Linear Weingarten Surfaces in $\mathbf{R}^3$

J. A. Gálvez* A. Martínez* F. Milán*

Departamento de Geometría y Topología, Universidad de Granada, 18071 Granada, Spain
e-mails: jagalvez@ugr.es, amartine@ugr.es, milan@ugr.es

Abstract. In this paper we study properties of linear Weingarten immersions and graphs related to non-existence problems and behaviour of its curvatures. The main results are obtained giving a harmonic representation of linear Weingarten surfaces and by proving optimal estimates of the height and curvatures that the immersion must satisfy, characterizing the spherical cap as the only ones achieving these bounds.

2000 Mathematics Subject Classification: 53A05

Key words: constant mean curvature, constant Gauss curvature, Weingarten surfaces, height, area.

1 Introduction

Surfaces with either constant mean curvature H, H-surfaces, or constant Gauss curvature K, K-surfaces, in the Euclidean 3-space, $\mathbb{R}^3$, arise as critical points of a functional which involves a linear combination of either the area or the total mean curvature, respectively, with the volume bounded by the immersion.

In 1853, Bonnet remarked that the study of K-surfaces could be as difficult as the study of H-surfaces. This is because any K-surface, $K > 0$, has a parallel surface of constant mean curvature $H = \sqrt{K}/2$ to a distance $1/\sqrt{K}$, which may have singularities. In this paper we shall consider any parallel surface to either a K-surface or a H-surface. In fact, we are going to study linear Weingarten surfaces in $\mathbb{R}^3$, namely, those immersions ψ from a surface S in $\mathbb{R}^3$ such that a linear combination of its mean curvature and Gauss curvature is constant, that is,

$$2\,a\,H + b\,K = c,$$

*Research partially supported by DGICYT Grant No. BFM2001-3318 and Junta de Andalucía CEC: FQM0203.

for some real numbers $a,\ b,\ c$, not all zero. These surfaces are critical points of a natural functional: a linear combination (with constant coefficients depending on $a,\ b$ and c) of the area of $\psi(S)$, the volume bounded by $\psi(S)$, and the total mean curvature of $\psi(S)$. They have been studied when S is closed by Chern, [3], Hopf, [8] and Hartman and Wintner, [6]. In [15], Rosenberg-Earp considered properly immersed (noncompact in general) surfaces. They extended the results of Meeks, [13], and Korevaar-Kusner-Solomon, [11], to surfaces $M(a,c)$ satisfying $aH+K=c$, $a\geq 0$, $c>0$. Here, the surfaces in which we are interesed have non-empty boundary.

The aim of this paper is to understand some geometric aspects of linear Weingarten surfaces, particularly those related with its Gauss map N, its natural conformal structure and some optimal estimates about the height and curvatures.

In §2 we treat the case of elliptic linear Weingarten (ELW) surfaces, this means that $a^2+bc>0$. By using the conformal structure induced by $a\psi-bN$, we derive two fundamental elliptic partial differential equations which involve the immersion and the Gauss map (Theorem 1). The result let us to recover the immersion from a harmonic local diffeomorphism into the unit sphere: its Gauss map, (Corollary 1).

In §3 we extend the results of Heinz [7] and Rosenberg-Earp, [15] and give optimal estimates of the height that a compact elliptic linear Weingarten surface can rise above a plane, (Theorem 2). We also prove optimal estimates for $2aH+bK$ of a linear Weingarten surface (Theorem 3) and of a general graph with planar boundary, (Theorem 4).

2 Elliptic Linear Weingarten Surfaces

Let S be an orientable surface (possibly with boundary) and $\psi:S\longrightarrow\mathbb{R}^3$ an immersion with Gauss map $N:S\longrightarrow\mathbb{S}^2$. It is said that ψ is a linear Weingarten immersion if a linear combination of its mean curvature H and its Gauss curvature K is constant on S, that is, there exist three real numbers $a,\ b,\ c$, not all zero, such that

$$2\,a\,H+b\,K\;=\;c \tag{1}$$

Moreover, the above equation is elliptic only when $a^2+bc>0$ (see [8, pp.128-129]). In that case we will say that the immersion ψ is elliptic linear Weingarten, in short ELW.

Some interesting examples of ELW immersions are given by the surfaces with constant mean curvature, that is $b=0$, and the surfaces with positive constant Gauss curvature, that is $a=0$.

Lemma 1 *Let $\psi:S\longrightarrow\mathbb{R}^3$ be an ELW immersion satisfying* (1) *then there exists a Gauss map $\eta:S\longrightarrow\mathbb{S}^2$ and two real numbers $\alpha,\ \beta$ such that*

$$2\,\alpha\,H+\beta\,K\;=\;\gamma\geq 0, \tag{2}$$

and $\alpha\, I + \beta\, II$ is a positive definite metric; being $I = \langle d\psi, d\psi\rangle$ and $II = \langle d\psi, -d\eta\rangle$ the first and second fundamental form of the immersion, respectively.

Proof: By changing the sign in (1), if it was necessary, we can assume that (2) is satisfied.

On the other hand, if $\{e_1, e_2\}$ is an orthonormal basis at a point p which diagonalizes $d\eta$, that is, $d\eta(e_i) = -k_i e_i$, $i = 1, 2$, we have

$$\begin{aligned} \sigma(e_1, e_1)\, \sigma(e_2, e_2) - \sigma(e_1, e_2)^2 &= (\alpha + \beta k_1)\, (\alpha + \beta k_2) = \\ &= \alpha^2 + \beta\, (2\alpha H + \beta K) = \alpha^2 + \beta\gamma > 0 \end{aligned} \tag{3}$$

with $\sigma = \alpha I + \beta II$, namely, σ is definite.

Moreover, if σ is negative definite then we replace η by $-\eta$ and one has $2(-\alpha)H + \beta K = \gamma$ and $(-\alpha)\, I + \beta\, II = -\sigma$ is positive definite, since the mean curvature and the second fundamental form change the sign with η. □

Thus, from now on, we will suppose that every ELW immersion satisfies the above result. Moreover, the Gauss map η given by Lemma 1 will be called its **associated Gauss map**.

Now, we obtain a condition to determinate the associated Gauss map of an ELW immersion.

Lemma 2 *Let $\psi : S \longrightarrow \mathbb{R}^3$ be an ELW immersion satisfying* (2) *with associated Gauss map η, then at a point p with Gauss curvature $K(p) > 0$ we have that $\eta(p)$ is the inner normal if and only if $\alpha \geq 0$ or $\beta \geq 0$.*

Proof: If η is not the inner normal at p then the principal curvatures $k_1(p)$, $k_2(p)$ are both negative and using (3),

$$0 \;>\; k_2(p) \;=\; \frac{1}{\beta}\left(-\alpha + \frac{\alpha^2 + \beta\gamma}{\alpha + \beta k_1(p)}\right) \;=\; \frac{\gamma - \alpha k_1(p)}{\alpha + \beta k_1(p)}$$

when $\beta \neq 0$.

Since $\sigma = \alpha I + \beta II$ is positive definite, $\alpha + \beta k_1(p) > 0$ and so $0 \leq \gamma < \alpha k_1(p)$, therefore $\alpha < 0$ and $\beta < 0$. If $\beta = 0$ the result is obvious.

Conversely, if $\alpha < 0$ and $\beta < 0$ then since $\alpha + \beta k_1(p) > 0$ we have that $k_1(p) < 0$, namely, $\eta(p)$ is the outter normal. □

Theorem 1 *Let $\psi : S \longrightarrow \mathbb{R}^3$ be an ELW immersion satisfying* (2) *with associated Gauss map η. Then*

$$\begin{aligned} \Delta^\sigma \psi \;&=\; \frac{\gamma + \beta K}{\alpha^2 + \beta\gamma}\eta, \\ \Delta^\sigma \eta \;&= 2\,\frac{\alpha K - \gamma H}{\alpha^2 + \beta\gamma}\eta, \end{aligned} \tag{4}$$

where Δ^σ denotes the Laplacian respect to the Riemannian metric $\sigma = \alpha\, I + \beta\, II$.

Proof: Let (u, v) be isothermal parameters for σ, that is,

$$\begin{aligned} I &= E_1\, du^2 + 2F_1\, dudv + G_1\, dv^2, \\ II &= E_2\, du^2 + 2F_2\, dudv + G_2\, dv^2, \\ \sigma &= (\alpha E_1 + \beta E_2)\, du^2 + 2(\alpha F_1 + \beta F_2)\, dudv + (\alpha G_1 + \beta G_2)\, dv^2 \;=\; \lambda\, (du^2 + dv^2). \end{aligned} \tag{5}$$

If we denote by $\wedge$ the usual cross product in $\mathbb{R}^3$ then, bearing in mind that $\eta \wedge \psi_u$ and $\eta \wedge \psi_v$ are a basis of the tangent plane, we can write

$$\begin{aligned} \alpha\, \psi_u - \beta\, \eta_u &= \mu_{11}\, \eta \wedge \psi_u + \mu_{12}\, \eta \wedge \psi_v, \\ \alpha\, \psi_v - \beta\, \eta_v &= \mu_{21}\, \eta \wedge \psi_u + \mu_{22}\, \eta \wedge \psi_v. \end{aligned}$$

for certain real functions μ_{11}, μ_{12}, μ_{21} and μ_{22}.

Now, making the inner product with ψ_u and ψ_v,

$$\begin{aligned} \lambda &= \sigma(\psi_u, \psi_u) = \mu_{12}\, \langle \eta \wedge \psi_v, \psi_u \rangle = -\mu_{12}\, |\psi_u \wedge \psi_v|, \\ 0 &= \sigma(\psi_u, \psi_v) = \mu_{11}\, \langle \eta \wedge \psi_u, \psi_v \rangle = \mu_{11}\, |\psi_u \wedge \psi_v|, \\ 0 &= \sigma(\psi_v, \psi_u) = \mu_{22}\, \langle \eta \wedge \psi_v, \psi_u \rangle = -\mu_{22}\, |\psi_u \wedge \psi_v|, \\ \lambda &= \sigma(\psi_v, \psi_v) = \mu_{21}\, \langle \eta \wedge \psi_u, \psi_v \rangle = \mu_{21}\, |\psi_u \wedge \psi_v|, \end{aligned}$$

and so, since $|\psi_u \wedge \psi_v| = \sqrt{E_1 G_1 - F_1^2}$, one has

$$\begin{aligned} \alpha\, \psi_u - \beta\, \eta_u &= \frac{-\lambda}{\sqrt{E_1 G_1 - F_1^2}}\, \eta \wedge \psi_v, \\ \alpha\, \psi_v - \beta\, \eta_v &= \frac{\lambda}{\sqrt{E_1 G_1 - F_1^2}}\, \eta \wedge \psi_u. \end{aligned} \tag{6}$$

By using that

$$\begin{aligned} \lambda^2 &= (\alpha E_1 + \beta E_2)\, (\alpha G_1 + \beta G_2) - (\alpha F_1 + \beta F_2)^2 \\ &= (\alpha^2 + \beta(2\alpha H + \beta K))\, (E_1 G_1 - F_1^2) = (\alpha^2 + \beta\gamma)\, (E_1 G_1 - F_1^2), \end{aligned} \tag{7}$$

it follows from (6) that

$$\begin{aligned} \alpha\, \psi_u - \beta\, \eta_u &= -\sqrt{\alpha^2 + \beta\gamma}\;\; \eta \wedge \psi_v, \\ \alpha\, \psi_v - \beta\, \eta_v &= \sqrt{\alpha^2 + \beta\gamma}\;\; \eta \wedge \psi_u, \end{aligned} \tag{8}$$

and

$$\begin{aligned} \alpha\, \psi_u \wedge \eta - \beta\, \eta_u \wedge \eta &= -\sqrt{\alpha^2 + \beta\gamma}\;\, \psi_v, \\ \alpha\, \psi_v \wedge \eta - \beta\, \eta_v \wedge \eta &= \sqrt{\alpha^2 + \beta\gamma}\;\, \psi_u. \end{aligned}$$

Thus, the derivative of the second equation with respect to u minus the derivative of the first equation with respect to v gives

$$(2\alpha H + 2\beta K)\,\psi_u \wedge \psi_v \;=\; \sqrt{\alpha^2+\beta\gamma}\;(\psi_{uu}+\psi_{vv})$$

that is,

$$\psi_{uu}+\psi_{vv} \;=\; \frac{\gamma+\beta K}{\sqrt{\alpha^2+\beta\gamma}}\,\psi_u \wedge \psi_v. \tag{9}$$

On the other hand, if we consider in (8) the derivative of the first equation with respect to u plus the derivative of the second equation with respect to v,

$$\alpha\,(\psi_{uu}+\psi_{vv}) - \beta\,(\eta_{uu}+\eta_{vv}) \;=\; 2H\sqrt{\alpha^2+\beta\gamma}\;\psi_u \wedge \psi_v.$$

Therefore, if $\beta \neq 0$,

$$\eta_{uu}+\eta_{vv} \;=\; 2\frac{\alpha K - \gamma H}{\sqrt{\alpha^2+\beta\gamma}}\,\psi_u \wedge \psi_v \tag{10}$$

and the Theorem follows from (7), (9) and (10). (The result is well-known if $\beta = 0$, see, for instance, [14]). $\square$

Remark 1 Since $H^2 \geq K$ on every surface, given an ELW immersion satisfying (2) the equality in the above inequality occurs when

$$K = \frac{\gamma^2}{(\alpha \pm \sqrt{\alpha^2+\beta\gamma})^2} \quad \text{if } \beta\gamma \neq 0, \qquad K = \frac{\gamma^2}{4\alpha^2} \quad \text{if } \beta = 0$$

$$\text{and} \quad K = 0,\ K = \frac{4\alpha^2}{\beta^2} \quad \text{if } \gamma = 0 \text{ and } \beta \neq 0.$$

Moreover, from (4) and (5)

$$2\,\frac{\alpha K - \gamma H}{\alpha^2+\beta\gamma} \;=\; \langle \Delta^{\sigma}\eta, \eta\rangle \;=\; \frac{1}{\lambda}\langle \eta_{uu}+\eta_{vv}, \eta\rangle \;=\; -\frac{1}{\lambda}\left(\langle \eta_u, \eta_u\rangle + \langle \eta_v, \eta_v\rangle\right).$$

Hence, $\alpha K - \gamma H \leq 0$ and since σ is positive definite

(A) If $\alpha \geq 0$, then $K \leq \dfrac{\gamma^2}{(\alpha+\sqrt{\alpha^2+\beta\gamma})^2}$.

(B) If $\alpha \leq 0$, then $K \geq \dfrac{\gamma^2}{(\alpha+\sqrt{\alpha^2+\beta\gamma})^2}$ when $\gamma \neq 0$ and $K \geq \dfrac{4\alpha^2}{\beta^2}$ when $\gamma = 0$.

As an immediate consequence of the above Theorem we obtain that η can be considered as a harmonic map, that is,

Corollary 1 *Let $\psi : S \longrightarrow \mathbb{R}^3$ be an ELW immersion satisfying* (2) *with associated Gauss map η. If we consider on S the conformal structure induced by $\sigma = \alpha\, I + \beta\, II$, then η is harmonic. Moreover, if $\gamma \neq 0$, ψ can be recovered as*

$$\psi \;=\; -\frac{\alpha}{\gamma}\,\eta + \frac{\sqrt{\alpha^2+\beta\gamma}}{\gamma}\;\int \eta\wedge\eta_v\, du - \eta\wedge\eta_u\, dv, \tag{11}$$

for (u, v) isothermal parameters on S.

Conversely, if S is a simply-connected Riemann surface and $\eta : S \longrightarrow \mathbb{S}^2$ is a harmonic map, then (11) *gives an ELW immersion (possibly degenerated at some points) such that the conformal structure on S is the induced one by $\alpha\, I + \beta\, II$.*

Proof: From (6) and (7)

$$\begin{aligned} \alpha\,\psi_u &= \beta\,\eta_u - \sqrt{\alpha^2+\beta\gamma}\;\eta\wedge\psi_v,\\ \alpha\,\psi_v &= \beta\,\eta_v + \sqrt{\alpha^2+\beta\gamma}\;\eta\wedge\psi_u \end{aligned}$$

and putting the second equation into the first one

$$\alpha^2\,\psi_u \;=\; \alpha\beta\,\eta_u - \beta\,\sqrt{\alpha^2+\beta\gamma}\;\eta\wedge\eta_v + (\alpha^2+\beta\gamma)\,\psi_u,$$

that is,

$$\beta\gamma\,\psi_u \;=\; -\alpha\beta\,\eta_u + \beta\,\sqrt{\alpha^2+\beta\gamma}\;\eta\wedge\eta_v.$$

Analogously,

$$\beta\gamma\,\psi_v \;=\; -\alpha\beta\,\eta_v - \beta\,\sqrt{\alpha^2+\beta\gamma}\;\eta\wedge\eta_u.$$

Hence, if $\beta \neq 0$, the immersion is recovered using (11). (If $\beta = 0$ the result is well-known, see [10]).

The converse is a straightforward computation. □

Remark 2 It is easy to see that there exists, up to an isometry, a unique totally umbilical immersion in the family of ELW immersions satisfying (2), given by

(i) a round sphere of radius $R = |\alpha + \sqrt{\alpha^2+\beta\gamma}|/\gamma$ if $\gamma \neq 0$,

(ii) a round sphere of radius $R = |\beta|/2|\alpha|$ if $\gamma = 0$ and $\alpha < 0$,

(iii) a plane if $\gamma = 0$ and $\alpha > 0$.

These surfaces are characterized as the simply-connected ones with a closed line of curvature. More precisely,

Corollary 2 *Let S be a closed topological disk and $\psi : S \longrightarrow \mathbb{R}^3$ an ELW immersion. If the image of the boundary of S, $\psi(\partial S)$, is a line of curvature then $\psi(S)$ lies on a round sphere or on a plane.*

Proof: Let us assume ψ satisfies (2) and consider S as a Riemann surface with the conformal structure induced by $\sigma = \alpha\, I + \beta\, II$. Then S is conformally equivalent to the closed unit disk $D = \{z \in \mathbb{C}/ \ |z| \leq 1\}$.

Thus, we can consider $S = D$ and choose polar coordinates (r, θ) given by $z = u + iv = re^{i\theta}$. Therefore,

$$\frac{\partial}{\partial r} = \cos\theta \frac{\partial}{\partial u} + \sin\theta \frac{\partial}{\partial v},$$
$$\frac{\partial}{\partial \theta} = -\sin\theta \frac{\partial}{\partial u} + \cos\theta \frac{\partial}{\partial v},$$

on ∂D. Since (u, v) are conformal parameters respect to σ, we have

$$0 = \alpha\, \langle \frac{\partial \psi}{\partial r}, \frac{\partial \psi}{\partial \theta} \rangle + \beta\, \langle \frac{\partial \psi}{\partial r}, -\frac{\partial \eta}{\partial \theta} \rangle$$

and using that $|z| = 1$ is a line of curvature with normal curvature k_n we have

$$0 = (\alpha + \beta\, k_n)\, \langle \frac{\partial \psi}{\partial r}, \frac{\partial \psi}{\partial \theta} \rangle$$

on $|z| = 1$.

Because of σ is positive definite, $\alpha + \beta\, k_n > 0$, and

$$\langle \frac{\partial \psi}{\partial r}, \frac{\partial \psi}{\partial \theta} \rangle = 0 = \langle \frac{\partial \psi}{\partial r}, -\frac{\partial \eta}{\partial \theta} \rangle$$

on ∂D, or equivalently,

$$0 = -\frac{1}{2} \sin 2\theta\ (E_i - G_i) + \cos 2\theta\ F_i, \qquad i = 1, 2 \tag{12}$$

on ∂D, where I and II are written as in (5).

On the other hand, from Theorem 1 the complex functions

$$f_1(z) = \langle \frac{\partial \psi}{\partial z}, \frac{\partial \psi}{\partial z} \rangle \quad \text{and} \quad f_2(z) = \langle \frac{\partial \psi}{\partial z}, -\frac{\partial \eta}{\partial z} \rangle$$

are holomorphic. In fact,

$$\frac{\partial f_1}{\partial \bar{z}} = 2\, \langle \frac{\partial^2 \psi}{\partial z \partial \bar{z}}, \frac{\partial \psi}{\partial z} \rangle = \frac{\lambda}{2}\, \langle \Delta^\sigma \psi, \frac{\partial \psi}{\partial z} \rangle = 0,$$
$$\frac{\partial f_2}{\partial \bar{z}} = \langle \frac{\partial^2 \psi}{\partial z \partial \bar{z}}, -\frac{\partial \eta}{\partial z} \rangle + \langle \frac{\partial \psi}{\partial z}, -\frac{\partial^2 \eta}{\partial z \partial \bar{z}} \rangle = \frac{\lambda}{4}\, \langle \Delta^\sigma \psi, -\frac{\partial \eta}{\partial z} \rangle + \frac{\lambda}{4}\, \langle \frac{\partial \psi}{\partial z}, -\Delta^\sigma \eta \rangle = 0.$$

Moreover, $g_i(z) = z^2 f_i(z)$ is also holomorphic with imaginary part, $Im(g_i) = \sin 2\theta\ (E_i - G_i) - 2\cos 2\theta\ F_i$, on ∂D and, from (12), g_i must be constant, i=1,2.

Using that $g_i(0) = 0$, we have $g_i \equiv 0 \equiv f_i$, i=1,2 and II is proportional to I. □

3 Estimates on Linear Weingarten Surfaces and Graphs

First, we give a bound of the height that a compact ELW surface can rise above a plane, generalizing the well-known bounds for constant mean curvature due to E. Heinz, positive constant Gauss curvature [14] and special linear Weingarten surfaces [15]. For that, we need the following result,

Lemma 3 *Let S be a compact surface with boundary ∂S and $\psi : S \longrightarrow \mathbb{R}^3$ a non-flat ELW graph on the plane $P = \{x_3 = 0\}$, satisfying* (2)*, with $\psi(\partial S) \subseteq P$. Then $\psi(S)$ is contained in a halfspace determined by P. Moreover, if we call h to the maximum height that $\psi(S)$ can rise above P and ψ_3 the third coordinate immersion then*

1. *if $\alpha \geq 0$ or $\beta \geq 0$ and $|\nabla \psi_3| \leq m$ along ∂S for a non-negative constant $m \leq 1$, we have*
$$h \leq R\left(1 - \sqrt{1 - m^2}\right),$$
2. *if $\alpha < 0$, $\beta < 0$ and $|\nabla \psi_3| \geq m$ along ∂S for a non-negative constant $m \leq 1$, we have*
$$h \geq R\left(1 - \sqrt{1 - m^2}\right),$$

where R is the radius of the unique sphere satisfying (2)*, given by Remark* 2*, and $\nabla \psi_3$ is the gradient of ψ_3 for the induced metric.*

Proof: Let us assume that $\alpha \geq 0$ or $\beta \geq 0$ and there exists a point p_1 such that $\psi(p_1) \in \{x_3 > 0\}$. If we take a hemisphere $\mathcal{H}$ with boundary on P such that $S \cap \{x_3 \geq 0\}$ is inside and we consider spherical caps with boundary $\mathcal{H} \cap P$, then a spherical cap meets to $\psi(S)$ the first time at a point q_1 with positive Gauss curvature and inner normal pointing down. Thus, from Lemma 2, the associated Gauss map η points down at q_1.

On the other hand, if there exists another point p_2 such that $\psi(p_2) \in \{x_3 < 0\}$, then reasoning as above there would exist q_2 such that η points up at q_2, which is a contradiction, since ψ is a graph. Therefore, $\psi(S)$ is contained in a halfspace determined by P.

Up to a reflexion, we can assume $\psi(S) \subseteq \{x_3 \geq 0\}$. Then from Theorem 1

$$\begin{aligned}
\Delta^{\sigma}\left(\frac{1}{R}\psi_3 + \eta_3\right) &= \left(\frac{1}{R}\,\frac{\gamma + \beta K}{\alpha^2 + \beta\gamma} + \frac{2(\alpha K - \gamma H)}{\alpha^2 + \beta\gamma}\right)\eta_3 \\
&= \frac{(2\alpha R + \beta)K + \gamma(1 - 2RH)}{R\left(\alpha^2 + \beta\gamma\right)}\,\eta_3
\end{aligned}$$

where η_3 is the third coordinate of the associated Gauss map.

By using that

$$2\alpha R + \beta = \gamma R^2 \tag{13}$$

we have

$$\Delta^\sigma \left(\frac{1}{R} \psi_3 + \eta_3 \right) = \frac{\gamma R}{\alpha^2 + \beta\gamma} \left(k_1 - \frac{1}{R} \right) \left(k_2 - \frac{1}{R} \right) \eta_3$$

for the principal curvatures k_1, k_2 of the immersion.

On the other hand, if $\beta \neq 0$, from (2) and (13), the two points (k_1, k_2) and $(1/R, 1/R)$ belong to the equilateral hyperbola $\alpha(x+y) + \beta xy = \gamma$ on the (x,y)-plane. And, since, $\alpha + \beta k_i > 0$, $i = 1, 2$, $\alpha + \beta/R > 0$, both points are on the same connected component of the hyperbola. Therefore,

$$\left(k_1 - \frac{1}{R} \right) \left(k_2 - \frac{1}{R} \right) \leq 0.$$

If $\beta = 0$, the above inequality is clear from (2) and (13).

Now, from Lemma 2, $\eta_3 \leq 0$ and

$$\Delta^\sigma \left(\frac{1}{R} \psi_3 + \eta_3 \right) \geq 0. \tag{14}$$

Bearing in mind that

$$\frac{1}{R} \psi_3 + \eta_3 = \eta_3 = -\sqrt{1 - |\nabla \psi_3|^2} \leq -\sqrt{1 - m^2}$$

along ∂S, we have

$$\psi_3 \leq R \left(-\eta_3 - \sqrt{1 - m^2} \right) \leq R \left(1 - \sqrt{1 - m^2} \right)$$

as we wanted to prove.

The case $\alpha < 0$, $\beta < 0$ is analogous to the first one, but (14) changes the sign. □

Thus, by using the classical Alexandrov reflection principle in a standard way (see, for instance, [14]) for the elliptic equation (1) in an ELW embedding and since $|\nabla \psi_3| \leq 1$ everywhere, one has from Remark 2 and Lemma 3, the following

Theorem 2 *Let S be a compact surface with boundary ∂S, P a plane and $\psi : S \longrightarrow \mathbb{R}^3$ a non-flat ELW embedding satisfying (2) with $\psi(\partial S) \subseteq P$. Then, if $\alpha \geq 0$ or $\beta \geq 0$ the maximum height that $\psi(S)$ can rise above P is*

$$\frac{2|\alpha + \sqrt{\alpha^2 + \beta\gamma}|}{\gamma}, \ \textit{with } \gamma \neq 0 \qquad \textit{and} \qquad \frac{|\beta|}{|\alpha|}, \ \textit{with } \gamma = 0.$$

Now, we obtain a balancing formula for linear Weingarten surfaces in $\mathbb{R}^3$. Let S be a compact surface with boundary ∂S and $\psi : S \longrightarrow \mathbb{R}^3$ a linear Weingarten immersion satisfying

$$2a\, H + b\, K = c$$

for a unit normal vector field $N : S \longrightarrow \mathbb{S}^2$. Then, if we consider local coordinates (u, v), we have

$$dN \wedge d\psi = (N_u \wedge \psi_v + \psi_u \wedge N_v)\, du \wedge dv = -2H\,(\psi_u \wedge \psi_v)\, du \wedge dv = -H\, d\psi \wedge d\psi$$

$$dN \wedge dN = 2\,(N_u \wedge N_v)\, du \wedge dv = 2K\,(\psi_u \wedge \psi_v)\, du \wedge dv = K\, d\psi \wedge d\psi. \tag{15}$$

Therefore,

$$-2a\, d\,(N \wedge d\psi) + b\, d\,(N \wedge dN) = c\, d\,(\psi \wedge d\psi)$$

and, by applying Stoker's theorem, we have the following balancing formula

$$-2a \int_{\partial S} N \wedge d\psi + b \int_{\partial S} N \wedge dN = c \int_{\partial S} \psi \wedge d\psi. \tag{16}$$

Now, we use the above balancing formula to study linear Weingarten surfaces spanned by a fixed planar Jordan curve. For that, we remind that the algebraic area of the curve $\psi(\partial S)$, given by the vector

$$\overline{A} = \frac{1}{2} \int_{\partial S} \psi \wedge d\psi,$$

only depends on the curve and not on its representation $\psi|_{\partial S}$. Moreover, if $\psi(\partial S)$ is a planar Jordan curve, then $|\overline{A}|$ is the area enclosed by $\psi(\partial S)$ on the plane.

Theorem 3 *Let S be a compact surface with boundary ∂S and $\psi : S \longrightarrow \mathbb{R}^3$ a linear Weingarten immersion satisfying*

$$2a\, H + b\, K = c$$

and such that $\psi(\partial S) \subseteq \{x_3 = 0\}$ is a convex Jordan curve. Then, if $|\nabla \psi_3| \leq m \leq 1$ along ∂S, one has

$$|2a\, H + b\, K| \leq \frac{|a|\, m\, L + |b|\, m^2\, \pi}{A},$$

where L and A are the length and enclosed area by the curve $\psi(\partial S)$, respectively.

Moreover, if S is a topological disk and the equality holds then $\psi(S)$ is planar or a spherical cap.

Proof: Let t be a unit tangent vector field along ∂S, then $-dN(t) = k_n t + \lambda N \wedge t$, where k_n is the normal curvature along ∂S. And the boundary is a line of curvature if and only if $\lambda \equiv 0$.

If we put $v_3 = (0,0,1)$, then from (16)

$$\begin{aligned} 2|c|\, A &= \left| c \int_{\partial S} \langle \psi \wedge d\psi, v_3 \rangle \right| = \left| -2a \int_{\partial S} \langle N \wedge t, v_3 \rangle + b \int_{\partial S} \langle N \wedge dN(t), v_3 \rangle \right| \\ &\leq \left| -2a \int_{\partial S} \langle N \wedge t, v_3 \rangle \right| + \left| -b \int_{\partial S} k_n \langle N \wedge t, v_3 \rangle \right| . \end{aligned} \tag{17}$$

On the other hand, it is well known that $k_n = k\langle N, t \wedge v_3 \rangle$, where k denotes the curvature of the curve $\psi(\partial S)$ (see, for instance, [4, p. 141]). Moreover, $\langle N \wedge t, v_3 \rangle^2 = |\nabla \psi_3|^2$; indeed, $\langle N, v_3 \rangle^2 = 1 - |\nabla \psi_3|^2$ and $N = \langle N, t \wedge v_3 \rangle\, t \wedge v_3 + \langle N, v_3 \rangle\, v_3$, therefore $\langle N, t \wedge v_3 \rangle^2 = 1 - \langle N, v_3 \rangle^2 = |\nabla \psi_3|^2$. Thus, by using that k does not change the sign

$$2|c|\, A \leq 2|a|\, m\, L + |b|\, m^2 \left| \int_{\partial S} k \right| = 2|a|\, m\, L + 2|b|\, m^2\, \pi$$

as we wanted to prove.

Moreover, if the equality holds $\langle N, v_3 \rangle$ and $\langle N \wedge t, v_3 \rangle$ are constant along ∂S. So, if we derive respect to t

$$0 = \langle -k_n t - \lambda N \wedge t, v_3 \rangle = -\lambda \langle N \wedge t, v_3 \rangle$$
$$0 = -\lambda \langle (N \wedge t) \wedge t, v_3 \rangle + \langle N \wedge k_g (N \wedge t), v_3 \rangle = \lambda \langle N, v_3 \rangle$$

where k_g is the geodesic curvature of ∂S [4, p. 248]. Therefore, $\lambda \equiv 0$ and ∂S is a line of curvature.

Hence, if S is a topological disk, we can distinguish three cases:

(i) $a^2 + bc > 0$. Then, from Corollary 2, $\psi(S)$ lies on a plane or it is a spherical cap.

(ii) $a^2 + bc < 0$. Since a point $p \in S$ is umbilical if and only if $H^2 = K$ at p, one has that there does not exist any umbilical point on S. Now, by the Poincaré-Hopf Theorem ([9, p. 135]), the sum of the indices of the singularities of a line field, on a compact surface with boundary, which is transversal along the boundary and has a finite number of singularities, must be the Euler charasteristic of the surface. Consequently, the Euler charasteristic of S is zero, since the field of line elements associated to the principal curvature different of k_n has not got any singularity and it is perpendicular to the boundary, which is a contradiction. Therefore, the equality is not possible in this case.

(iii) $a^2 + bc = 0$. Then, from (3), $(a + bk_1)(a + bk_2) = 0$, being k_1 and k_2 the principal curvatures of the immersion. Therefore, $a + bk_i = 0$ for some $i = 1, 2$, but if the equality holds, from (17), the signs of a and bk_n are the same and $a + bk_n \neq 0$ everywhere (otherwise $a = 0$ and $K \equiv 0$, that is, the immersion lies on a plane).

Thus, ∂S is free of umbilical points and the principal curvature k_n is different to the constant principal curvature $-a/b$. Hence, if p is an interior point on S and q is the nearest point on ∂S, then the minimizing geodesic γ from p to q meets ortogonally to the boundary and, using [16, Lemma 3], there is no umbilical point on γ and p is a non-umbilical point. Consequently, the immersion is umbilically free and, reasoning as the above case, the equality is not possible.

$\square$

Observe that the hypothesis about convexity of $\psi(\partial S)$ is non-necessary if $b = 0$, that is, for constant mean curvature. In that case, the result was first proved by E. Heinz [7] (whithout any assumption about $|\nabla \psi_3|$, that is, $m = 1$).

Now, we obtain a similar result about the behaviour of the curvatures of a general graph with planar boundary.

Theorem 4 *Let S be a compact surface and $\psi : S \longrightarrow \mathbb{R}^3$ a graph with connected convex planar boundary $\psi(\partial S) \subseteq \{x_3 = 0\}$ and such that for a positive number m, $|\nabla \psi_3| \leq m \leq 1$ along the boundary. Then, given non-negative real numbers a, b, not both zero, one has*

$$\min_{p \in S}(2a\, H + b\, K) \leq \frac{amL + bm^2\pi}{A}$$

for any Gauss map N on S, where L and A denote the length and enclosed area by the curve $\psi(\partial S)$.

Moreover, if the equality holds then $\psi(S)$ is a spherical cap.

Proof: Let c be the minimum of the function $2aH + bK$ on S for a Gauss map N on S, then if c is non-positive the result is obvious. Otherwise, we have $2aH + bK \geq c > 0$ and taking $\sigma = a\, I + b\, II$,

$$\begin{aligned} \sigma(e_1, e_1)\, \sigma(e_2, e_2) - \sigma(e_1, e_2)^2 &= (a + bk_1)\, (a + bk_2) = \\ &= a^2 + b\, (2aH + bK) \geq a^2 + bc > 0, \end{aligned}$$

for an orthonormal basis $\{e_1, e_2\}$ which diagonalizes dN at a point $p \in S$, that is, σ is definite.

Since $\psi(S)$ is not contained in $\{x_3 = 0\}$, up to a reflexion, we can assume that there exists a point in $\{x_3 > 0\}$. Then, reasoning as in Lemma 3 there exists a point q on $S \cap \{x_3 > 0\}$ with positive Gauss curvature satisfying $a + bk_i > 0$, $i = 1, 2$, for the inner normal pointing down η, that is, σ is positive definite for η.

On the other hand, if $N \neq \eta$ then the mean curvature H_N for N is $-H_\eta$ and $2aH_N + bK < 2aH_\eta + bK$ at q. Moreover, this inequality is true everywhere, otherwise, the equality should hold at some point p. Consequently, $H(p)=0$ and, since, $H^2 \geq K$ one has $K(p) \leq 0$ and c should be non-positive. So, it is sufficient to prove the result for η.

By using (15),

$$0 \geq c \int_S \langle d\psi \wedge d\psi, v_3 \rangle \geq -2a \int_S \langle d\eta \wedge d\psi, v_3 \rangle + b \int_S \langle d\eta \wedge d\eta, v_3 \rangle$$

and by applying Stoker's theorem for a well-oriented tangent unit vector field t along ∂S

$$\begin{aligned} 0 > -2c\,A &= c \int_{\partial S} \langle \psi \wedge d\psi, v_3 \rangle \geq -2a \int_{\partial S} \langle \eta \wedge d\psi, v_3 \rangle + b \int_{\partial S} \langle \eta \wedge d\eta, v_3 \rangle \\ &= -\int_{\partial S} (2a + bk_n) \langle \eta \wedge t, v_3 \rangle \geq -m \int_{\partial S} (2a + bk_n) \\ &\geq -m \int_{\partial S} (2a + bk \langle \eta \wedge t, v_3 \rangle) \geq -(2amL + 2bm^2\pi), \end{aligned}$$

being k_n and k the normal curvature and curvature of $\psi(\partial S)$, respectively, where we have used that $2a + bk_n = a + \sigma(t,t) > 0$.

Moreover, if the equality holds $\min_{p\in S}(2aH + bK) = \max_{p\in S}(2aH + bK)$ and the immersion is linear Weingarten. Thus, the result follows from Theorem 3. □

It is important to remark that the convexity of the boundary is not necessary in the above Theorem if $b = 0$. In that way, we obtain

Corollary 3 *For any compact graph with connected planar boundary, one has*

$$\min_{p\in S} H \leq \frac{L}{2A} \qquad \text{and} \qquad \min_{p\in S} K \leq \frac{\pi}{A}$$

where L and A are the length and enclosed area by the curve $\psi(\partial S)$.

Moreover, the equality holds if and only if $\psi(S)$ is a hemisphere.

Proof: If $\min_{p\in S} K > 0$ then the boundary is a convex curve. Hence, since $|\nabla\psi_3| \leq 1$ everywhere, taking $m = 1$ in Theorem 4, we obtain the first inequality for $a = 1/2,\ b = 0$ and the second one for $a = 0,\ b = 1$. □

References

[1] Aledo J A, Gálvez J A (2001) Remarks on Compact Linear Weingarten Surfaces in Space Forms. Preprint.

[2] Chern S S (1945) Some new characterizations of the Euclidean sphere. Duke Math J **12**: 279-290.

[3] Chern S S (1955) On special W-surfaces. P Am Math Soc **6**: 783-786.

[4] do Carmo M P (1976) Differential Geometry of Curves and Surfaces, Englewood Cliffs, New Jersey: Prentice-Hall, Inc.

[5] Gálvez J A, Martínez A (2000) Estimates in surfaces with positive constant Gauss Curvature. P Am Math Soc **128**: 3655-3660.

[6] Hartman P, Winter W (1954) Umbilical points and W-surfaces. Am J Math **76**: 502–508.

[7] Heinz E (1969) On the Nonexistence of a Surface of Constant Mean Curvature with Finite Area and Prescribed Rectificable Boundary. Arch Rational Mech Anal **35**: 249-252.

[8] Hopf H (1983) Differential Geometry in the Large, Lecture Notes in Math, vol 1000, Berlin: Springer-Verlag.

[9] Hirsch M W (1976) Differential Topology, New York: Springer-Verlag.

[10] Kenmotsu K (1979) Weierstrass formula for surfaces of prescribed mean curvature. Math Ann **245**: 89-99.

[11] Korevaar N, Kusner R, Solomon B (1989) The structure of complete embedded surfaces with constant mean curvature. J Differ Geom **30**: 465-503.

[12] Li H (1997) Global Rigidity Theorems of Hypersurfaces. Ark Mat **35**: 327-351.

[13] Meeks W H (1988) The topology and geometry of embedded surfaces of constant mean curvature. J Differ Geom **27**: 539-552.

[14] Rosenberg H (1993) Hypersurfaces of constant curvature in space forms. B Soc Math Fr 2^e série **117**: 211-239.

[15] Rosenberg H, Sa Earp R (1994) The Geometry of properly embedded special surfaces in $\mathbb{R}^3$; e. g., surfaces satisfying $aH + bK = 1$, where a and b are positive. Duke Math J **73**: 291-306.

[16] Shiohama K, Takagi R (1970) A Characterization of a Standard Torus in E^3. J Differ Geom **4**: 477–485.