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# The Affine Cauchy Problem <sup>1</sup>

Juan A. Aledo <sup>a</sup>, Antonio Martínez<sup>b</sup> and Francisco Milán<sup>c</sup>

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<sup>a</sup> Departamento de Matemáticas, Universidad de Castilla-La Mancha, E-02071 Albacete, Spain.

e-mail: juanangel.aledo@uclm.es

<sup>b</sup>, <sup>c</sup> Departamento de Geometría y Topología, Universidad de Granada, E-18071 Granada, Spain.

e-mail: amartine@ugr.es ; milan@ugr.es

Date: March 31, 2008

Keywords: affine area, maximal surfaces, Cauchy problem, helicoidal affine maximal surfaces, singularities.

## Abstract

The aim of this paper is to solve the Cauchy problem for locally strongly convex surfaces which are extremal for the equiaffine area functional. These surfaces are called affine maximal surfaces and here, we give a new complex representation which let us describe the solution to the corresponding Cauchy problem. As applications, we obtain a generalized symmetry principle, characterize when a curve in  $\mathbb{R}^3$  can be a geodesic or pre-geodesic of a such surface and study the helicoidal affine maximal surfaces. Finally, we investigate the existence and uniqueness of affine maximal surfaces with a given analytic curve in its singular set.

## 1 Introduction

There has been a growing interest in recent years in geometric functionals whose Euler-Lagrange equations are nonlinear fourth order partial differential equations. Well known examples are the Willmore functional [GLW, Sim], the functional proposed by Calabi in [Cal2, WZ] and the equiaffine area functional [Cal, TrWa] .

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<sup>1</sup>Research partially supported by Ministerio de Educación y Ciencia Grant No. MTM2007-65249, Junta de Andalucía Grant No. FQM325 and la Junta de Comunidades de Castilla-La Mancha Grant No. PCI-08-0023

Perhaps for being the most classical, the equiaffine area functional has attracted the interest of a considerable group of geometers as evidenced by the amount of works that it has generated.

In affine surfaces theory, Blaschke (see [Bla2]) found that the corresponding Euler-Lagrange equation is of fourth order and nonlinear. He also showed that this equation is equivalent to the vanishing of the affine mean curvature, which led to the notion of "affine minimal surface" without a previous study of the second variation formula. But sixty years later Calabi proved in [Cal] that, for locally strongly convex surfaces, the second variation is always negative and since then, locally strongly convex surfaces with vanishing affine mean curvature are called "*affine maximal surfaces*".

After Calabi's work this class of surfaces has become a fashion research topic and it has received many interesting contributions that help us to understand its geometry. So far, some important facts are known:

- Affine maximal surfaces have got affine Weierstrass formulas that, along with methods from complex function theory, have provided an important tool in their study (see [Cal, Cal3, Li, Te]).
- Entire solutions of the fourth order affine maximal surface equation

$$L[\phi] := \phi_{yy}\omega_{xx} - 2\phi_{xy}\omega_{xy} + \phi_{xx}\omega_{yy} = 0, \quad \omega = (\det(\nabla^2\phi))^{-3/4}, \quad (1.1)$$

where  $\nabla^2\phi$  is the positive definite Hessian matrix of  $\phi$ , are always quadratic polynomials, [TrWa].

- Every Affine complete affine maximal surface must be an elliptic paraboloid, [LiJi, TrWa3].
- There is a formulation of the Affine Plateau Problem as a geometric variational problem for the equiaffine area functional for which the existence and regularity of maximizers have been proved, [TrWa4].

These results have opened two research lines. One of them deals with their extension to different nonlinear fourth order equations (see [LiJi2, TrWa2]). The other one concerns to study the validity of the results in affine maximal surfaces with some natural singularities that may arise (see [AMaM, GMaMir, Ma]). In the last direction and for the particular case of improper affine spheres, a previous study of the corresponding Cauchy problem has been very useful to understand and motivate the problem (see [ACG]).

In the present work we deal with the general Cauchy Problem for affine maximal surfaces. To be more precise, we are going to solve the following Affine Cauchy Problem:

Let  $\beta : I \longrightarrow \mathbb{R}^3$  be a regular analytic curve, and let  $Y : I \longrightarrow \mathbb{R}^3$  be an analytic vector field along  $\beta$  such that  $\beta' \times Y \neq 0$ . Find all affine maximal surfaces containing  $\beta$  with  $Y$  the affine normal along  $\beta$ .

This problem can be considered as a generalization to the Cauchy problem for the equation (1.1) and has been inspired by the classical Björling problem for minimal surfaces in  $\mathbb{R}^3$ , proposed by E.G. Björling in 1844 and solved by H.A. Schwarz in 1890. More details and some research on this topic may be consulted in [ACMi, DHKW, GMi, GMi2, MiPa].

Blaschke was the first who considered a Björling type problem in affine differential geometry. However, he just considered the case of non convex affine minimal surfaces, that is, when the Berwald-Blaschke metric is indefinite, [Bla].

After some notation, we discuss in Section 3 the existence and uniqueness of solutions to the above affine Cauchy problem. We also construct the solutions in terms of the data  $\beta, Y$ . These constructions set up, in all the cases, new conformal representations for affine maximal surfaces.

Section 4 is devoted to applications in several directions. First, we use our conformal representation to prove uniqueness of the Cauchy problem for the equation (1.1) and give explicitly its solution. Second, we obtain a generalized symmetry principle and characterize when a curve in  $\mathbb{R}^3$  can be geodesic or pre-geodesic of an affine maximal surface. In the rest of the section we apply our results to the study of affine maximal surfaces invariant under a one-parameter equiaffine isometry group.

Finally, in Section 5 we get down the problem of finding affine maximal surfaces with a singular set which contains a prescribed analytic curve. The results in this section will motivate a forthcoming study of affine maximal surfaces with singularities, [AMaM2].

## 2 Basic notations

Consider  $\psi : \Sigma \rightarrow \mathbb{R}^3$  a locally strongly convex immersion of a surface  $\Sigma$ , oriented so that the second fundamental form,  $\sigma_e$ , is positive definite everywhere. Denote by  $K_e$  and  $dA_e$  its Gaussian curvature and the element of euclidean area, respectively. The most elementary unimodular affine invariants of the immersion are the Berwald-Blaschke metric,  $g$ , the equiaffine area element,  $dA$ , and the Blaschke normal or affine normal  $\xi$  given by the following objects:

$$\begin{aligned} g &= K_e^{-\frac{1}{4}} \sigma_e, \\ dA &= K_e^{\frac{1}{4}} dA_e, \\ \xi &= \frac{1}{2} \Delta_g \psi, \end{aligned}$$

where  $\Delta_g$  is the Laplace-Beltrami operator associated to  $g$ .

The *affine conormal* field  $N := K_e^{-1/4} N_e$ , where  $N_e$  is the unit normal to the immersion, satisfies

$$\langle N, \xi \rangle = 1, \quad \langle N, d\psi(v) \rangle = 0, \quad v \in T_p \Sigma, \quad (2.1)$$

where  $\langle \cdot, \cdot \rangle$  denotes the standard inner product in  $\mathbb{R}^3$ .

On  $\Sigma$  we have a conformal structure representable by the common conformal equivalence class of  $g$ . In terms of a local complex parameter  $z$ , the holomorphic conditions are expressed as follows

$$[\psi_z, \psi_{\bar{z}}, \psi_{zz}] = [\psi_z, \psi_{\bar{z}}, \psi_{\bar{z}\bar{z}}] = 0, \quad -i[\psi_z, \psi_{\bar{z}}, \psi_{z\bar{z}}] > 0,$$

where  $[X, Y, Z]$  denotes the determinant functional of any ordered triple of vectors  $X, Y, Z$ , subscripts are partial derivations with respect to the indicate parameters and by bar we denote the usual conjugation.

Using this parameter, the above affine invariants can be written as

$$\begin{aligned} g &= -\langle dN, d\psi \rangle = 2\rho|dz|^2, & \rho &= (-i[\psi_z, \psi_{\bar{z}}, \psi_{z\bar{z}}])^{\frac{1}{2}} \\ dA &= i\rho dz \wedge d\bar{z}, \\ \xi &= \frac{1}{\rho}\psi_{z\bar{z}}. \end{aligned} \tag{2.2}$$

Moreover

$$\begin{aligned} N &= \frac{-i}{\rho}\psi_z \times \psi_{\bar{z}}, \\ \psi_z &= iN \times N_z, \\ N_z &= i\xi \times \psi_z, \end{aligned} \tag{2.3}$$

where by  $\times$  we denote the cross product in  $\mathbb{C}^3$ . Furthermore, the metric factor  $\rho$  can be expressed in terms of the affine conormal as

$$\rho = -i[N, N_z, N_{\bar{z}}]. \tag{2.4}$$

On the other hand, [Bla2, Cal], the Euler-Lagrange equation for the affine area functional of locally strongly convex immersions

$$\int dA = \int K_e^{\frac{1}{4}} dA_e,$$

leads to the following system of PDE's:

$$\Delta N = 0.$$

So, when  $\Sigma$  is simply-connected,  $\frac{1}{2}N$  is locally the real part of a holomorphic curve  $\Phi : \Omega \subset \Sigma \rightarrow \mathbb{C}^3$  determined by  $\psi$  up to a real translation which satisfies

$$N = \Phi + \bar{\Phi} \tag{2.5}$$

$$0 > i[\Phi + \bar{\Phi}, \Phi_z, \bar{\Phi}_z] = -\rho. \tag{2.6}$$

Conversely, the expressions in (2.3) allow us (via the Lelievre formula, [Cal4, Li]), to recover  $\psi$  from its affine conormal field and the conformal class of the Blaschke metric,

$$\psi = 2 \operatorname{Re} \int i N \times N_z dz, \tag{2.7}$$

which, along with (2.5) and (2.6), say that  $\psi$  is uniquely determined, up to a real translation, by a holomorphic curve  $\Phi$  satisfying (2.6) (see also, [Cal3, Cal4]). To be precise,

$$\psi = 2 \operatorname{Re} \int i (\Phi + \bar{\Phi}) \times \Phi_z dz = -i \left( \Phi \times \bar{\Phi} - \int \Phi \times d\Phi + \int \bar{\Phi} \times d\bar{\Phi} \right).$$

### 3 The Affine Björling problem

Motivated from the classical Schwarz's formula we will obtain a local representation of affine maximal surfaces in terms of holomorphic data, which let us solve the "affine Björling problem" of finding affine maximal surfaces containing a prescribed analytic strip.

Consider  $\psi : \Sigma \rightarrow \mathbb{R}^3$  an affine maximal surface with Blaschke normal  $\xi$  and affine conormal  $N$ . Let  $I$  be an interval and  $\beta : I \rightarrow \Sigma$  a regular analytic curve. If  $\alpha = \psi \circ \beta$ ,  $Y = \xi \circ \beta$  and  $U = N \circ \beta$ , then, from (2.1), (2.2) and (2.3), we have that along the curve  $\alpha$

$$\left. \begin{aligned} 0 &= \langle \alpha', U \rangle, \\ 1 &= \langle Y, U \rangle, \\ 0 &= \langle Y', U \rangle, \\ 0 < \lambda &= -\langle \alpha', U' \rangle = \langle \alpha'', U \rangle, \end{aligned} \right\} \quad (3.1)$$

where by prime we indicate derivation respect to  $s$ , for all  $s \in I$ .

**Remark 1.** *From the fourth condition in (3.1) it is clear that  $\alpha''(s)$  does not vanish anywhere. Therefore, throughout this paper we will assume that the curves  $\alpha(s)$  have non zero curvature everywhere.*

Let  $\alpha, Y, U : I \rightarrow \mathbb{R}^3$  be regular analytic curves. We say the pair  $\{Y, U\}$  is an *analytic equiaffine normalization* of  $\alpha$  if there is an analytic positive function  $\lambda : I \rightarrow \mathbb{R}^+$  such that all the equations in (3.1) hold on  $I$ .

**Theorem 3.1.** *Let  $\{Y, U\}$  be an analytic equiaffine normalization of  $\alpha : I \rightarrow \mathbb{R}^3$ . Then there exists a unique affine maximal surface  $\psi$  containing  $\alpha(I)$  and such that the affine conormal field and the Blaschke normal along  $\alpha$  are  $U$  and  $Y$ , respectively.*

*We shall say that  $\psi$  is the affine maximal surface along  $\alpha$  generated by  $\{Y, U\}$ .*

*Proof.* Assume that  $\psi : \tilde{\Omega} \subseteq \mathbb{C} \rightarrow \mathbb{R}^3$  is an affine maximal surface containing  $\alpha(I)$  with  $U$  and  $Y$  their affine conormal field and Blaschke normal along  $\alpha$ . By the Inverse Function Theorem it is not difficult to prove (see [ACMi]) that there exists a conformal parameter  $z := s + it$ , with  $s \in I$ .

If  $N$  is the affine conormal field of  $\psi$ , then from (2.3) we have that, along  $\alpha$ ,

$$N_z = \frac{1}{2} (N_s - i N_t) = \frac{1}{2} (U' + i Y \times \alpha'),$$

and the Identity Principle shows that on a neighbourhood  $\Omega \subseteq \tilde{\Omega}$  of  $I$  in  $\mathbb{C}$ , the holomorphic curve  $N_z$  is given by

$$N_z = \frac{1}{2} (U_z + i Y \times \alpha_z), \quad z \in \Omega \quad (3.2)$$

where  $U(z)$ ,  $Y(z)$  and  $\alpha(z)$  denote the holomorphic extensions of the analytic curves  $U$ ,  $Y$  and  $\alpha$ , respectively. Thus, the immersion can be recovered from (2.7) and (3.2) in terms of  $U$ ,  $Y$  and  $\alpha$ , which proves the uniqueness.

For the existence, we consider the holomorphic curve  $\Phi : \Omega \rightarrow \mathbb{C}^3$  given by

$$\Phi(z) = \frac{1}{2} \left( U + i \int_{s_0}^z Y \times \alpha_\zeta d\zeta \right), \quad z \in \Omega, \quad s_0 \in I, \quad (3.3)$$

where  $\Omega$  is a simply-connected domain containing  $I$ , and  $U$ ,  $Y$  and  $\alpha$  are extended in a holomorphic way. If we set

$$\psi = \alpha(s_0) + 2 \operatorname{Re} \int_{s_0}^z i(\Phi + \bar{\Phi}) \times \Phi_\zeta d\zeta, \quad (3.4)$$

then

$$\left. \begin{aligned} \psi_z &= i(\Phi + \bar{\Phi}) \times \Phi_z \\ \psi_z \times \psi_{\bar{z}} &= [\Phi + \bar{\Phi}, \Phi_z, \bar{\Phi}_z] (\Phi + \bar{\Phi}) \\ \psi_{zz} &= i(\Phi + \bar{\Phi}) \times \Phi_{zz}, \\ \psi_{z\bar{z}} &= -i \Phi_z \times \bar{\Phi}_z, \end{aligned} \right\} \quad (3.5)$$

and

$$[\psi_z, \psi_{\bar{z}}, \psi_{zz}] = [\psi_z, \psi_{\bar{z}}, \psi_{z\bar{z}}] = 0, \quad i[\psi_z, \psi_{\bar{z}}, \psi_{zz}] = [\Phi + \bar{\Phi}, \Phi_z, \bar{\Phi}_z]^2. \quad (3.6)$$

From (3.1) and (3.3) we have that, along  $\alpha$ ,

$$\begin{aligned} -i[\Phi + \bar{\Phi}, \Phi_z, \bar{\Phi}_z] &= -\frac{i}{4} [U, U' + i Y \times \alpha', U' - i Y \times \alpha'] \\ &= -\frac{1}{2} \langle U \times U', Y \times \alpha' \rangle = \frac{\lambda}{2} > 0. \end{aligned}$$

This fact jointly with (3.5) and (3.6), proves that  $\psi$  is an immersion on a simply-connected neighbourhood of  $I$  and its conormal field  $N$  is given by  $N = \Phi + \bar{\Phi}$  which is an extension of  $U$  along  $\alpha$ .

Moreover, from (2.2), (3.1), (3.3) and (3.5), the Blaschke normal  $\xi$  of  $\psi$  along  $\alpha$  is given by

$$\xi = \frac{-i}{\rho} (\Phi_z \times \bar{\Phi}_z) = \frac{-1}{\lambda} (U' \times (Y \times \alpha')) = \frac{-1}{\lambda} \langle U', \alpha' \rangle Y = Y.$$

The proof is completed by showing that the immersion contains the curve  $\alpha(I)$ . But this is clear from (3.1), (3.3) and (3.5) because

$$\begin{aligned} \psi_z &= i U \times \Phi_z = \frac{-1}{2} U \times (Y \times \alpha') + \frac{i}{2} U \times U' \\ &= \frac{1}{2} \alpha' + \frac{i}{2} U \times U' \end{aligned} \quad (3.7)$$

along  $\alpha$ . □

**Corollary 3.2.** *Let  $\alpha, Y : I \rightarrow \mathbb{R}^3$  be two regular analytic curves satisfying*

$$[ Y', \alpha', Y ] [ Y', \alpha', \alpha'' ] > 0, \quad \text{on } I. \quad (3.8)$$

*Then there exists a unique affine maximal surface  $\psi$  containing the curve  $\alpha(I)$  and such that its Blaschke normal along  $\alpha$  is  $Y$ .*

*Moreover, the immersion  $\psi$  can be written as (3.4) in a simply-connected neighbourhood  $\Omega$  of  $I$  in  $\mathbb{C}$ , where the holomorphic curve  $\Phi$  is given by*

$$\Phi(z) = \frac{Y_z \times \alpha_z}{2[ Y_z, \alpha_z, Y ]} + \frac{i}{2} \int_{s_0}^z Y \times \alpha_\zeta d\zeta, \quad z \in \Omega, \quad s_0 \in I,$$

*$Y(z)$  and  $\alpha(z)$  being holomorphic extensions of  $Y$  and  $\alpha$ , respectively.*

*Proof.* From (3.1) and the condition (3.8), there exists a unique  $U$ ,

$$U = \frac{Y' \times \alpha'}{[ Y', \alpha', Y ]}, \quad (3.9)$$

such that the pair  $\{Y, U\}$  is an analytic equiaffine normalization of  $\alpha$ . Then, the result follows from Theorem 1, (3.3) and (3.9).  $\square$

**Remark 2.** *It is clear from the proof that if  $[ Y', \alpha', Y ] [ Y', \alpha', \alpha'' ] < 0$  on  $I$ , then there exists a unique affine maximal surface  $\psi$  containing the curve  $\alpha(I)$  and such that its Blaschke normal along  $\alpha$  is  $-Y$ .*

**Corollary 3.3.** *Let  $\alpha, Y : I \rightarrow \mathbb{R}^3$  be two regular analytic curves satisfying*

$$[ Y, \alpha', \alpha'' ] \neq 0, \quad Y' \times \alpha' = 0, \quad \text{on } I. \quad (3.10)$$

*Then, for a given positive analytic function  $\lambda : I \rightarrow \mathbb{R}^+$ , there exists a unique affine maximal surface  $\psi$  containing the curve  $\alpha(I)$ , such that its Blaschke normal along  $\alpha$  is  $Y$  and  $g(\alpha', \alpha') = \lambda$ .*

*Moreover, the immersion  $\psi$  can be written as (3.4) in a simply-connected neighborhood  $\Omega$  of  $I$  in  $\mathbb{C}$ , where now  $\Phi$  is given as*

$$\Phi(z) = \frac{(-\alpha_{zz} + \lambda Y) \times \alpha_z}{2[ \alpha_z, \alpha_{zz}, Y ]} + \frac{i}{2} \int_{s_0}^z Y \times \alpha_\zeta d\zeta, \quad z \in \Omega, \quad s_0 \in I,$$

*$Y(z)$ ,  $\alpha(z)$  and  $\lambda(z)$  being holomorphic extensions of  $Y$ ,  $\alpha$  and  $\lambda$ , respectively.*

*Proof.* From (3.1) and the condition (3.10), we can prove that there is a unique  $U$ ,

$$U = \frac{(-\alpha'' + \lambda Y) \times \alpha'}{[ \alpha', \alpha'', Y ]}, \quad (3.11)$$

such that the pair  $\{Y, U\}$  is an analytic equiaffine normalization of  $\alpha$ . Then the result is an easy consequence of Theorem 1, (3.3) and (3.11).  $\square$

Using Proposition 3.1 in [ACG] and the above Corollary, it follows:

**Corollary 3.4.** *If the Blaschke normal  $\xi$  of a connected affine maximal surface  $\psi : \Sigma \rightarrow \mathbb{R}^3$  is constant along an analytic curve  $\beta : I \rightarrow \Sigma$  and  $[\xi \circ \beta, (\psi \circ \beta)', (\psi \circ \beta)'] \neq 0$  in some point, then  $\xi$  is constant, that is,  $\psi$  is an improper affine sphere.*

**Remark 3.** *Let  $\alpha, Y : I \rightarrow \mathbb{R}^3$  be two regular analytic curves. From (3.1):*

1. *If  $Y' \times \alpha' \neq 0$  on  $I$ , then there exists an affine maximal surface  $\psi$  containing  $\alpha(I)$ , with Blaschke normal  $Y$  (resp.  $-Y$ ) along  $\alpha$  if, and only if,  $[Y', \alpha', Y][Y', \alpha', \alpha''] > 0$  (resp.  $[Y', \alpha', Y][Y', \alpha', \alpha''] < 0$ ).*
2. *If  $Y' \times \alpha' = 0$ ,  $[Y, \alpha', \alpha''] = 0$  and there is an affine maximal surface  $\psi$  containing  $\alpha(I)$  such that the Blaschke normal along  $\alpha$  is  $Y$ , then*

$$\alpha'' = \nu\alpha' + \lambda Y$$

*for  $\nu, \lambda$  analytic functions,  $\lambda > 0$ , and there exist infinitely many affine maximal surfaces containing  $\alpha(I)$ , with  $Y$  as Blaschke normal along  $\alpha$ . In fact, a pair  $\{Y, U\}$  is an analytic equiaffine normalization of  $\alpha$  if, and only if, the pair  $\{Y, U + \mu Y \times \alpha'\}$  is also an analytic equiaffine normalization of  $\alpha$ , for any analytic function  $\mu$ .*

## 4 Applications

Next, we will apply the affine Björling-type representation given in Section 3 via the formulas (3.3) and (3.4) in order to obtain the solution of the corresponding Cauchy problem, some symmetry properties, affine maximal surfaces containing a prescribed geodesic and affine maximal surfaces which are invariants under a one parametric group of equiaffine transformations.

### 4.1 The Cauchy problem

Consider  $\psi : \Omega \rightarrow \mathbb{R}^3$  the graph of a locally strictly convex function  $\phi(x, y)$ ,  $(x, y)$  in a planar simply-connected domain  $\Omega$ . The Euler-Lagrange equation for the affine area functional

$$\mathcal{A}(\phi) = \int (\det(\nabla^2\phi))^{1/4} dx dy = \int K_e^{1/4} dA_e,$$

is the following fourth order non-linear equation

$$\phi_{yy}\omega_{xx} - 2\phi_{xy}\omega_{xy} + \phi_{xx}\omega_{yy} = 0, \quad \omega = (\det(\nabla^2\phi))^{-3/4}.$$

In this situation one can check that the Berwald-Blaschke metric, the Blaschke normal and the affine conormal field of  $\psi$  are given by

$$\begin{aligned} g_\phi &= \sqrt[3]{\omega} (\phi_{xx} dx^2 + 2\phi_{xy} dx dy + \phi_{yy} dy^2), \\ \xi &= \left( \varphi_y, -\varphi_x, \frac{1}{\sqrt[3]{\omega}} - \phi_y \varphi_x + \phi_x \varphi_y \right), \\ N &= \sqrt[3]{\omega} (-\phi_x, -\phi_y, 1), \end{aligned} \quad (4.1)$$

where

$$\varphi_x = \frac{1}{3} (\phi_{xy} \omega_x - \phi_{xx} \omega_y), \quad \varphi_y = \frac{1}{3} (\phi_{yy} \omega_x - \phi_{xy} \omega_y).$$

Using the above expression and Theorem 3.1 we can solve the Cauchy problem for the equation of an affine maximal surface

$$\begin{cases} \phi_{yy} \omega_{xx} - 2\phi_{xy} \omega_{xy} + \phi_{xx} \omega_{yy} = 0, & \omega = (\det(\nabla^2 \phi))^{-3/4} \\ \phi(x, 0) = a(x), \\ \phi_y(x, 0) = b(x), \\ \phi_{yy}(x, 0) = c(x), & c(x)a''(x) - b'(x)^2 > 0 \\ \phi_{yyy}(x, 0) = d(x), \end{cases} \quad (4.2)$$

where  $a, b, c, d$  are analytic functions defined on an interval  $I$ , and  $\phi$  is defined on a simply-connected planar domain  $\Omega$  containing  $I \times \{0\}$ . We are assuming that  $c(x)a''(x) - b'(x)^2 > 0$  because  $\det(\nabla^2 \phi)$  must be positive. In particular, changing the orientation if necessary, we can also assume that  $a''(x) > 0$  on  $I$ .

From (4.1) and Theorem 3.1 it follows easily the following

**Theorem 4.1.** *There exists a unique solution  $\phi(x, y)$  to the Cauchy Problem (4.2) such that*

$$(x, y, \phi(x, y)) = (s_0, 0, a(s_0)) + 2 \operatorname{Re} \int_{s_0}^z i (\Phi + \bar{\Phi}) \times \Phi_\zeta d\zeta, \quad z = s + it,$$

where  $\Phi$  is the holomorphic extension of the analytic curve

$$\Phi(s) = \frac{1}{2} \left( U(s) + i \int_{s_0}^s Y(u) \times A(u) du \right),$$

being

$$\begin{aligned} U(s) &= (c(s)a''(s) - b'(s)^2)^{-1/4} (-a'(s), -b(s), 1), \\ A(s) &= (1, 0, a'(s)), \\ Y(s) &= \frac{1}{4} (c(s)a''(s) - b'(s)^2)^{-7/4} (b'(da'' + 3cb'') - 2b'^2c' - c(c'a'' + ca'''), \\ &\quad b'(3c'a'' + ca''') - 2b'^2b'' - a''(da'' + cb''), \\ &\quad + 4b'^4 - 2b'^2(a'c' + 4ca'' + bb'') - a''((-4c^2 + bd)a'' + bcb'') \\ &\quad - ca'(c'a'' + ca''') + b'(a'(da'' + 3cb'') + b(3c'a'' + ca'''))). \end{aligned}$$

## 4.2 Symmetry and geodesics

Consider  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  the equiaffine transformation given by

$$T(v) = Av + b, \quad v \in \mathbb{R}^3$$

where  $A \in \text{SL}(3, \mathbb{R})$  is a  $3 \times 3$  matrix with determinant 1, and  $b \in \mathbb{R}^3$  is a fixed vector. Given an analytic equiaffine normalization  $\{Y, U\}$  of an analytic regular curve  $\alpha : I \rightarrow \mathbb{R}^3$ , we will say that  $T$  is a symmetry of the equiaffine normalization if there exists an analytic diffeomorphism  $\Gamma : I \rightarrow I$  such that  $\alpha \circ \Gamma = T \circ \alpha$ ,  $Y \circ \Gamma = AY$  and  $U \circ \Gamma = (A^t)^{-1}U$ .

The following result is a generalization of Theorem 4.2 in [ACMi] and it can be proved analogously to the corresponding ones in [GMi2]:

**Theorem 4.2.** (*Generalized symmetry principle*). *Any symmetry of an analytic equiaffine normalization induces a global symmetry of the affine maximal surface generated by the equiaffine normalization.*

The results in Section 3 let us also characterize when curves in  $\mathbb{R}^3$  can be geodesics or pre-geodesics of affine maximal surfaces. Indeed, we have

**Theorem 4.3.** *Let  $\Sigma$  be a Riemann surface and  $\psi : \Sigma \rightarrow \mathbb{R}^3$  an affine maximal surface with Blaschke normal  $\xi$  and affine conormal  $N$ . If  $\beta : I \rightarrow \Sigma$  is a regular analytic curve from an interval  $I$ ,  $\alpha = \psi \circ \beta$ ,  $Y = \xi \circ \beta$  and  $U = N \circ \beta$ , then  $\alpha$  is a pre-geodesic for the Blaschke metric if and only if*

$$[\alpha', \alpha'', Y] + [U, U', U''] = 0 \quad \text{on } I. \quad (4.3)$$

*Proof.* As we have seen in Theorem 3.1, there exists a conformal parameter  $z = s + it$  for the Berwald-Blaschke metric  $g$ , defined in a neighbourhood containing  $I$ , and such that  $\psi(s, 0) = \alpha(s)$ .

It is well-known that  $\alpha$  is a pregeodesic if, and only if,  $\nabla_{\alpha'(s)}\alpha'(s)$  is proportional to  $\alpha'(s)$ , where  $\nabla$  is the Levi-Civita connection of  $g$ , or equivalently,

$$0 = g \left( \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right) = \frac{1}{2} \frac{\partial}{\partial t} g \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial s} \right) = -\frac{\partial}{\partial t} g \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right)$$

along  $\alpha(s)$ . That is, the imaginary part of

$$\frac{\partial}{\partial z} g \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right)$$

vanishes identically for all  $z = s \in I$ . But, from (2.5) and (2.6),

$$\begin{aligned} \frac{\partial}{\partial z} g \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right) &= i [N, N_{zz}, N_{\bar{z}}] = \frac{i}{4} [U, U'' + i(Y' \times \alpha' + Y \times \alpha''), U' - iY \times \alpha'] \\ &= -\frac{1}{4} ([U, U'', Y \times \alpha'] + [U, Y' \times \alpha' + Y \times \alpha'', U']) \\ &\quad -\frac{i}{4} ([\alpha', \alpha'', Y] + [U, U', U'']) \end{aligned}$$

along  $\alpha$ , where we are using (3.2) and taking  $U(s) = N(s, 0)$ . Then, the result is clear.  $\square$

As a consequence we have

**Corollary 4.4.** *Let  $I$  be an interval and  $\alpha : I \rightarrow \mathbb{R}^3$  be a regular analytic curve. Then  $\alpha$  is the geodesic of some affine maximal surface if and only if there exists an equiaffine normalization  $\{Y, U\}$  of  $\alpha$  satisfying (4.3) and  $\langle \alpha'', U \rangle = c$  for a positive constant  $c$ .*

**Planar geodesics or pre-geodesics:** Let us take a planar analytic curve  $\alpha(s)$  whose curvature  $k(s)$  does not vanish at any point. Let us call  $\Pi$  the plane where  $\alpha$  is contained. If we choose  $\{Y, U\}$  an analytic equiaffine normalization of  $\alpha$  such that both  $Y$  and  $U$  are also contained in  $\Pi$ , then the condition (4.3) is fulfilled trivially and so  $\alpha$  is a pre-geodesic (geodesic if  $\langle \alpha'', U \rangle$  is a positive constant) of the corresponding affine maximal surface given in Theorem 3.1 which, by Theorem 4.2, will have  $\Pi$  as a plane of symmetry. Observe that it is always possible to choose a such equiaffine normalization of  $\alpha$ . For instance we can take  $Y = U = n$ , where  $n(s)$  is the unit normal vector field of  $\alpha(s)$ . In this case,  $\alpha(s)$  is a geodesic as long as  $\alpha$  has constant curvature  $k_0 = \langle \alpha'', U \rangle$ .

Thus we have

**Corollary 4.5.** *Every planar analytic curve whose curvature does not vanish at any point is pre-geodesic of an affine maximal surface which has the plane containing the curve as a symmetry plane.*

**Remark 4.** *The curve  $\alpha(s) = (\cos(s), \sin(s), 0)$  cannot be the geodesic of an improper affine sphere (see [ACG]). However,  $\alpha$  is geodesic of a large family of affine maximal surfaces. In fact, from (3.1) one deduces that if  $\{Y, U\}$  is an analytic equiaffine normalization of  $\alpha$ , then*

$$U = (-c \cos(s), -c \sin(s), \mu(s)) \quad (4.4)$$

for some regular analytic function  $\mu$  and a positive constant  $c$ . Thus, if  $Y = (Y_1, Y_2, Y_3)$  then from (4.3) it follows that

$$Y_3 = -\det(U, U', U'') = -c^2(\mu + \mu'')$$

and using again (3.1) we conclude that

$$Y = \frac{1}{c}(\cos(s)(\mu Y_3 - 1) - \sin(s)\mu' Y_3, \cos(s)\mu' Y_3 + \sin(s)(\mu Y_3 - 1), c Y_3). \quad (4.5)$$

The expressions (4.4) and (4.5) give a wide family of analytic equiaffine normalizations of  $\alpha$  such that the corresponding affine maximal surfaces which they generate have  $\alpha$  as a geodesic.

**Non planar geodesics or pregeodesics:** Now, let us take an analytic curve  $\alpha(s)$  whose curvature  $k(s)$  and torsion  $\tau(s)$  do not vanish at any point. If we take  $Y(s)$  as the unit normal vector field  $n(s)$  of  $\alpha(s)$ , we have that

$$[Y', \alpha', Y] = -\tau \neq 0 \quad \text{and} \quad [Y', \alpha', \alpha''] = -k\tau \neq 0$$

and so (3.8) is satisfied. Then there exists a unique affine maximal surface containing the curve  $\alpha(s)$  such that its Blaschke normal along  $\alpha$  is  $Y$ , and the affine conormal is

$$U = \frac{Y' \times \alpha'}{[Y', \alpha', Y]} = n.$$

It is easy to check that (4.3) is satisfied if and only if  $k/\tau$  is constant, that is, if  $\alpha$  is a helix.

In particular

**Corollary 4.6.** *Every analytic helix is pre-geodesic of an affine maximal surface.*

### 4.3 Helicoidal Affine Maximal Surfaces

Here we shall show how to obtain the affine maximal surfaces which are invariant under a one-parametric group of equiaffine transformations.

We are going to identify the group  $\mathcal{A}$  of equiaffine transformation of  $\mathbb{R}^3$  with a subgroup of matrices of  $\mathrm{SL}(4, \mathbb{R})$  in the following way:  $T(v) = Av + b$  will be identified to the matrix  $\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(4, \mathbb{R})$ . Under this identification,

$$\mathcal{A} = \left\{ \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} : A \in \mathrm{SL}(3, \mathbb{R}), b \in \mathbb{R}^3 \right\}$$

and its Lie algebra  $\mathfrak{a}$  is given by

$$\mathfrak{a} = \left\{ \begin{pmatrix} C & d \\ 0 & 0 \end{pmatrix} : \mathrm{Trace} C = 0, d \in \mathbb{R}^3 \right\}.$$

Since the one-parameter groups of equiaffine transformations are obtained as  $\exp(sG)$ ,  $s \in \mathbb{R}$ ,  $G \in \mathfrak{a}$ , the Jordan matrix decomposition Theory let us obtain the following result:

**Proposition 4.7.** *Up to a conjugation in  $\mathcal{A}$ , the one-parametric groups of equiaffine transformations can be identified to the following subgroups of  $\mathrm{SL}(4, \mathbb{R})$ :*

$$\begin{aligned}
G_{1,a} &= \begin{pmatrix} 1 & as & \frac{as^2}{2} & \frac{as^3}{6} \\ 0 & 1 & s & \frac{s^2}{2} \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{pmatrix} & G_{2,a} &= \begin{pmatrix} \cos(s) & \sin(s) & 0 & 0 \\ -\sin(s) & \cos(s) & 0 & 0 \\ 0 & 0 & 1 & as \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
G_{3,a} &= \begin{pmatrix} e^s & 0 & 0 & 0 \\ 0 & e^{-s} & 0 & 0 \\ 0 & 0 & 1 & as \\ 0 & 0 & 0 & 1 \end{pmatrix} & G_{4,a} &= \begin{pmatrix} 1 & as & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
G_{5,a} &= \begin{pmatrix} e^{as} & e^{as}s & 0 & 0 \\ 0 & e^{as} & 0 & 0 \\ 0 & 0 & e^{-2as} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & G_{6,a} &= \begin{pmatrix} e^{as} & 0 & 0 & 0 \\ 0 & e^s & 0 & 0 \\ 0 & 0 & e^{-as-s} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
G_{7,a} &= \begin{pmatrix} e^{as} \cos(s) & e^{as} \sin(s) & 0 & 0 \\ -e^{as} \sin(s) & e^{as} \cos(s) & 0 & 0 \\ 0 & 0 & e^{-2as} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

where  $a \in \mathbb{R}$ .

Let  $T_s = \begin{pmatrix} A_s & b_s \\ 0 & 1 \end{pmatrix}$ ,  $s \in \mathbb{R}$ , be a one-parametric subgroup of  $\mathcal{A}$ , which can be seen as an affine transformation in  $\mathbb{R}^3$  as

$$T_s(v) = A_s v + b_s,$$

where  $A_s \in \mathrm{SL}(3, \mathbb{R})$  and  $b_s \in \mathbb{R}^3$ .

From Theorem 3.1 and Theorem 4.2, we know that an affine maximal surface invariant under  $T_s$ ,  $s \in \mathbb{R}$ , is locally given as the surface generated by the following  $\{T_s\}$ -symmetric analytic equiaffine normalization  $\{Y, U\}$ , along the orbit  $\alpha_p(s) = T_s(p)$  of a fixed point  $p$ , where  $Y(s) = A_s Y_p$ ,  $U(s) = (A_s^t)^{-1} U_p$  and  $Y_p, U_p \in \mathbb{R}^3$  satisfy the necessary conditions for (3.1) holds. It is remarkable that in this situation the Berwald-Blaschke metric must be constant along  $\alpha_p$ .

From this fact, the local classification of helicoidal affine maximal surfaces, that is, affine maximal surfaces which are invariant under a one-parametric group of equiaffine transformations, comes from applying Theorem 3.1 to the study of the orbits of a point  $p$  under the groups described in Proposition 4.7 and the corresponding symmetric analytic equiaffine normalization. As the process involves straightforward computations, we are going to apply our representation to classify the affine maximal surfaces invariant under the two first groups in Proposition 4.7. The total classification can be done analogously.

•  $G_{1,a}$ -Invariant affine maximal surfaces:

In this case the orbit of a point  $p = (p_1, p_2, p_3)$  is given by

$$\alpha_p(s) = \left( p_1 + p_2as + p_3a\frac{s^2}{2} + a\frac{s^3}{6}, p_2 + p_3s + \frac{s^2}{2}, p_3 + s \right)$$

and the Blaschke normal along  $\alpha_p$  can be written as

$$Y_p(s) = \left( y_1 + y_2as + y_3a\frac{s^2}{2}, y_2 + y_3s, y_3 \right).$$

Thus, up to a change of parameter, it is clear from this expression that we can assume either  $y_2 = 0$  or  $y_3 = 0$ .

**Case I:**  $y_2 = 0 \neq y_3$ . In this case  $Y'_p \times \alpha'_p \neq 0$  on  $\mathbb{R}$  and from Remark 3, there exists the corresponding affine maximal surface if and only if

$$[Y'_p, \alpha'_p, Y_p][Y'_p, \alpha'_p, \alpha''_p] = ap_3y_3^2(y_1 - ap_2y_3) > 0.$$

Under this assumption one can obtain from Corollary 3.2 and (3.4) that

$$U_p(s) = \frac{1}{y_4} \left( -1, as, ap_2 - \frac{as^2}{2} \right),$$

and the corresponding helicoidal affine maximal surface is given, up to a translation, by

$$\begin{aligned} \psi(s, t) = & \left( p_2as + p_3a\frac{s^2 + t^2}{2} + a\frac{s^3}{6} - as\frac{t^2}{2} + \frac{1}{30}a^2p_3^2y_3^2(t^5 + 5s^2t^3 + 10p_2t^3) \right. \\ & + \frac{a^2}{6y_4^2}(t^3 + 3s^2t + 6p_2t) + \frac{a^2p_3y_3}{6y_4}(-t^4 + 3s^2t^2 - 6p_2t^2) + \frac{1}{3}ap_3y_3y_4t^3(s - p_3) \\ & + \frac{1}{3}t^3y_4^2, p_3s + \frac{1}{2}(s^2 - t^2) + \frac{1}{3}p_3y_3t^3(ap_3y_3s + y_4) + \frac{ast}{y_4^2}(1 - y_4p_3y_3t), \\ & \left. s + \frac{a(-1 + p_3ty_3y_4)^3}{3p_3y_3y_4^3} \right), \end{aligned}$$

where  $y_4 = -y_1 + ap_2y_3$ . From (2.2), the Berwald-Blaschke metric in this case is given by

$$g = P_1[t] (ds^2 + dt^2),$$

where

$$\begin{aligned} P_1[t] = & -\frac{ap_3}{y_4} + \left( ap_3^2y_3 - y_4 - \frac{a^2}{y_4^3} \right) t + \left( \frac{2a^2p_3y_3}{y_4^2} \right) t^2 \\ & - \left( \frac{4a^2p_3^2y_3^2}{3y_4} \right) t^3 + \left( \frac{a^2p_3^3y_3^3}{3} \right) t^4. \end{aligned}$$

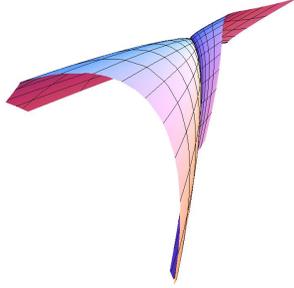


Figure 1: helicoidal affine maximal surface  $G_{1,a}$ -invariant.

Thus,  $g$  degenerates along  $\beta_{t_1}(s) = (s, t_1)$ , for any root  $t_1$  of  $P_1[t]$ .

The Figure 1, gives a representation of  $\psi$  when  $p_1 = 0$ ,  $p_2 = 0$ ,  $p_3 = 1$ ,  $y_1 = 1$ ,  $y_3 = 1$ , and  $a = 1$ .

**Case II:**  $y_3 = 0$ . In this case,  $Y_p = (y_1 + asy_2, y_2, 0)$ . We distinguish two subcases:

If  $a = 0$  (or  $y_2 = 0$ ),  $Y_p$  is constant along  $\alpha_p$  and from (3.1) we can write, up to a translation,

$$\begin{aligned}\alpha_p(s) &= \left(0, p_2 - \frac{us}{\lambda} + \frac{s^2}{2}, -\frac{u}{\lambda} + s\right), \\ Y_p(s) &= \left(y, \frac{1 - yu_1}{\lambda}, 0\right), \\ U_p(s) &= (u_1, \lambda, -s\lambda + u)\end{aligned}$$

and then, from (3.3) and (3.4), the corresponding  $G_{1,a}$ -invariant affine maximal surface is given by

$$\psi(s, t) = \left(\lambda^2 t + \lambda t^2 y + \frac{t^3 y^2}{3} \frac{s^2 + t^2}{2} - \lambda t u_1 - \frac{su}{\lambda} + \frac{t^3 y}{3\lambda} - t^2 u_1 y - \frac{t^3 u_1 y^2}{3\lambda}, s\right).$$

In this case the Berwald-Blaschke metric is given by

$$g = (\lambda + ty)(ds^2 + dt^2)$$

and all the surfaces are  $G_{1,0}$ -invariant improper affine spheres (see Figure 2)

If  $a \neq 0$ , then

$$Y'_p \times \alpha'_p \neq 0$$

on  $\mathbb{R}$  and again, from Remark 3, the corresponding affine maximal surface exists if and only if

$$[Y'_p, \alpha'_p, Y_p][Y'_p, \alpha'_p, \alpha''_p] = a^2 y_2^3 > 0.$$

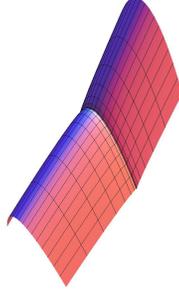


Figure 2:  $G_{1,0}$ -invariant improper affine sphere

So,  $a \neq 0$ ,  $y_2 > 0$  and from Corollary 3.2 and (3.4),

$$U_p(s) = \frac{1}{y_2} (0, 1, -p_3 - s),$$

and up to a translation, the helicoidal affine maximal surface is given by

$$\begin{aligned} \psi(s, t) = & \left( p_2 a s + p_3 a \frac{s^2 - t^2}{2} + a \frac{s^3}{6} + a s \frac{t^2}{2} + \frac{t}{y_2} + \frac{t^2 y_1}{y_2} + \frac{1}{30} a^2 t^5 y_2^2 \right. \\ & + \frac{t^3}{6} (2y_1^2 - 2ap_3 y_1 y_2 + 2a s y_1 y_2 + 2a^2 p_2 y_2^2 + a^2 s^2 y_2^2), \\ & \left. p_3 s + \frac{s^2 + t^2}{2} + \frac{t^3}{3} (y_1 y_2 + a s y_2^2) s + \frac{a t^3}{3} y_2^2 \right). \end{aligned}$$

In this case the Berwald-Blaschke metric is given by

$$g = P_2[t] (ds^2 + dt^2),$$

where

$$P_2[t] = \frac{1}{y_2} + (y_1 - ap_3 y_2) t - \left( \frac{1}{3} a^2 y_2^3 \right) t^4.$$

Thus,  $g$  degenerates along the curves  $\beta_{t_2}(s) = (s, t_2)$ , for any root  $t_2$  of  $P_2[t]$ .

The Figure 3, gives a representation of  $\psi$  when  $p_1 = p_2 = p_3 = y_1 = 0$ ,  $a = y_2 = 1$ .

- **$G_{2,a}$ -invariant affine maximal surfaces.**

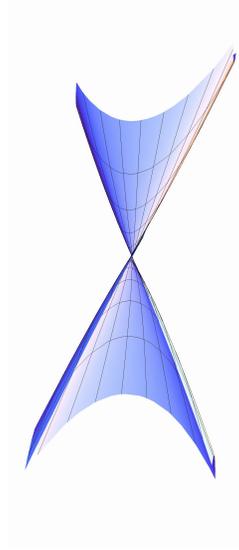


Figure 3: helicoidal affine maximal surface  $G_{1,a}$ -invariant.

For this one-parametric group the orbit of a point  $p = (p_1, p_2, p_3)$  is given by

$$\alpha_p(s) = (p_1 \cos(s) + p_2 \sin(s), -p_1 \sin(s) + p_2 \cos(s), p_3 + as),$$

and every  $G_{2,a}$ -symmetric analytic equiaffine normalization  $\{Y_p, U_p\}$  along  $\alpha_p$  can be written as

$$\begin{aligned} Y_p(s) &= (y_1 \cos(s) + y_2 \sin(s), -y_1 \sin(s) + y_2 \cos(s), y_3), \\ U_p(s) &= (u_1 \cos(s) + u_2 \sin(s), -u_1 \sin(s) + u_2 \cos(s), u_3). \end{aligned}$$

Up to a change of parameter if necessary, it is clear from these expressions that we can assume without loss of generality, that  $y_2 = 0$ . Thus from (3.1),

$$\begin{aligned} y_1 u_1 + y_3 u_3 &= 1, & p_2 u_1 - p_1 u_2 &= -a u_3, \\ p_1 u_1 + p_2 u_2 &= -\lambda, & u_2 y_1 &= 0. \end{aligned} \tag{4.6}$$

After applying the equiaffine transformation

$$T = \begin{pmatrix} \frac{u_1}{\sqrt{u_1^2 + u_2^2}} & \frac{u_2}{\sqrt{u_1^2 + u_2^2}} & 0 & 0 \\ \frac{-u_2}{\sqrt{u_1^2 + u_2^2}} & \frac{u_1}{\sqrt{u_1^2 + u_2^2}} & 0 & 0 \\ 0 & 0 & 1 & -p_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and using (4.6), one has,

$$\begin{aligned} \alpha_p(s) &= \left( -\frac{\lambda}{u} \cos(s), \frac{\lambda}{u} \sin(s), as \right), \\ Y_p(s) &= \left( \frac{1 - u_3 y_3}{u} \cos(s), -\frac{1 - u_3 y_3}{u} \sin(s), y_3 \right), \\ U_p(s) &= (u \cos(s), -u \sin(s), u_3), \end{aligned}$$

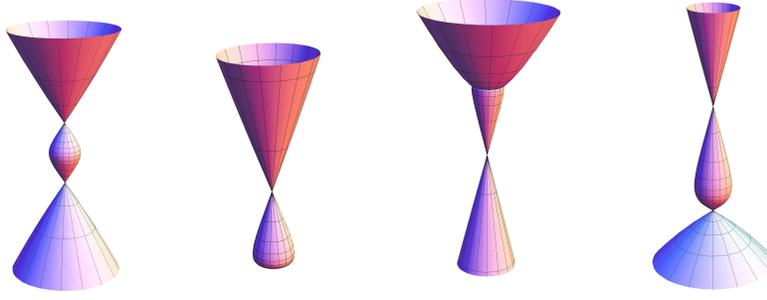


Figure 4: Rotational affine maximal surfaces.

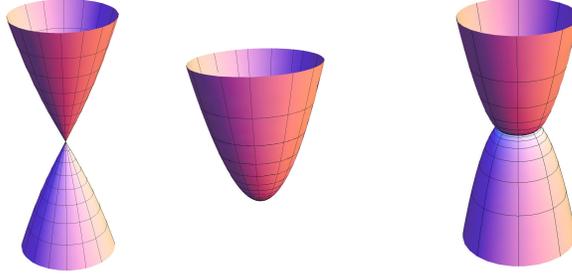


Figure 5: Rotational improper affine spheres.

where  $u = \sqrt{u_1^2 + u_2^2}$  and  $au_3 = 0$ .

If  $a = 0$  we obtain the **Rotational affine maximal surfaces** and, from Theorem 3.1, they can be described up to an equiaffine transformation as

$$\psi(s, t) = \left( R(t) \cos(s), R(t) \sin(s), \frac{1}{4u^2} (2tu^4 - 2\lambda^2 ty_3^2 + 2\lambda u^2 y_3 \cosh(2t) + (u^4 + \lambda^2 y_3^2) \sinh(2t)) \right),$$

where

$$R(t) = \frac{1}{u^3} \left( \cosh(t) (\lambda (u^2 + \lambda t y_3 (-1 + u_3 y_3))) + \sinh(t) (u_3 u^4 + \lambda t u^2 (-1 + u_3 y_3) + \lambda^2 (y_3 - u_3 y_3^2)) \right).$$

In this case the Berwald-Blaschke metric is given by  $g = E(t)(ds^2 + dt^2)$ , where

$$E(t) = \frac{1}{2u^4} \left( \lambda u^2 (u^2 + \lambda t y_3 (-1 + u_3 y_3)) \cosh(t)^2 + (u_3 u^6 + \lambda^2 u^2 y_3 (2 - u_3 y_3) + \lambda t u^4 (-1 + u_3 y_3) + \lambda^3 t y_3^2 (-1 + u_3 y_3)) \cosh(t) \sinh(t) + \lambda y_3 (u_3 u^4 + \lambda^2 y_3 (1 - u_3 y_3) + \lambda t u^2 (-1 + u_3 y_3)) \sinh(t)^2 \right).$$

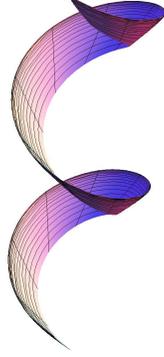


Figure 6: Non rotational  $G_{2,a}$ -invariant affine maximal surfaces

A straightforward computation let us see that  $E(t)$  vanishes at most for two values  $t_1, t_2 \in \mathbb{R}$ , and consequently, the Berwald-Blaschcke metric degenerates along  $\beta_i(s) = (s, t_i)$ ,  $i = 1, 2$  (see Figures 4 and 5).

When  $u_3 = 0$  and  $a \neq 0$  we obtain  $G_{2,a}$ -invariant affine maximal surfaces which are not rotational (see Figure 6). In this case, again from Theorem 3.1, the immersion  $\psi = (\psi_1, \psi_2, \psi_3)$  is given up to an equiaffine transformation by

$$\begin{aligned} \psi_1(s, t) &= \frac{\lambda}{u^3} (\cosh(t) ((-u^2 + \lambda t y_3) \cos(s) + at \sin(s)) \\ &\quad + \sinh(t) ((tu^2 - \lambda y_3) \cos(s) - a\lambda \sin(s)), \\ \psi_2(s, t) &= \frac{\lambda}{u^3} (\cosh(t) ((u^2 - \lambda t y_3) \sin(s) + at \cos(s)) \\ &\quad + \sinh(t) ((-tu^2 + \lambda y_3) \sin(s) - a\lambda \cos(s)), \\ \psi_3(s, t) &= \frac{1}{4u^2} (-2a^2t + 4asu^2 + 2tu^4 - 2\lambda^2 t y_3^2 + 2\lambda u^2 y_3 \cosh(2t) \\ &\quad + (a^2 + u^4 + \lambda^2 y_3^2) \sinh(2t)). \end{aligned}$$

## 5 A class of affine maximal surfaces with singularities

Helicoidal examples show the existence of an important amount of affine maximal surfaces glued by analytic curves where the affine metric is degenerated but the affine normal and the affine conormal are well defined.

In other words, they can be represented as in (3.4), where  $\Phi$  is a well-defined holomorphic regular curve on the Riemann surface  $\Sigma$  such that  $[\Phi + \bar{\Phi}, \Phi_z, \bar{\Phi}_z]|dz|^2$  vanishes only on some analytic curves.

When a map  $\psi : \Sigma \longrightarrow \mathbb{R}^3$  admits a representation as in (3.4) for a certain holomorphic curve  $\Phi$  which satisfies that  $[\Phi + \bar{\Phi}, \Phi_z, \bar{\Phi}_z]|dz|^2$  does not vanish identically, we say that  $\psi$  is an *affine maximal map*. A deeper study of this class of surfaces with singularities can be found in [AMaM2]. Their study may be motivated by the following two results:

**Theorem 5.1.** *Let  $I$  be an interval and  $\alpha : I \longrightarrow \mathbb{R}^3$  a regular analytic curve with non-vanishing torsion. Then, for any non-vanishing regular analytic function  $h : I \longrightarrow \mathbb{R}$  there exists a unique affine maximal map  $\psi_h$  containing  $\alpha(I)$  in its set of singularities.*

*Proof.* Assume that  $\psi : \Sigma \rightarrow \mathbb{R}^3$  is an affine maximal map which can be represented as in (3.4) for some holomorphic curve  $\Phi : \Sigma \longrightarrow \mathbb{C}^3$ , with  $\alpha(I)$  in its singular set.

From (3.5) and (3.7) the affine conormal  $N = \Phi + \bar{\Phi}$  of  $\psi$  does not vanish along  $\alpha$ . We shall denote by  $U$  and  $Y$  the affine conormal field and Blaschke normal along  $\alpha$ , respectively. As in Theorem 3.1, we can take a conformal parameter  $z := s + it$ , with  $s \in I$ .

As  $\alpha$  is a curve of singularities, from (3.1) we have that

$$0 = -\langle \alpha', U' \rangle = \langle \alpha'', U \rangle. \quad (5.1)$$

Since the torsion of  $\alpha$  does not vanish,  $[\alpha', \alpha'', \alpha'''] \neq 0$  on  $I$ . From (3.1), (5.1) and (2.3) we have that

$$\begin{aligned} N(s, 0) &= U(s) = h(s)\alpha'(s) \times \alpha''(s), \\ N_t(s, 0) &= -Y(s) \times \alpha'(s), \end{aligned}$$

where  $h(s)$  is a non-vanishing regular analytic function, and so

$$N_s(s, 0) \times N_t(s, 0) = 0. \quad (5.2)$$

Since the affine normal  $\xi$  of  $\psi$  satisfies that  $\langle N, \xi \rangle = 1$ , we get

$$Y(s) = \xi(s, 0) = \frac{N_s \times N_t}{[N_s, N_t, N]}(s, 0) = \frac{0}{0}.$$

By applying L'Hopital Theorem, and taking into account that  $N_{st} = N_{ts}$  and  $N_{tt} = -N_{ss}$  (because  $N$  is harmonic), we have

$$\xi(s, 0) = \frac{N_{ts} \times N_t - N_s \times N_{ss}}{[N_{ts}, N_t, N] - [N_s, N_{ss}, N]}(s, 0). \quad (5.3)$$

On the other hand, from (5.2) we can write  $N_t(s, 0) = m(s)N_s(s, 0)$  for a differentiable function  $m(s)$ . Hence

$$N_{ts}(s, 0) = m'(s)N_s(s, 0) + m(s)N_{ss}(s, 0)$$

and so (5.3) becomes

$$Y(s) = \xi(s, 0) = \frac{N_{ss} \times N_s}{[N_{ss}, N_s, N]}(s, 0)$$

which gives an analytic curve, because

$$[N_{ss}, N_s, N](s, 0) = h(s)^3[\alpha', \alpha'', \alpha''']^2(s) \neq 0.$$

As in the proof of Theorem 3.1 the Identity Principle shows that, on a neighborhood  $\Omega \subseteq \Sigma$ ,  $N_z$  is given as in (3.2) and the immersion can be recovered from (2.7) in terms of the holomorphic extensions of the analytic curves  $U$ ,  $Y$  and  $\alpha$ , which proves the uniqueness and the existence.  $\square$

**Theorem 5.2.** *Let  $I$  be an interval and  $\alpha : I \rightarrow \mathbb{R}^3$  a planar regular analytic curve. Then, for any regular analytic function  $h : I \rightarrow \mathbb{R}$  there exists a unique affine maximal map  $\psi_h$  containing  $\alpha(I)$  in its set of singularities.*

*Proof.* Assume that  $\psi : \tilde{\Omega} \subseteq \mathbb{C} \rightarrow \mathbb{R}^3$  is an affine maximal surface containing  $\alpha(I)$  as a curve of singularities, with  $U$  and  $Y$  their affine conormal field and Blaschke normal along  $\alpha$ . Again, we can take a conformal parameter  $z := s + it$ , with  $s \in I$ .

Since  $\alpha(s)$  is planar, we can assume without loss of generality that  $\alpha(s) = (f(s), g(s), 0)$ . Then, from (3.1) we can take

$$U(s) = (0, 0, 1), \quad Y(s) = (Y_1, Y_2, 1).$$

Consequently

$$N_t(s, 0) = -Y(s) \times \alpha'(s) = (g'(s), -f'(s), h(s)), \quad h(s) = (f'Y_2 - g'Y_1)(s),$$

where  $N$  is the affine conormal field of  $\psi$ .

Now, by applying L'Hopital we get

$$\xi(s, 0) = \left( \frac{f'h' - f''h}{g'f'' - g''f'}, \frac{g'h' - g''h}{g'f'' - g''f'}, 1 \right) (s)$$

and reasoning as in Theorem 5.1 we finish the proof.  $\square$

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