J.A. Gálvez, A. Martínez, F. Milán[1]

The geometry of convex affine maximal graphs

Abstract. Locally strongly convex surfaces which are extremal for the first variation of the equiaffine area integral have been investigated on several occasions. Here we are interested in the description on their behaviour at infinity. We consider an affine maximal annular end which is a graph of vertical flux and give a detailed representation of it when its affine conormal map has a good behaviour at infinity.

1 Introduction

In the beginning of the century Blaschke, see [B], studied the first variation of the equiaffine area integral. He found that the Euler-Lagrange equation is of fourth order and nonlinear, but it is equivalent to the vanishing of the affine mean curvature.

When Calabi discovered in 1982 that for extremal locally strongly convex surfaces, the second variation is negative, he proposed to call this class of surfaces **affine maximal** surfaces.

Although the elliptic paraboloid is still the only known example which is Euclidean complete, there are many properly embedded affine maximal annular ends. In fact, the improper affine spheres are an important class of affine maximal surfaces that, up an equiaffine transformation, are, locally, graphs of solutions of the following Monge-Ampère equation

(1) $$Det(\nabla^2 f) = 1, \qquad on \quad \Omega.$$

In [FMM1] and [FMM2], was proved that the graph of a solution of (1) on $\Omega = \{\zeta \in \mathbb{C} \,|\, |\zeta| > 1\}$, has always a properly embedded affine maximal end whose behaviour at infinity depends on five real numbers. Affine maximal surfaces of rotation, which were

[1]J.A. Gálvez, A. Martínez and F. Milán: Departamento de Geometría y Topología.
Facultad de Ciencias. Universidad de Granada. 18071 GRANADA. SPAIN.
(e-MAIL:jagalvez@goliat.ugr.es; amartine@goliat.ugr.es; milan@goliat.ugr.es)
Mathematics Subject Classification (1991): 53A15
Research partially supported by DGICYT Grant No. PB97-0785.

described in [K], have also Euclidean complete annular ends.

Our aim in this paper is to study affine maximal graphs in a neighborhood of its end. We shall see that if $x_f \equiv (x_1, x_2, f(x_1, x_2))$ with f proper is a affine maximal annular end of vertical flux and such that its affine conormal map is, locally, a vertical graph, then the end of x_f resembles to the end of an affine maximal surface of revolution.

The paper is organized as follows. In §2 we give a (brief) description about affine maximal surfaces, their fundamental equations and properties. The interested reader may consult [C3] and [LSZ] for a deep discussion.

In §3 we introduce the concept of regular-balanced ends and see that a well-defined improper affine sphere at infinity can be associated with them. After to give some examples of affine maximal annular ends we prove the elliptic paraboloid is the only Euclidean complete affine maximal surface with a regular-balanced end (Theorem 1) and obtain a general representation of regular-balanced ends (Theorem 2).

2 Affine maximal surfaces

Let S be a smooth surface and $x : S \longrightarrow \mathbb{R}^3$ be an immersion with **positive** Gauss curvature K. Since S is orientable (the mean curvature vector field orientates it) we have an unit normal vector field $\vec{n}$ on S such that the Blaschke metric, g,

$$g_p(v, w) = -K^{-1/4} < d\vec{n}_p(v), w >, \qquad p \in S, \quad v, w \in T_pS,$$

is a Riemannian metric, where $<,>$ denotes the usual inner product in $\mathbb{R}^3$. Since g is invariant under the equiaffine transformation group, we also call it **equiaffine metric**. From now on, S will be considered as a Riemann surface with the **conformal structure** induced by g.

By considering the first variation of the equiaffine area integral of x one can find (see [B], [C2], [C3]) that the Euler-Lagrange equation for this variational problem is equivalent to the following system of differential equations

$$\triangle_g(K^{-1/4}\vec{n}) = 0, \tag{2}$$

where $\triangle_g$ is the Laplace-Beltrami operator associated to g.

The immersion $U = K^{-1/4}\vec{n} : S \longrightarrow \mathbb{R}^3$ is called the **affine conormal map** of x.

From (2), $x(S)$ is affine maximal if and only if U is a harmonic immersion. Associated to the affine conormal map U we also have the conjugate affine conormal map U^* which is well-defined only on some covering $\widetilde{S}$ of S. The relation $dU^* = -dU \circ Rot_{\pi/2}$ shows that U and U^* are locally immersions and $Z = U + iU^*$ is holomorphic (where

$Rot_{\pi/2}$ denotes the operator on vector fields which rotates $\pi/2$ each tangent plane in the positive direction).

The equiaffine invariant vector field,

$$\xi = \frac{1}{2} \triangle_g x, \tag{3}$$

is always transversal to the immersion and it is called the **affine normal** vector field of x. The pair $\{U, \xi\}$ is a relative normalization invariant under equiaffine transformations (see [LSZ]), it is called the **equiaffine normalization** of x.

Choose $\zeta = u + iv$ a conformal parameter such that $g = E|d\zeta|^2$. Then a straight computation gives:

$$E^2 = Det(x_u, x_v, x_{uu}) = Det(x_u, x_v, x_{vv}), \qquad Det(x_u, x_v, x_{uv}) = 0, \tag{4}$$

$$E = Det(U_u, U_v, U) = Det(x_u, x_v, \xi), \tag{5}$$

$$\begin{array}{lll} < U, x_u >= 0, & < U_u, x_u >= -E, & < U_v, x_u >= 0, \\ < U, x_v >= 0, & < U_u, x_v >= 0, & < U_v, x_v >= -E, \\ < U, \xi >= 1, & < U_u, \xi >= 0, & < U_v, \xi >= 0, \end{array} \tag{6}$$

and

$$dx = -U \wedge dU^*, \qquad dU^* = \xi \wedge dx, \tag{7}$$

where $\wedge$ denotes the cross product, $Det(.,.,.)$ is the usual determinant form, $(.)_u$ and $(.)_v$ are, respectively, partial derivatives respect to u and v.

From (2), (6) and (7), if $x(S)$ is affine maximal, then $\xi \wedge dx$ is a closed one-form on S and for closed curves γ on S,

$$\int_\gamma \xi \wedge ds$$

is an homology-invariant vector.

Definition 1 The **flux** along γ is defined as the vector quantity

$$Flux([\gamma]) = \int_\gamma \xi \wedge ds. \tag{8}$$

From (7), U^* is well-defined if and only if $Flux([\gamma]) = 0$ for any closed curve γ.

When the affine normal ξ is a constant vector field the immersion is called **improper affine sphere**. From (5), (6) and (7) x is an improper affine sphere if and only if it is affine maximal and $U(S)$ lies on a plane. It is clear from (8) that improper affine spheres have vanishing flux.

3 Regular-balanced affine maximal graphs

We will follow the same notation that in §2.

Let $x : S \longrightarrow \mathbb{R}^3$ be a locally strongly convex affine maximal immersion and $\Sigma \subseteq S$ such that $x(\Sigma)$ is an annular end of $x(S)$ with compact boundary $x(\gamma)$.

Definition 2 $x(\Sigma)$ is called a **regular-balanced** end (in short, RB-end) if there exists a plane Π in $\mathbb{R}^3$ with unit normal vector A such that the immersions x and U satisfy:

(R1) $x(\Sigma)$ is a graph on a domain in Π and $< x, A >: \Sigma \longrightarrow \mathbb{R}$ is a proper map. (Regularity condition).

(R2) The orthogonal projection of $U(\Sigma)$ on Π is a local diffeomorphism. (Regularity condition)

(B) $A \wedge F = 0$, where F is the flux of $x(\Sigma)$ along γ. (Balanced condition).

Remark 1 If $x : S \longrightarrow \mathbb{R}^3$ is an Euclidean complete affine maximal immersion, then, from Hadamard Theorem (see [W]), the conditions (R1) and (B) are always satisfied.

Proposition 1 *Let $x(\Sigma)$ be a RB-end of x. Then Σ is conformally a punctured disk and $Y : \Sigma \longrightarrow \mathbb{R}^3$,*

$$Y = U^* \wedge A + < x, A > A, \tag{9}$$

is a well-defined improper affine sphere with end on the boundary of a convex set in $\mathbb{R}^3$ (that is, Y is regular at infinity in the sense of [FMM1]).

Proof: From (7), (8) and (B), it is clear that Y is well-defined.

Let $\zeta = u + iv$ be a conformal parameter of x such that $g = E|d\zeta|^2$. From (9),

$$Y_u = -U_v \wedge A + < x_u, A > A, \qquad Y_v = U_u \wedge A + < x_v, A > A. \tag{10}$$

Thus, $< Y_u \wedge Y_v, A >= Det(U_u, U_v, A)$ that, from (R2), does not vanish on Σ and Y is an immersion with transversal vector A.

From (5), (6), (7) and (10), we obtain,

$$\begin{aligned} Det(Y_u, Y_v, Y_{uu}) &= Det(Y_u, Y_v, Y_{vv}) = Det(U_u, U_v, A)^2, \\ Det(Y_u, Y_v, Y_{uv}) &= 0. \end{aligned}$$

It is not a restriction to assume $Det(U_u, U_v, A) > 0$. Then, from (4), (5), (6) and the above expressions, we have Y is a locally strongly convex immersion with Blaschke metric,

$$h = Det(U_u, U_v, A)|d\zeta|^2 =< \xi, A > E|d\zeta|^2. \tag{11}$$

Since $Y_{uu} + Y_{vv} =< x_{uu} + x_{vv}, A > A$, from (3), (10) and (11) we conclude that Y is an improper affine sphere with affine normal A.

By using that x is a graph and $< x, A >$ a proper map, the level curves $x(\gamma_c) = x(\Sigma) \cap \{< x, A >= c\}$ must be strictly convex Jordan curves for c large enough. Moreover from (7), (9) and (10), if T is a tangent vector along $x(\gamma_c)$, $< \xi, A > T$ is a tangent vector along $Y(\gamma_c)$. Thus, the end of $Y(\Sigma)$ is also fibred by strictly convex Jordan curves.

Since from (11), the affine metric h of Y and g are conformal metrics and Y is regular at infinity we conclude, see [FMM1], that Σ must be conformally a punctured disk. $\square$

Definition 3 The improper affine sphere Y given by (9) is called the **tangent improper affine sphere at infinity** of $x(\Sigma)$.

3.1 Some examples

Let us consider f a solution of

$$\det(\nabla^2 f) = 1 \qquad on \quad \widetilde{\Omega} = \{\zeta \in \mathbb{C} | \, |\zeta| > 1\}.$$

We also use the notation $\widetilde{\Omega}_R = R\widetilde{\Omega}$. Then the graph x_f of f is an improper affine sphere and its end is a RB-end conformal to $\widetilde{\Omega}$. Moreover, see [FMM2], it can be represented as

$$(12) \qquad x_f \equiv \left(\frac{G + \overline{F}}{2}, \frac{1}{8}|G|^2 - \frac{1}{8}|F|^2 + \frac{1}{4}\Re(GF) - \frac{1}{2}\Re \int F dG \right), \qquad on \quad \widetilde{\Omega}_R,$$

for some $R > 1$, with

$$(13) \qquad G(\zeta) = \zeta, \qquad F(\zeta) = \mu\zeta + \nu + \sum_{n=1}^{\infty} \frac{a_n}{\zeta^n}, \qquad \zeta \in \widetilde{\Omega}_R,$$

where by $\Re$ we denote the real part and μ, ν, $a_n \in \mathbb{C}$, for $n \geq 2$, $a_1 \in \mathbb{R}$ and $|\mu| < 1$.

The affine metric and the affine conormal map of x_f are given, respectively, by

$$(14) \qquad h_f = \frac{1}{4}\left(|dG|^2 - |dF|^2\right),$$

$$(15) \qquad U_f = \left(\frac{\overline{F} - G}{2}, 1 \right).$$

Now, let $a : \widetilde{\Omega}_R \longrightarrow \mathbf{R}$ be a bounded harmonic function and $a^* : \widetilde{\Omega}_R \longrightarrow \mathbf{R}$ be a conjugated harmonic of a (which always exists because a is bounded). Assume that

$Residue[a+ia^*,\infty]=0$. If we consider the following harmonic immersion,

$$N(\zeta)=\left(\frac{\overline{F(\zeta)}-\zeta}{2},a(\zeta)+b\log|\zeta|\right), \tag{16}$$

with $b\in\mathbb{R}$, $b\geq 0$ and $\lim_{|\zeta|\to\infty}a(\zeta)>0$ if $b=0$, then from (16) one has

$$\begin{aligned}4Det(N,N_u,N_v) &= (a(u,v)+b\log(u^2+v^2))(1-|F'|^2)+\\ &\quad (a_u+\frac{2bu}{u^2+v^2})((1+F_{2v})(F_1-u)-(v+F_2)F_{1v})+\\ &\quad (a_v+\frac{2bv}{u^2+v^2})((v+F_2)(F_{1u}-1)-F_{2u}(F_1-u)),\end{aligned}$$

where $F=F_1+iF_2$ and $\zeta=u+iv\in\widetilde{\Omega}_R$. Hence, it is clear, from the above assumptions, that there exists $R_1>R$, such that

$$Det(N,N_u,N_v)>0$$

for all $\zeta=u+iv\in\widetilde{\Omega}_{R_1}$. Consequently, using (7),

$$X(\zeta)=-\int_{\zeta_0}^{\zeta}N\wedge dN^*,\qquad \zeta\in\widetilde{\Omega}_{R_1}, \tag{17}$$

is an affine maximal surface.

Proposition 2 *X is a well-defined affine maximal surface with a RB-end. Moreover,*

$$X(\zeta)=\left(\frac{\overline{F}(\zeta)+\zeta}{2}(a(\zeta)+b\log|\zeta|^2)-b\zeta+2\int_{\zeta_0}^{\zeta}\gamma(w)dw+2\int_{\zeta_0}^{\zeta}\overline{\rho}(w)d\overline{w},\right. \tag{18}$$

$$\left.\frac{1}{8}|\zeta|^2-\frac{1}{8}|F(\zeta)|^2+\frac{1}{4}\Re(\zeta F(\zeta))-\frac{1}{2}\Re(\int_{\zeta_0}^{\zeta}F(w)dw)\right)$$

$\zeta=u+iv\in\widetilde{\Omega}_{R_2}$*, for some* $R_2>R_1$*, and*

$$\gamma(\zeta)=-\frac{\zeta}{2}\frac{\partial a}{\partial\zeta},\qquad \rho(\zeta)=-\frac{F(\zeta)}{2}\frac{\partial a}{\partial\zeta}-\frac{b}{2}\frac{F(\zeta)}{\zeta},$$

being $\frac{\partial}{\partial\zeta}=\frac{1}{2}\left(\frac{\partial}{\partial u}-i\frac{\partial}{\partial v}\right)$.

Proof: Since F is an holomorphic function, from (16) and (17), we obtain that the coordinate functions, (X_1, X_2, X_3), of X satisfy,

$$\frac{\partial}{\partial\zeta}\left(X_1 - \frac{F_1+u}{2}(a+b\log|\zeta|^2)+bu\right) = -\frac{F+\zeta}{2}a_\zeta - \frac{b}{2}\frac{F}{\zeta} = \gamma+\rho.$$

$$\frac{\partial}{\partial\zeta}\left(X_2 + \frac{F_2-v}{2}(a+b\log|\zeta|^2)+bv\right) = -i\frac{F-\zeta}{2}a_\zeta - i\frac{b}{2}\frac{F}{\zeta} = i(-\gamma+\rho).$$

Consequently,

$$X_1 = \frac{F_1+u}{2}(a+b\log|\zeta|^2) - bu + \int(\gamma+\rho)dw + \int\overline{(\gamma+\rho)}d\overline{w},$$

$$X_2 = \frac{v-F_2}{2}(a+b\log|\zeta|^2) - bv + i\int(-\gamma+\rho)dw + i\int\overline{(\gamma-\rho)}d\overline{w},$$

and

$$X_1+iX_2 = \frac{\overline{F}+\zeta}{2}(a+b\log|\zeta|^2) - b\zeta + 2\int\gamma dw + 2\int\overline{\rho}d\overline{w}.$$

A straight computation gives that $X_{1u}X_{2v}-X_{1v}X_{2u}$ is of order like $(a+b\log|\zeta|^2)(-2b+a+b\log|\zeta|^2)$. Thus, when R_3 is large enough, X_1+iX_2 is a local diffeomorphism on $\widetilde{\Omega}_{R_3}$ and also a covering map.
Now, using that

$$\frac{\overline{F}}{2}(a+b\log|\zeta|^2) + 2\int\gamma dw + 2\int\overline{\rho}d\overline{w}$$

is bounded on $\widetilde{\Omega}_{R_3}$ by a constant C and the fact that for $|\zeta| = R_4$, with R_4 large,

$$g(\zeta) = \left(\frac{a+b\log|\zeta|^2}{2} - b\right)\zeta$$

winds around the origin once and $|g(\zeta)| > C$, we conclude that X_1+iX_2 is one-to-one on $\widetilde{\Omega}_{R_4}$ and $X(\widetilde{\Omega}_{R_4})$ must be a graph on $\Pi \equiv x_3 = 0$.
Analogously, from (16) and (17),

$$\frac{\partial}{\partial u}\left(X_3 + \frac{|F|^2}{8} - \frac{u^2}{8}\right) = \frac{1}{4}(uF_{1u} - vF_{2u} - F_1),$$

$$\frac{\partial}{\partial v}\left(X_3 + \frac{|F|^2}{8} - \frac{v^2}{8}\right) = \frac{1}{4}(uF_{1v} - vF_{2v} + F_2),$$

and,

$$X_3 = \frac{1}{8}|\zeta|^2 - \frac{1}{8}|F(\zeta)|^2 + \frac{1}{4}\Re(\zeta F(\zeta)) - \frac{1}{2}\Re(\int F(w)dw).$$

From (6) and having in mind that

$$\lim_{|(u,v)|\to\infty} N_u(u,v) = (-1,0,0), \qquad \lim_{|(u,v)|\to\infty} N_v(u,v) = (0,-1,0), \tag{19}$$

we can choose $R_2 > R_4$ large enough in order to get that $<\xi,(0,0,1)>> 0$, that is (R2) also is satisfied.
Finally from (8) and (16), $F = (0,0,2\pi b)$ and the end of X is a RB-end. □

Remark 2 The tangent improper affine sphere at infinity of (18) is the graph x_f given by (12).

Remark 3 Every end of an elliptic revolution affine maximal surface can be represented as in (18) by taking $a \equiv$ *constant* and $F(\zeta) = c/\zeta$ for some $c \in \mathbb{R}$.

Remark 4 There exists $R > 1$ such that the immersion $x : \widetilde{\Omega}_R \longrightarrow \mathbb{R}^3$ given by

$$x(\zeta) = (4\zeta(1-\log(|\zeta|), 2|\zeta|^2 + 8\Re(\zeta)(1-\log(|\zeta|)) \tag{20}$$

is a well-defined affine maximal vertical graph which has not a RB-end because the condition (B) fails.

3.2 On the affine Bernstein problem

Theorem 1 *Let $x : S \longrightarrow \mathbb{R}^3$ be an Euclidean complete affine maximal surface with a RB-end. Then $x(S)$ is an elliptic paraboloid.*

Proof: From the Euclidean completeness $x(S)$ is the boundary of a unbounded convex set in $\mathbb{R}^3$ and using (R1) we have that it must be a global graph on a convex domain Ω in Π.

Since x has a RB-end $x(\Sigma)$, then, from Proposition 1, Σ is conformally a punctured disk and consequently S must be conformally $\mathbb{C}$. Now, as x is a graph, then one can assume that $< U, A >$ is a positive harmonic function on $\mathbb{C}$. Thus, $< U, A >$ must be constant and x is an improper affine sphere. The result follows from Jörgens Theorem, see [J]. □

3.3 The general case

Theorem 2 *Let $x : S \longrightarrow \mathbb{R}^3$ be a affine maximal surface with a RB-end. Then, there exists a representation of its end as in (18).*

Proof: It is not restriction to assume that $\Pi \equiv x_3 = 0$ and $A = (0,0,1)$.

Let $Y : \Sigma \longrightarrow \mathbb{R}^3$ be the tangent improper affine sphere at infinity of x. From Proposition 1, Σ is conformally equivalent to $\widetilde{\Omega}_R = \{\zeta \in \mathbb{C} | \, |\zeta| > R\}$. Moreover, see [FMM1], there exists a conformal representation of Y as

$$(21) \qquad Y(\zeta) = \left(\frac{\zeta + \overline{F(\zeta)}}{2}, \frac{1}{8}|\zeta|^2 - \frac{1}{8}|F(\zeta|^2 + \frac{1}{4}\Re(\zeta F(\zeta)) - \frac{1}{2}\Re \int F(\zeta) dz \right),$$

with

$$F(\zeta) = \mu\zeta + \nu + \sum_{n=1}^{\infty} \frac{a_n}{\zeta^n}, \qquad \zeta \in \widetilde{\Omega}_R.$$

By using (9) and (21), we can obtain that the affine conormal map of $x : \Sigma \longrightarrow \mathbb{R}^3$ is given by,

$$(22) \qquad U(\zeta) = \left(\frac{\overline{F(\zeta)} - \zeta}{2}, U_3 \right),$$

where $U_3 =< U, A >$ is a positive harmonic function on $\widetilde{\Omega}_R$ because of x is a graph.

So, $H = U_3 - < F, A > \log(|\zeta|^2)$ is the real part of an holomorphic function well-defined on $\widetilde{\Omega}_R$. Since H is harmonic, we have the following Laurent expansion at infinity,

$$0 < U_3(\zeta) =< F, A > \log(|\zeta|^2) + a(\zeta) + \Gamma(\zeta),$$

where $a(\zeta)$ is harmonic and bounded on $\widetilde{\Omega}_R$ and $\Gamma(\zeta)$ is harmonic in the finite plane $\mathbb{C}$. If n is any positive entire number greater than $< F, A >$, it follows that for $|\zeta| > R_1 > R$,

$$1 \leq M \left| \zeta^n \exp^{G(\zeta)} \right|,$$

where M is a suitable constant and $G(\zeta)$ is an entire function with real part $\Gamma(\zeta)$. Thus, the entire function $\zeta^n \exp^{G(\zeta)}$ has not a essential singularity at infinity and it is a polinomial. Hence, $G(\zeta)$ must be constant and from (22), the affine conormal map of the end of x is given as in (16), which concludes the proof. $\square$

Remark 5 From (19) and Theorem 2, we have that if an annular affine maximal end $x(\Sigma)$ is a RB-end, then

I) It is conformally a punctured disk and the affine Gauss map $\frac{\xi}{|\xi|}$ extends to the puncture as the unit vector A.

II) The affine conormal map $U(\Sigma)$ lies in a half-space of $\mathbb{R}^3$.

III) $F \wedge A = 0$, where F is the flux of x.

Conversely one can prove that if $x(\Sigma)$ satisfies I), II) and III), then $x(\Sigma)$ is a RB-end.

References

[A] Ahlfors, A.: Complex Analysis, McGraw-Hill, New York, 1979.

[B] Blaschke, W.: *Vorlesungen über Differentialgeometrie, II.* , Berlin, Springer, 1923.

[C1] Calabi, E.:*Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens*, Mich. Math. J. **5**, 108-126 (1958).

[C2] Calabi, E.:*Hypersurfaces with maximal affinely invariant area*, Amer. J. Math. **104**, 91-126 (1984).

[C3] Calabi, E.:*Convex affine maximal surfaces*, Result in Math. **13**, 209-223 (1988).

[CY] Cheng, S.Y. and Yau, S. T.:*Complete affine hypersurfaces, Part.I. The completeness of affine metrics*, Comm. Pure Appl. Math. **39**, 839-866 (1986).

[FMM1] Ferrer, L., Martínez, A. and Milán, F.:*Symmetry and Uniqueness of Parabolic Affine Spheres*, Math. Ann. **305**, 311-327 (1996).

[FMM2] Ferrer, L., Martínez, A. and Milán, F.:*An extension of a theorem by K. Jörgens and a maximum principle at infinity for parabolic affine spheres*, Math. Z. **230**, 471-486 (1999).

[J] Jörgens, K.:*Über die Lösungen der Differentialgleichung* $rt - s^2 = 1$, Math. Ann. **127**, 130-134 (1954).

[K] Krauter, P.:*Affine minimal hypersurfaces of rotation,* Geometriae Dedicata, **51**, 287-303 (1994).

[LSZ] Li, A.M., Simon U. and Zhao, G.: Global affine differential geometry of hypersurfaces, Walter de Gruyter, Berlin-New York, 1993.

[NS] Nomizu, K. and Sasaki, T.: Affine Differential Geometry, Cambridge Univ. Press, 1994.

[W] Wu, H.H.:*The spherical images of convex hypersurfaces*, J. Differential Geometry, **9**, 279-290 (1974).