1\right), \qquad b(v) = -(v, 0, 0).$$

Thus, from (3.2), we check the condition (2.10) since

$$-[N, N_u, N_v] = -[a(u) + b(v), a'(u), b'(v)] = f_1 f_2' - f_1' f_2 = 1.$$

Remark 2. *From (3.1), (3.3) and (3.4), if we take the curves*

$$\begin{array}{rcl} \beta(s) & = & \psi(s,s) = (s, s\, f_1(s) + g_1(s), s\, f_2(s) + g_2(s))\,, \\ Y(s) & = & \xi(s,s) = (0, f_1', f_2')\,, \\ \eta(s) & = & N(s,s) = (-s + g_1' f_2 - g_2' f_1,\ -f_2,\ f_1)\,. \end{array}$$

Then, from (3.2), we get

$$\delta\eta = Y' \times \beta', \qquad \delta = [Y', \beta', Y]$$

and ψ is in the family $M(0,\gamma)$ of the ruled improper affine spheres or the pair $\{\beta, Y\}$ determines $\eta(s)$ when $\delta(s) \neq 0$.

In both cases, see (2.9), the curves $\widetilde{a}(s)$ and $\widetilde{b}(s)$ given by

$$\begin{array}{rcl} 2\,\widetilde{a}' & = & \eta' - Y \times \beta' = 2\,(g_1' f_2' - g_2' f_1', -f_2', f_1')\,, \\ 2\,\widetilde{b}' & = & \eta' + Y \times \beta' = (-2, 0, 0) \end{array}$$

determine the affine conormal (3.4) and the ruled affine maximal surface (3.1), with $a(u) = \widetilde{a}(u)$ and $b(v) = \widetilde{b}(v)$.

However, see Remark 1, the above does not work if we take an asymptotic curve.

4 The non-characteristic Case

We use the Blaschke's representation in the study of the geometric Cauchy problem for the class of indefinite affine maximal surfaces. We want to find the surfaces of this class which contain a regular curve $\beta : I \longrightarrow \mathbb{R}^3$ with prescribed affine normal $Y : I \longrightarrow \mathbb{R}^3$ and affine conormal $\eta : I \longrightarrow \mathbb{R}^3$ along it.

Of course, from the above remarks, see also [15], the situation is different of the definite case studied in [3, 5], specially when $\langle d\beta, d\eta\rangle$ vanishes.

Hence, see (2.3) and (2.4), we will say that $\{\eta, Y\}$ is a non-characteristic admissible pair along β if we have

$$\left.\begin{array}{rcl} 0 & = & \langle \beta'(s), \eta(s)\rangle, \\ 1 & = & \langle Y(s), \eta(s)\rangle, \\ 0 & = & \langle Y'(s), \eta(s)\rangle \end{array}\right\} \tag{4.1}$$

and

$$\lambda(s) = \langle \beta''(s), \eta(s)\rangle = -\langle \beta'(s), \eta'(s)\rangle \neq 0, \qquad \forall s \in I. \tag{4.2}$$

Note, from (4.1), that η is determined by β' and Y, except when $[\beta', Y, Y'] = 0$.

Moreover, from (4.2), we can obtain a solution ψ, such that β is never tangent to its asymptotic (also known as characteristic) curves. Actually, the solution is unique around $\beta(I)$, that is, any two solutions agree on an open set containing $\beta(I)$.

Theorem 4.1. *Let $\{\eta, Y\}$ be a non-characteristic admissible pair of regular curves along $\beta : I \longrightarrow \mathbb{R}^3$. Then, in a neighborhood of $\beta(I)$, there exists a unique indefinite affine maximal surface ψ containing $\beta(I)$ and such that the affine conormal and the affine normal along β are η and Y, respectively.*

Proof. For the uniqueness, we consider a solution ψ of the above problem. Then, from (4.1) and (4.2), the curves a and b given by (2.8) and (2.9) satisfy

$$4[a+b,da,db] = 2[\eta, d\eta, Y\times d\beta] = 2\langle d\eta, d\beta\rangle \neq 0,$$

along β. Hence, see (2.6) and (2.10),

$$u = \int |da|\,, \qquad v = \int |db|$$

are asymptotic parameters of ψ around $\beta(I)$, where, by the inverse function theorem, we have

$$\beta(s) = \psi(u(s),v(s)), \qquad \eta(s) = N(u(s),v(s)), \qquad Y(s) = \xi(u(s),v(s))$$

and

$$2\rho(u(s),v(s))\; u'(s)\; v'(s) = -\langle \beta'(s), \eta'(s)\rangle. \tag{4.3}$$

In this case, from (4.2) and (4.3), $u(s)$ and $v(s)$ are diffeomorphisms onto their images and we can take the change of asymptotic parameters $(u(\widetilde{u}), v(\widetilde{v}))$.

Thus, $\widetilde{\psi}: I\times I \longrightarrow \mathbb{R}^3$ defined by

$$\widetilde{\psi}(\widetilde{u},\widetilde{v}) = \psi(u(\widetilde{u}), v(\widetilde{v}))$$

is a solution, with

$$\beta(s) = \widetilde{\psi}(s,s), \qquad \eta(s) = \widetilde{N}(s,s), \qquad Y(s) = \widetilde{\xi}(s,s).$$

Moreover, from (2.9) and (2.11), if we take the curves $\widetilde{a}(s)$ and $\widetilde{b}(s)$ given by

$$2\,\widetilde{a}' = \eta' - Y\times\beta', \qquad 2\,\widetilde{b}' = \eta' + Y\times\beta', \tag{4.4}$$

then

$$\widetilde{N}(\widetilde{u},\widetilde{v}) = \widetilde{a}(\widetilde{u}) + \widetilde{b}(\widetilde{v})$$

and $\widetilde{\psi}$ is uniquely determined by $\{\beta,\eta,Y\}$ around $\beta(I)$.

Also, up to the inverse change of asymptotic parameters, $\psi(u,v) = \widetilde{\psi}(\widetilde{u}(u), \widetilde{v}(v))$ is uniquely determined by $\{\beta,\eta,Y\}$ around $\beta(I)$.

For the existence, we consider the above curves $\widetilde{a}(s)$ and $\widetilde{b}(s)$. Now, from (4.1), (4.2) and (4.4), we have

$$4[\widetilde{a}(s)+\widetilde{b}(s), \widetilde{a}'(s), \widetilde{b}'(s)] = 2[\eta(s), \eta'(s), Y(s)\times\beta'(s)] = 2\langle\beta'(s), \eta'(s)\rangle \neq 0.$$

Thus, for any diffeomorphisms $u(s)$ and $v(s)$, with inverses $s(u)$ and $s(v)$, the regular curves $a(u)$ and $b(v)$ given by

$$a(u) = \widetilde{a}(s(u)), \qquad b(v) = \widetilde{b}(s(v)) \tag{4.5}$$

satisfy (2.10), since

$$[a(u)+b(v), a'(u), b'(v)] = [\widetilde{a}(s)+\widetilde{b}(s), \widetilde{a}'(s), \widetilde{b}'(s)]\; s'(u)\; s'(v) \neq 0.$$

So, from (2.8) and (2.11), they provide an indefinite affine maximal surface

$$\psi : u(I) \times v(I) \longrightarrow \mathbb{R}^3,$$

containing $\beta(I)$ with affine conormal $N = a + b = \eta$ and affine normal Y along β. In fact,

$$d\psi = (a+b) \times (da - db) = -\left(\eta \times (Y \times d\beta)\right) = d\beta$$

and

$$\xi = \frac{-1}{\rho} a' \times b' = \frac{2}{\langle \beta', \eta' \rangle} \widetilde{a}' \times \widetilde{b}' = \frac{1}{\langle \beta', \eta' \rangle} \eta' \times \left(Y \times \beta'\right) = Y.$$

□

As first consequence, we have a similar result for the associated PDE.

Theorem 4.2. *There exists a unique solution $f(x, y)$ to the Cauchy Problem for the equation of the indefinite affine maximal surfaces*

$$\begin{cases} f_{yy}\omega_{xx} - 2f_{xy}\omega_{xy} + f_{xx}\omega_{yy} = 0, \qquad \omega = \left|f_{xx}f_{yy} - f_{xy}^2\right|^{-3/4}, \\ f(x,0) = A(x), \qquad A''(x) \neq 0, \\ f_y(x,0) = B(x), \\ f_{yy}(x,0) = C(x), \qquad C(x)A''(x) - B'(x)^2 < 0, \\ f_{yyy}(x,0) = D(x), \end{cases} \tag{4.6}$$

where A, B, C, D are regular functions defined on an interval I and f is defined on a planar domain Ω containing $I \times \{0\}$.

Proof. The result follows from (2.1), (2.2) and Theorem 4.1 with

$$\begin{aligned} \beta(s) &= (s, 0, A(s)), \\ \eta(s) &= \left(B'(s)^2 - C(s)A''(s)\right)^{-1/4}\left(-A'(s), -B(s), 1\right), \\ Y(s) &= \frac{-1}{4}\left(B'^2 - CA''\right)^{-7/4}\big(B'(DA'' + 3CB'') - 2B'^2C' - C(C'A'' + CA'''), \\ &\quad B'(3C'A'' + CA''') - 2B'^2B'' - A''(DA'' + CB''), \\ &\quad +4B'^4 - 2B'^2(A'C' + 4CA'' + BB'') - A''((-4C^2 + BD)A'' + BCB'') \\ &\quad - CA'(C'A'' + CA''') + B'(A'(DA'' + 3CB'') + B(3C'A'' + CA'''))\big)(s). \end{aligned}$$

□

In the following consequences, we use that generically the affine conormal is determined by the curve and the affine normal along it.

Corollary 4.3. *Let $\beta, Y : I \longrightarrow \mathbb{R}^3$ be two regular curves satisfying*

$$[\, Y', \beta', Y \,][\, Y', \beta', \beta'' \,] \neq 0, \qquad on \quad I. \tag{4.7}$$

Then there exists a unique indefinite affine maximal surface ψ containing the curve $\beta(I)$ and such that its affine normal along β is Y.

Proof. From (4.1), (4.2) and the condition (4.7), there exists a unique

$$\eta = \frac{Y' \times \beta'}{[\ Y', \beta', Y\ ]}, \tag{4.8}$$

such that the pair $\{\eta, Y\}$ is a non-characteristic admissible pair of regular curves along β. Then, the result follows from Theorem 4.1. □

Corollary 4.4. *Let $\beta, Y : I \longrightarrow \mathbb{R}^3$ be two regular curves satisfying*

$$[\ Y, \beta', \beta''\ ] \neq 0, \qquad Y' \times \beta' = 0, \qquad \textit{on} \quad I. \tag{4.9}$$

Then, for a given regular function $\lambda : I \longrightarrow \mathbb{R} - \{0\}$, there exists a unique indefinite affine maximal surface ψ containing the curve $\beta(I)$, such that its affine normal along β is Y and $h(\beta', \beta') = \lambda$.

Proof. Again, from (4.1), (4.2) and the condition (4.9), there exists a unique

$$\eta = \frac{(-\beta'' + \lambda Y) \times \beta'}{[\ \beta', \beta'', Y\ ]}, \tag{4.10}$$

such that the pair $\{\eta, Y\}$ is a non-characteristic admissible pair of regular curves along β. □

Remark 3. *In particular, if Y is constant in Corollary 4.4, then ψ is the unique indefinite improper affine sphere associated to the above $\{\eta, Y\}$ along β and we have the Theorem 4.1 of [17].*

Remark 4. *If $Y' \times \beta' = 0$, $[\ Y, \beta', \beta''\ ] = 0$ and there is an indefinite affine maximal surface ψ containing $\beta(I)$ with affine normal Y (and affine conormal η) along β, then*

$$\beta'' = \nu\beta' + \lambda Y$$

for ν, λ regular functions, $\lambda \neq 0$, and there exist a family of indefinite affine maximal surfaces containing $\beta(I)$ with affine normal Y (and affine conormal $\eta + \mu Y \times \beta'$) along β, for any regular function μ.

5 The characteristic Case

Again from Remark 1, we know that a pair $\{\eta, Y\}$ along β generates many indefinite affine maximal surfaces ψ, when $\langle d\beta, d\eta\rangle$ vanishes identically. Hence, we will assume that (4.1) holds and that $\langle \beta'(s), \eta'(s)\rangle$ only vanishes at isolated points, $s_0 \in I$.

In this case, from [2, 15] and the proof of Theorem 4.1, the key point is the relation

$$\langle \beta'(s), \eta'(s)\rangle = -2\rho(u(s), v(s))\ u'(s)\ v'(s)$$

which suggests the following definition.

We say that $s_0 \in I$ is a *characteristic point with sign* if

$$\langle \beta'(s_0), \eta'(s_0)\rangle = 0$$

and $\langle \beta'(s), \eta'(s) \rangle$ does not change sign around s_0.

Equivalently, $u'(s_0)v'(s_0) = 0$ and both $u'(s)$ and $v'(s)$ do not change sign around s_0, since $(u'(s_0), v'(s_0)) \neq (0,0)$ by the regularity of $\beta(s) = \psi(u(s), v(s))$.

In particular, $u(s)$ and $v(s)$ are diffeomorphisms onto their images and, similarly to [2, 15] and Theorem 4.1, we can obtain the following results of uniqueness and existence.

Theorem 5.1. *Let $\{\eta, Y\}$ be an admissible pair of regular curves along β such that all their characteristic points are isolated. Then, two solutions to the corresponding geometric Cauchy problem agree on a domain which contains $\beta(I)$ except its characteristic points without sign.*

Theorem 5.2. *Let $\{\eta, Y\}$ be an admissible pair of regular curves along β such that all their characteristic points s_0 are isolated. If the traces of the curves $\widetilde{a}$ and $\widetilde{b}$ given by (4.4) are regular, then $\{\beta, \eta, Y\}$ generates an indefinite affine maximal surface ψ if, and only if*

i) The limit

$$\lim_{s \to s_0} \frac{\langle \beta'(s), \eta'(s) \rangle}{u'(s)\, v'(s)}$$

exists and is non-zero, when s_0 has sign, for some parametrizations $u(s)$ of $\widetilde{a}(I)$ and $v(s)$ of $\widetilde{b}(I)$.

ii) The curve $\widetilde{a}$ or $\widetilde{b}$ that is singular at s_0 has the same image on $]s_0 - \varepsilon, s_0[$ and $]s_0, s_0 + \varepsilon[$, for some $\varepsilon > 0$, when $\langle \beta'(s), \eta'(s) \rangle$ changes sign around s_0.

6 Geodesics and symmetry

Similarly to [3, 17], the above results let us characterize when curves in $\mathbb{R}^3$ can be geodesics of an indefinite affine maximal surface.

Theorem 6.1. *Let $\psi : \Sigma \longrightarrow \mathbb{R}^3$ be an indefinite affine maximal surface and $\alpha : I \longrightarrow \Sigma$ a regular curve. If $\eta = N \circ \alpha$ and $Y = \xi \circ \alpha$, then $\beta = \psi \circ \alpha$ is a geodesic if and only if*

$$\langle \beta'', \eta \rangle = m, \qquad [\beta', \beta'', Y] = [\eta, \eta', \eta''] \qquad \text{on } I, \tag{6.1}$$

for a constant m.

Proof. If we consider $\beta(s) = \psi(u(s), v(s))$, then the geodesic equations for the affine metric (2.6) are

$$\rho u''(s) + \rho_u u'(s)^2 = 0 = \rho v''(s) + \rho_v v'(s)^2,$$

that is, from (2.10) and (4.5), we have

$$[\widetilde{a} + \widetilde{b},\ \widetilde{a}'',\ \widetilde{b}'] = 0 = [\widetilde{a} + \widetilde{b},\ \widetilde{a}',\ \widetilde{b}'']$$

and we obtain (6.1), from (4.1) and (4.4). □

Actually, extending [15], a geodesic is an asymptotic straight line if $m = 0$ and ψ is ruled in a neighborhood or a convex curve if $m \neq 0$. In this case, see [16, 21], we can assume that $\beta(s) = (\beta_1(s), \beta_2(s), 0)$ is parametrized by its affine arc length, that is,

$$[\beta', \beta'', (0,0,1)] = \beta_1'\beta_2'' - \beta_1''\beta_2' = 1, \qquad \forall s \in I, \tag{6.2}$$

which implies that

$$\beta''' + \kappa\beta' = 0,$$

where $\kappa = [\beta'', \beta''', (0,0,1)]$ is the affine curvature of β.

Thus, from (4.1), (6.1) and (6.2), we have

$$\eta = (-m\beta_2', m\beta_1', \mu)$$

for a regular function μ,

$$Y_3 = [\beta', \beta'', Y] = m^2(\mu\kappa + \mu'') \tag{6.3}$$

and

$$mY_1 = \mu' Y_3 \beta_1' + (1 - \mu Y_3)\beta_1'', \qquad mY_2 = \mu' Y_3 \beta_2' + (1 - \mu Y_3)\beta_2''. \tag{6.4}$$

As consequence, β is geodesic of a family $M(\kappa, m, \mu)$ of indefinite affine maximal surfaces. (Note that κ determines β).

Moreover, this family contains an indefinite improper affine sphere if and only if the affine normal Y along β is constant. Then, from (6.2), (6.3) and (6.4) we get

$$\mu Y_3 = 1 - mY_2\beta_1' + mY_1\beta_2'$$

and

$$Y_3^2 = m^2(\mu Y_3 \kappa + \mu'' Y_3) = m^2\kappa.$$

Corollary 6.2. *The family $M(\kappa, m, \mu)$ contains an indefinite improper affine sphere if and only if κ is a non negative constant.*

In particular, the curve $\beta(s) = (cos(s), sin(s), 0)$ in the plane $\Pi \equiv z = 0$ is geodesic of the family $M(1, m, \mu)$, which contains the indefinite improper affine sphere $M(1,1,1)$, see Figure 1 and [15], with

$$\eta(s) = (-cos(s), -sin(s), 1) \notin \Pi, \qquad Y(s) = (0,0,1) \notin \Pi$$

and given by

$$\begin{aligned}
\psi_1(u,v) &= \frac{1}{2}\left(cos(u) + cos(v) + sin(u) - sin(v)\right), \\
\psi_2(u,v) &= \frac{1}{2}\left(cos(v) - cos(u) + sin(u) + sin(v)\right), \\
\psi_3(u,v) &= \frac{1}{2}\left(u - v + cos(u - v)\right).
\end{aligned}$$

We remark that Π is not a plane of symmetry of the revolution surface $M(1,1,1)$. However, we have this symmetry for the indefinite affine maximal surface $M(1,1,0)$, see Figure 2, with

$$\eta(s) = (-cos(s), -sin(s), 0) \in \Pi, \qquad Y(s) = (-cos(s), -sin(s), 0) \in \Pi$$

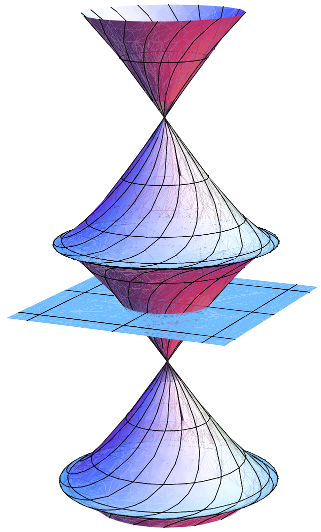

Figure 1: M(1, 1, 1) and Π

and given by

$$\psi_1(u,v) = \frac{1}{4}\left((u-v)(sin(u)-sin(v))+2cos(u)+2cos(v)\right),$$
$$\psi_2(u,v) = \frac{1}{4}\left((u-v)(cos(v)-cos(u))+2sin(u)+2sin(v)\right),$$
$$\psi_3(u,v) = \frac{1}{4}\left(u-v+sin(u-v)\right).$$

Note that the symmetries in the above examples are consequence of the extension of Theorem 5.1 in [17] that follows from Theorems 4.1 and 5.1,

Theorem 6.3. *Any symmetry of an admissible pair, such that all their characteristic points are isolated and have sign, induces the corresponding symmetry of the indefinite affine maximal surface generated by it.*

As consequence, we also have the symmetry respect to the plane $z=0$ for the indefinite affine maximal surface $M(-1,1,0)$, see Figure 3, with $\beta(s)=(cosh(s),-sinh(s),0)$,

$$\eta(s)=(cosh(s),sinh(s),0), \qquad Y(s)=(cosh(s),-sinh(s),0)$$

and given by

$$\psi_1(u,v) = \frac{1}{4}\left((u-v)(sinh(v)-sinh(u))+2cosh(u)+2cosh(v)\right),$$
$$\psi_2(u,v) = \frac{1}{4}\left((u-v)(cosh(u)-cosh(v))-2sinh(u)-2sinh(v)\right),$$
$$\psi_3(u,v) = \frac{1}{4}\left(u-v+sinh(u-v)\right).$$

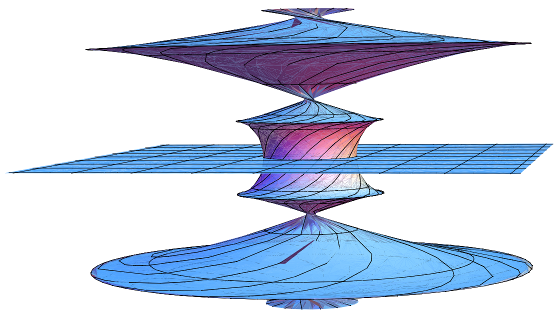

Figure 2: M(1, 1, 0) and Π

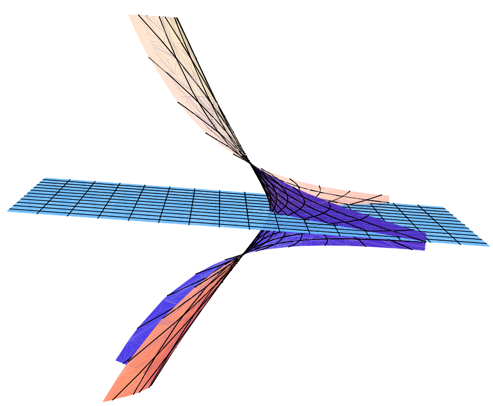

Figure 3: M(-1, 1, 0) and Π

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