Non-Removable Singularities of a Fourth Order Nonlinear Partial Differential Equation[1]

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Abstract

The aim of this paper is to give a classification of the solutions $\phi = \phi(x, y)$ of the fourth order nonlinear equation $\phi_{yy}\rho_{xx} - 2\phi_{xy}\rho_{xy} + \phi_{xx}\rho_{yy} = 0$, $\rho = \left(det\left(\nabla^2\phi\right)\right)^{-3/4}$ in a punctured domain which are $\mathcal{C}^2$ at the singularity. We prove that this kind of solutions either have a removable singularity or they are asymptotic to rotationally symmetric solutions near the singularity. A geometric description of solutions which are not $\mathcal{C}^1$ at the singularity is also given.

1 Introduction

In this work we apply methods suggested by Differential Geometry and Complex Analysis to study isolated singularities of strictly convex solutions of the following fourth order nonlinear equation

$$L[\phi] := \phi_{yy}\rho_{xx} - 2\phi_{xy}\rho_{xy} + \phi_{xx}\rho_{yy} = 0, \qquad \rho = \left(det\left(\nabla^2\phi\right)\right)^{-3/4}, \tag{1.1}$$

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in a planar domain, where $\nabla^2\phi > 0$ is the Hessian matrix of ϕ.

The equation (1.1) is the Euler-Lagrange equation of the affine area functional

$$\mathcal{A}(\phi) = \int \left(det\left(\nabla^2\phi\right)\right)^{1/4} dxdy = \int K^{1/4} d\sigma, \tag{1.2}$$

where K is the Euclidean Gauss curvature of the graph of ϕ and $d\sigma$ its volume element.

The study of (1.1) may help to understand well-known functionals involving curvatures of a hypersurface whose Euler-Lagrange equation is also a fourth order partial differential equation, such as the Willmore's functional ([GLW],[Si]) and the Calabi's functional ([Ca3], [Ch], [WZ]).

The equation (1.1) has been widely studied from a global point of view. For instance, Trudinger and Wang [TW1] proved that entire convex solution of (1.1) are quadratic polynomials. This result has solved the so called *Affine Bernstein Problem* conjectured by Chern [Che] in 1978 and has motivated its extension to more general classes of fourth order elliptic equations, [LJ], [TW2]. It generalizes a celebrate result by Jörgens [Jo1] which states that all the solutions of the classical Monge-Ampère equation

$$\det(\nabla^2\phi) := \phi_{xx}\phi_{yy} - \phi_{xy}^2 = 1, \tag{1.3}$$

globally defined on $\mathbb{R}^2$, are quadratic polynomials (see also [Ca1]).

The aim of this paper is to investigate the behavior of solutions of (1.1) around isolated singularities. To be more precise, we shall consider the following problem:

$$L[\phi] = 0, \qquad \text{in a punctured domain } \mathcal{U}^\star = \mathcal{U} \setminus \{(0,0)\}, \tag{1.4}$$

where $\mathcal{U} \subseteq \mathbb{R}^2$ ia a planar domain containing the origin.

To the best of our knowledge, no results are known for (1.4) apart from those satisfying $\det(\nabla^2\phi) = \text{const} > 0$ which are trivially solutions of that equation. In this sense Jörgens [Jo2] studied the equation (1.3) proving the removability of isolated singularities under the additional assumption that one of the first derivatives ϕ_x or ϕ_y (or some directional derivative) has a continuous extension to the singular point. Recent advances in the understanding and classification of isolated singularities of (1.3) can be found in [GMMir] and [ACG].

Jörgens' result has been extended to the following general equation

$$A\phi_{xx} + 2B\phi_{xy} + C\phi_{yy} + \phi_{xx}\phi_{yy} - \phi_{xy}^2 = E$$

where the coefficients A, B, C and E are regular enough and the uniform ellipticity condition

$$AC - B^2 + E \geq \text{const} > 0$$

is satisfied (see [Be1], [Be2], [HB], [SW] and references therein).

Some results about isolated singularities for fully nonlinear elliptic equations (see [Be2], [La]) allow us to have a certain control of some locally strictly convex functions in $\mathcal{U}^\star$ near the singularity. But the classification of non removable singularities for this kind of equations is nowadays an open problem only solved for some particular equations (see [ACG], [GM], [GMMir]) which, as the Equation (1.3), can be solved in terms of holomorphic data (see , [FMM1] and [FMM2] for more details).

Bearing in mind the results on isolated singularities of elliptic partial differential equations of second order, one may conjecture that for a fourth order partial differential equation as (1.4) the singularity may be removed if the solution is $\mathcal{C}^3$ at the origin. However, it is natural to ask for the existence of solutions of (1.4) with a non removable singularity which are $\mathcal{C}^2$ at the origin.

The main objective of this work is to give a classification of non removable isolated singularities of (1.4) which are $\mathcal{C}^2$ at the singularity. In particular, we shall prove that the singularity can be removed if the solution is $\mathcal{C}^3$ at the singular point. We also give general representation's formulas for solutions of (1.4) whose first partial derivatives not extend continuously to the singularity. To do that, we follow the Jörgens' approach of introducing conformal parameters with respect to the Blaschke metric.

The paper is organized as follows. In Section 2, we introduce the notation and recall basic facts about affine surfaces that allow to solve (1.1) by meromorphic data. Section 3 is devoted to analyze in detail the behavior of a large family of solutions of (1.4) that we will call *canonical examples*, which are $\mathcal{C}^2$ at the origin but such that the singularity is not removable. Indeed, we will se that all of them are asymptotic to rotational symmetric solutions near the singularity. In Section 4 we prove that the canonical examples are precisely, up to equiaffine transformations, the only solutions to (1.4) which are $\mathcal{C}^2$ at the origin and have a non removable singularity. Finally, in Section 5 we give a geometrical construction procedure for solutions of (1.4) which are not $\mathcal{C}^1$ at the singular point.

2 Notation and Basic Facts

Let $\phi \in \mathcal{C}^4(\Omega)$ be a locally strongly convex function in a planar domain Ω. We can assume without loss of generality that $\nabla^2\phi$ is a positive definite matrix. Then, the Berwald-Blaschke metric

$$g = \frac{1}{\left(det\left(\nabla^2\phi\right)\right)^4}\left(\phi_{xx}dx^2 + 2\phi_{xy}dxdy + \phi_{yy}dy^2\right)$$

induces on Ω a Riemann surface structure that we shall call the *underlying conformal structure* of ϕ.

With respect to this conformal structure, the affine conormal vector field of the graph of ϕ,

$$N = (N_1, N_2, N_3) = \frac{1}{\left(det\left(\nabla^2\phi\right)\right)^4}\left(-\phi_x, -\phi_y, 1\right), \tag{2.1}$$

is harmonic if and only if ϕ is a solution of (1.1).

On the other hand, let $\mathbb{D}$ be a simply connected planar domain and $N(u, v)$ a vector field, $(u,v)\in\mathbb{D}$, satisfying

$$\left.\begin{array}{ll} 0 = N_{uu} + N_{vv} & \text{in } \mathbb{D}, \\ 0 < [N, N_u, N_v] & \text{in } \mathbb{D}, \end{array}\right\} \tag{2.2}$$

where by $[A,B,C]$ we denote the determinant of the vectors $\{A,B,C\}$. Then, there exists, up to a translation, a unique locally strongly convex affine maximal immersion $\psi:\mathbb{D}\longrightarrow\mathbb{R}^3$ having N as its affine conormal vector field. The terminology *maximal* comes from the fact ψ is a maximum of the affine area functional (1.2) (see [Ca2]).

The immersion ψ can be recovered from N in the following way,

$$\psi = \int N\times N_v du - N\times N_u dv, \tag{2.3}$$

where $\times$ denotes the usual cross product in $\mathbb{R}^3$. We remark that if $N_3>0$, then ψ is locally a vertical graph of a convex solution ϕ of the equation (1.1), and

$$g = [N, N_u, N_v](du^2+dv^2) \tag{2.4}$$

is its Berwald-Blaschke metric (see [Ca2], [GMMil] and Section 4.2 in [LSZ] for more details).

When $det\,(\nabla^2\phi) = \text{const} > 0$, the immersion ψ is called *improper affine sphere.* In this case the affine normal of the immersion, given by

$$\xi = \frac{1}{2}\Delta_g\psi, \tag{2.5}$$

becomes constant, where by Δ_g we denote the Laplace-Beltrami operator with respect to the metric g.

From now on, we will denote by $\mathcal{U}$ a planar domain containing the origin. If ϕ is a locally strongly convex solution of (1.4), we have that ϕ has a bounded gradient $\nabla\phi=(\phi_x,\phi_y)$ at least in a neighborhood of the origin (see [Be2]) and then, ϕ has a continuous extension to $\mathcal{U}$ that we will also denote by ϕ. Thus, the convex graph $\psi(x,y)=(x,y,\phi(x,y))$ on $\mathcal{U}^\star$ extends continuously to $(0,0)$ and, up to a suitable translation, we can assume that $\psi(0,0)=(0,0,0)$.

Let us consider the Legendre transform L^ψ of $\psi=(\psi_1,\psi_2,\psi_3)$, (see [LSZ] pp. 89),

$$\begin{aligned} L^\psi &= (L_1^\psi, L_2^\psi, L_3^\psi) = -\left(\frac{N_1}{N_3}, \frac{N_2}{N_3}, \psi_1\frac{N_1}{N_3} + \psi_2\frac{N_2}{N_3} + \psi_3\right) \\ &= (\phi_x, \phi_y, x\phi_x + y\phi_y - \phi)\,. \end{aligned} \tag{2.6}$$

It is not difficult to check that L^ψ is also a locally strongly convex immersion with Euclidean normal

$$n^L = \frac{1}{\sqrt{1+\psi_1^2+\psi_2^2}}(-\psi_1,-\psi_2,1) = \frac{1}{\sqrt{1+x^2+y^2}}(-x,-y,1). \tag{2.7}$$

Since the limit of L_3^ψ when (x,y) tends to $(0,0)$ is 0, then (L_1^ψ, L_2^ψ) tends to a curve Γ which is limit of the convex sections $L_3^\psi = \varepsilon$ when ε tends to 0.

From the convexity of ψ, $L^\psi(\mathcal{U}^\star)$ can be parametrized as the graph of a convex function ϕ^L on a domain $\mathcal{V}$ in the exterior of Γ, such that Γ is a boundary component of the closure of $\mathcal{V}$. It is clear from (2.6) and (2.7) that ϕ^L and its gradient $\nabla\phi^L$ extend continuously to Γ. Actually, $\phi^L \equiv 0$ and $\nabla\phi^L \equiv 0$ on Γ.

3 Canonical Examples

We begin with a rotational example that may help us to understand the general description of a large family of solutions which are $\mathcal{C}^2$ at the singularity.

Let $\mathcal{D}^\star = \{(u,v) \in \mathbb{R}^2 \mid 0 < u^2+v^2 < 1\}$ be the punctured unit disk and a a positive real number. Let us take the harmonic vector field $N^a : \mathcal{D}^\star \longrightarrow \mathbb{R}^3$ given by

$$N^a(u,v) = \left(u, v, -a\log\sqrt{u^2+v^2} - a\right), \qquad (u,v) \in \mathcal{D}^\star.$$

It is clear from the above expression that N^a satisfies the first condition in (2.2) and as $[N^a, N^a_u, N^a_v] = -a\log(\sqrt{u^2+v^2})$, the second condition also holds. Then, from (2.3), it defines an affine maximal immersion

$$\psi^a : \widetilde{\mathcal{D}^\star} \longrightarrow \mathbb{R}^3,$$

where $\widetilde{\mathcal{D}}$ is the conformal universal covering of $\mathcal{D}^\star$. The immersion ψ^a is actually well defined in $\mathcal{D}^\star$. In fact, it is easy to see that

$$\psi^a(u,v) = \left(au\log\sqrt{u^2+v^2}, av\log\sqrt{u^2+v^2}, \frac{u^2+v^2}{2}\right), \quad (u,v) \in \mathcal{D}^\star.$$

Consequently, ψ^a is a rotational embedding which extends continuously to the origin (see Figure 1).

When N_3 is positive, we have that ψ^a is a global graph of a function $\phi^a(x,y)$. Indeed, if we consider polar coordinates $u = R\cos(t)$, $v = R\sin(t)$, then

$$\phi^a(x,y) = \frac{R^2}{2}, \qquad x = aR\log(R)\cos(t), \qquad y = aR\log(R)\sin(t),$$

for $0 < R < 1/\mathrm{e}$, $0 \leq t < 2\pi$, and ϕ^a is a solution of (1.1).

As $-aR\log(R)$ is a monotone function of $]0, 1/\mathrm{e}[$ into $]0, a/\mathrm{e}[$, if we denote by $\mathcal{D}^\star_\varepsilon = \{(u,v) \in \mathbb{R}^2 \mid 0 < u^2+v^2 < \varepsilon^2\}$ the punctured disk of radius ε, we have that ϕ^a is a regular rotational convex solution of (1.1) in $\mathcal{D}^\star_{a/\mathrm{e}}$ which is $\mathcal{C}^2$ at the origin but the singularity can not be removed. By construction, ϕ^a has the underlying conformal structure of a punctured disk.

Now we can extend the above example by considering the large family $\mathcal{N}$ of harmonic vector fields $N = (N_1, N_2, N_3) \in \mathcal{N}$ which admit, in a punctured disk $\mathcal{D}^\star_\varepsilon$, a series

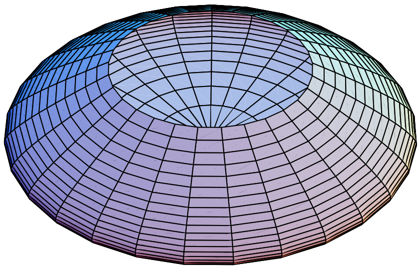

Figure 1: Rotational example with $N = (u, v, -2 - \log(u^2 + v^2))$

development of the form

$$
\left.
\begin{array}{l}
N_1(u,v) = N_1(R\cos(t), R\sin(t)) = R\cos(t) + \sum_{m\geq 2} R^m A_{1m}(t), \\
N_2(u,v) = N_2(R\cos(t), R\sin(t)) = R\sin(t) + \sum_{m\geq 2} R^m A_{2m}(t), \\
N_3(u,v) = N_3(R\cos(t), R\sin(t)) = -a\log(R) + \sum_{m\geq 0} R^m A_{3m}(t),
\end{array}
\right\} \tag{3.1}
$$

where $A_{jm}(t) = a_{jm}\cos(mt) + b_{jm}\sin(mt)$, $a_{jm},\ b_{jm}, a \in \mathbb{R}$, $j = 1,2,3$, $a > 0$, $a_{30} \neq 0$, $0 < R < \varepsilon$ and $0 \leq t < 2\pi$. It is remarkable that $N \in \mathcal{N}$ if and only if $N_3 > 0$ near the origin and N_1, N_2 are harmonic functions in the whole disk D_ε and asymptotic to the linear harmonic functions u and v, respectively.

Indeed, we have the following existence result:

Theorem 1 *For each $N \in \mathcal{N}$ with a series development as in (3.1) there exists a neighborhood $\mathcal{U}$ of the origin in $\mathbb{R}^2$ and a solution ϕ_N of (1.4) such that it can be written as*

$$
\begin{array}{rcl}
\phi_N(x,y) & = & \dfrac{R^2}{2} + h_3(R,t), \\
x & = & aR\log(R)\cos(t) + h_1(R,t), \\
y & = & aR\log(R)\sin(t) + h_2(R,t),
\end{array} \tag{3.2}
$$

for some regular functions h_j in a small disk $\mathcal{D}_{\varepsilon(N)}$, $j = 1,2,3$, satisfying

$$
\lim_{R\to 0}\frac{h_1}{R} \;=\; \lim_{R\to 0}\frac{h_2}{R} \;=\; \lim_{R\to 0}\frac{h_3}{R^2} \;=\; 0. \tag{3.3}
$$

Moreover, ϕ_N is regular in $\mathcal{U}^\star$, $\mathcal{C}^2$ at the origin and has the underlying conformal structure of a punctured disk and the singularity can not be removed.

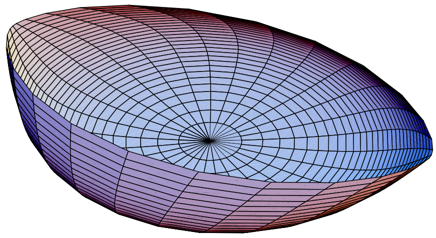

Figure 2: Non-rotational example with $N = (u, v, -\log(u^2 + v^2) + u^2 - v^2)$

Proof: From (3.1), we have that $[N, N_u, N_v] = -a\log(R) + o(u, v))$ for some bounded function $o(u, v)$ in $\mathcal{D}_\varepsilon$. Thus, there exists ε', $0 < \varepsilon' < \varepsilon$, such that N satisfies the conditions in (2.2). From (2.3), we recover a locally convex affine maximal immersion

$$\psi_N : \widetilde{\mathcal{D}^\star_{\varepsilon'}} \longrightarrow \mathbb{R}^3,$$

where $\widetilde{\mathcal{D}^\star_{\varepsilon'}}$ is the conformal universal covering of $\mathcal{D}^\star_{\varepsilon'}$. The immersion ψ_N is, in fact, well defined in $\mathcal{D}^\star_{\varepsilon'}$ and after a straightforward computation we obtain that it can be written as

$$\begin{aligned}\psi_N(u, v) \quad = \quad & \left(au\log\left(\sqrt{u^2+v^2}\right), av\log\left(\sqrt{u^2+v^2}\right), \frac{u^2+v^2}{2}\right) \\ & + (h_1(u, v), h_2(u, v), h_3(u, v)),\end{aligned}$$

$(u, v) \in \mathcal{D}^\star_{\varepsilon'}$, where $h_j : \mathcal{D}_{\varepsilon'} \longrightarrow \mathbb{R}$, $j = 1, 2, 3$, are regular functions satisfying (3.3). Consequently, ψ_N is asymptotic to ψ_a at the origin (see Figure 2).

Now, a standard argument let us prove that there is a small punctured disk $\mathcal{D}^\star_{\varepsilon(N)}$ where ψ_N is globally the graph of a function $\phi_N(x, y)$ which is written as in (3.2).

Using (3.2) and (3.3) it follows that there is a neighborhood $\mathcal{U}$ of $(0, 0)$ in $\mathbb{R}^2$, such that ϕ_N is a solution of (1.4) which is $\mathcal{C}^2$ at the origin. Moreover, by construction, ϕ_N has the underlying conformal structure of a punctured disk and the origin is a non removable singularity of ϕ_N.

□

Definition 2 We will call *canonical examples* to the ones described in Theorem 1.

4 Main Result

This Section is devoted to prove the following result:

Theorem 3 *Let ϕ be a solution of (1.4) which is $\mathcal{C}^2$ at the origin. If ϕ has a non removable singularity at the origin, then the graph ψ of ϕ is, up to an equiaffine transformation, the graph of one of the canonical examples.*

To begin, we will discuss about solutions of (1.1) with the underlying conformal structure of the punctured disk $\mathcal{D}^\star$. The following argument is also successful, by restricting the corresponding functions to the unit disk, if the underlying conformal structure is the one of $\mathbb{C}\setminus\{(0,0)\}$.

Let us take (u,v) conformal parameters in $\mathcal{D}^\star$ for the Berwald-Blaschke metric, which can be written as in (2.4). Since the affine conormal $N=(N_1,N_2,N_3)$ of the graph ψ of ϕ is a harmonic vector field in $\mathcal{D}^\star$, there exist holomorphic functions $F,G,H:\mathcal{D}^\star\longrightarrow\mathbb{C}$ and real constants a, b and c such that

$$\begin{aligned} N_1(z) &= \operatorname{Re}(F(z))+b\log|z|,\\ N_2(z) &= \operatorname{Re}(G(z))+c\log|z|,\\ N_3(z) &= \operatorname{Re}(H(z))+a\log|z|, \end{aligned}$$

where $z=u+\mathrm{i}\,v$. Since ψ is a vertical graph, N_3 is a positive function in $\mathcal{D}^\star$ and then H is holomorphic in the whole disk $\mathcal{D}$ (see [ABR] for more details). Consequently, bearing in mind that neither N_1/N_3 nor N_2/N_3 go to infinity when z tends to 0 (see (2.6)), both F and G have a removable singularity at $z=0$ and therefore they are also holomorphic functions in $\mathcal{D}$. In addition:

- If $a=0$, then $b=c=0$ and

$$\lim_{z\to 0}\left(\frac{N_1}{N_3},\frac{N_2}{N_3}\right)=\left(\frac{F(0)+\overline{F}(0)}{H(0)+\overline{H}(0)},\frac{G(0)+\overline{G}(0)}{H(0)+\overline{H}(0)}\right).$$

- If $a\neq 0$, then

$$\lim_{z\to 0}\left(\frac{N_1}{N_3},\frac{N_2}{N_3}\right)=\left(\frac{b}{a},\frac{c}{a}\right).$$

In particular, whichever the case the unit Euclidean normal vector field is well-defined at the origin. Thus, up to an equiaffine transformation we can assume that the unit normal of the graph ψ at the origin is $(0,0,1)$, that is, $b=c=0$. Moreover, if $a=0$ then $F(0),G(0)\in\mathrm{i}\,\mathbb{R}$.

Let us study separately the cases $a=0$ and $a\neq 0$.

Lemma 4 *If $a=0$, then ϕ has a removable singularity at the origin.*

Proof: If $a=0$, then $b=c=0$ and N is harmonic in the whole disk $\mathcal{D}$. Furthermore,

$$N_0:=N(0)=(0,0,\operatorname{Re}(H(0)))\neq(0,0,0),$$

because $H+\overline{H}$ is a positive harmonic function in $\mathcal{D}$ and so it cannot attain its minimum at $z=0$. We can assume that $H(0)=1$; otherwise we would argue in a similar way.

Let us take
$$\widetilde{\psi} = N^\star \times N_0 + \langle N_0, \psi\rangle N_0,$$
where $\langle\, ,\, \rangle$ denotes the usual metric in $\mathbb{R}^3$ and $N^\star$ the harmonic conjugate of the affine conormal N in $\mathcal{D}$. As it can be seen in [GMMil], $\widetilde{\psi}$ is an improper affine sphere with affine normal N_0 and Berwald-Blaschke metric
$$\widetilde{g} = \langle \xi, N_0\rangle [N, N_u, N_v](du^2 + dv^2) = \langle \xi, N_0\rangle g,$$
where ξ is the affine normal of ψ (see (2.5)). Since $\widetilde{\psi}$ is the graph of a solution of (1.3) with the underlying conformal structure of a puncture disk, it has a removable singularity at the origin (see, for instance, [Jo2], [GMMir]). Consequently $[N, N_u, N_v] > 0$ in $\mathcal{D}$, ψ is regular in the whole disk $\mathcal{D}$ and ϕ has a removable singularity at the origin.

□

Lemma 5 *If $a \neq 0$ then, up to equiaffine transformations, the affine conormal N of ψ is in the family $\mathcal{N}$.*

Proof: Let us take polar coordinates (R, t) in $\mathcal{D}$ such that $u = R\cos t$ and $v = R\sin t$. Since F, G and H are holomorphic functions in $\mathcal{D}$, near the origin we can write (see [ABR])
$$N = (A_{1\,0} + \sum_{m\geq p} R^m A_{1m}(t), A_{2\,0} + \sum_{m\geq q} R^m A_{2m}(t), A_{3\,0} + \sum_{m\geq k} R^m A_{3m}(t) + a\log(R)),$$
where
$$A_{jm}(t) = a_{jm}\cos(mt) + b_{jm}\sin(mt), \qquad j = 1, 2, 3,$$
$$A_{1p} \neq 0,\ A_{2q} \neq 0,\ A_{3k} \neq 0.$$

Observe that, up to equiaffine transformations, we can assume that one of the following cases happens:

- $A_{2\,0} = A_{3\,0} = 0$ (or, symmetrically, $A_{1\,0} = A_{3\,0} = 0$).
- $A_{1\,0} = A_{2\,0} = 0$, $A_{3\,0} \neq 0$.

Let us see that the first one is not possible. In what follows, we will denote by $o(l)$, $l \in \mathbb{Z}$, $l \geq 0$, a function depending on R and t which can be written as $o(l) = R^l f(R, t)$ for a certain function $f(R, t)$ bounded in a neighborhood of the origin. It is easy to check that, if $A_{2\,0} = A_{3\,0} = 0$, $A_{1\,0} \neq 0$, then
$$[N, N_u, N_v] = \frac{1}{R}[N, N_R, N_t] = R^{q-2}(-aA_{1\,0}A'_{2q}(t) + o(1))$$
and so the sign of $[N, N_u, N_v]$ changes depending on the angle t, for R small enough. But this is impossible because g is a metric in $\mathcal{D}^\star$ and so $[N, N_u, N_v] \geq 0$ in $\mathcal{D}$.

Hence, we can assume that $A_{1\,0} = A_{2\,0} = 0$, $A_{3\,0} \neq 0$. By putting
$$N = \big(R^p A_{1p}(t) + o(p+1), R^q A_{2q}(t) + o(q+1), A_{3\,0} + R^k A_{3k}(t) + a\log(R) + o(k+1)\big)$$

we get

$$[N, N_u, N_v] = R^{p+q-2}(-a\log(R)(pA_{1p}(t)A'_{2q}(t) - qA_{2q}(t)A'_{1p}(t)) + o(0)).$$

Observe that if

$$\begin{aligned} pA_{1p}(t)A'_{2q}(t) - qA_{2q}(t)A'_{1p}(t) &= pq\,((a_{1p}a_{2q} + b_{1p}b_{2q})\sin((p-q)t) \\ &\quad + (a_{1p}b_{2q} - b_{1p}a_{2q})\cos((p-q)t)) \end{aligned}$$

is not constant, then the sign of $[N, N_u, N_v]$ changes depending on the angle t for R small enough, which is not possible. Therefore, it must be $p = q$. Thus we can take, up to an equiaffine transformation, $a_{1p} = 1 = b_{2p}$, $b_{1p} = 0 = a_{2p}$, and N becomes

$$\begin{aligned} N &= (R^p\cos(pt), R^p\sin(pt), A_{3\,0} + a\log(R)) \\ &\quad + \left(\sum_{m\geq p+1} R^m A_{1m}(t), \sum_{m\geq q+1} R^m A_{2m}(t), \sum_{m\geq k} R^m A_{3m}(t)\right). \end{aligned} \tag{4.1}$$

From (2.3), the graph ψ of ϕ satisfies $\psi_u = N \times N_v$ and $\psi_v = -N \times N_u$, or, equivalently, $\psi_R = (1/R)N \times N_t$ and $\psi_t = -RN \times N_R$. Hence, from (4.1) and after long but easy computations, we obtain

$$\begin{aligned} \psi_R &= (-apR^{p-1}\log(R)\cos(pt), -apR^{p-1}\log(R)\sin(pt), pR^{2p-1}) \\ &\quad +(o(p-1), o(p-1), o(2p)) \\ \psi_t &= (apR^p\log(R)\sin(pt), -apR^p\log(R)\cos(pt), 0) + (o(p), o(p), o(2p+1)) \end{aligned}$$

and so

$$\begin{aligned} \psi &= (-aR^p\log(R)\cos(pt), -aR^p\log(R)\sin(pt), R^{2p}/2) \\ &\quad +(o(p), o(p), o(2p+1)). \end{aligned} \tag{4.2}$$

Using now that $\psi(x, y) = (x, y, \phi(x, y))$ we get that

$$x + \mathrm{i}\,y = -aR^p\log(R)e^{\mathrm{i}\,pt} + o(p).$$

Hence, $x + \mathrm{i}\,y$ moves around the origin p times (see Figure 3 for $p = 2$). Consequently, since ψ is the graph of ϕ, it is embedded and so $p = 1$. The proof concludes from (3.1) and (4.2)

$\square$

Lemma 6 *Let ϕ be a solution of (1.4) with a non removable singularity at the origin. Then ϕ has the underlying conformal structure of either $\mathbb{C}\setminus\{(0,0)\}$ or a punctured disk if and only if*

$$\lim_{(x,y)\to(0,0)} \det\left(\nabla^2\phi\right) = 0. \tag{4.3}$$

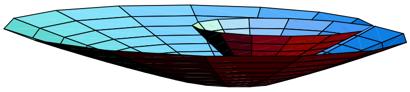

Figure 3: Non-embedded example with $N = (u^2 - v^2, uv, 2 + u + \log(u^2 + v^2))$

Proof: From Lemmas 4 and 5, if ϕ has the underlying conformal structure of either a punctured disk or $\mathbb{C} \setminus \{(0,0)\}$, then the condition (4.3) holds.

Otherwise, if ϕ has the underlying conformal structure of an annulus, there exists a holomorphic function H_3 in $\Delta_-^{r_0} = \{z = u + \mathrm{i}\, v \,:\, -r_0 < \mathrm{Im}(z) < 0\}$ such that $N_3 = \mathrm{Re}(H_3)$. Thus, if (4.3) holds, then $1/H_3(z)$ goes to 0 when $\mathrm{Im}(z)$ tends to 0. But this is not possible because $1/H_3(z)$ is holomorphic.

$\square$

Proof of Theorem 3: Since ϕ is $\mathcal{C}^2$ at the singularity,

$$\lim_{(x,y)\to(0,0)} \det\left(\nabla^2\phi\right)$$

is well-defined.

If this limit is equal to 0, then the proof follows immediately from Lemmas 4, 5 and 6. Otherwise, from Lemma 6, ϕ has the underlying conformal structure of an annulus, and reasoning as in Section 4 of [Be1] we see that this is not possible.

Corollary 7 *Let ϕ be a solution of (1.4) which is $\mathcal{C}^3$ at $(0,0)$. Then ϕ has a removable singularity at the origin.*

5 Examples with the Underlying conformal Structure of an Annulus

Let ϕ be a solution of (1.4) satisfying the following condition

$$\lim_{(x,y)\to(0,0)} \det\left(\nabla^2\phi\right) = E_0, \tag{5.1}$$

where E_0 is a positive real number or ∞.

If ϕ has not a removable singularity at the origin, we know from Lemma 6 that ϕ has the underlying conformal structure of an annulus. Hence, the graph $\psi(x,y) = (x, y, \phi(x,y))$, $(x,y) \in \mathcal{U}^\star$, can be parametrized as

$$\widetilde{\psi}(z) = (x(z), y(z), \phi(z)), \quad z \in \Delta_-^{r_0} = \{z = u + \mathrm{i}\, v \mid -r_0 < v < 0\},$$

and

$$\widetilde{\psi}(z) = (0,0,0) \qquad \text{when } \operatorname{Im}(z) = 0. \tag{5.2}$$

Moreover $\widetilde{\psi}$ is well defined in the annulus $\Delta_-^{r_0}/(2\pi\mathbb{Z})$, that is, $\widetilde{\psi}(z+2\pi) = \widetilde{\psi}(z)$, $z \in \Delta_-^{r_0}$.

The affine conormal vector field $N = (N_1, N_2, N_3)$ satisfies the conditions (2.2) in $\Delta_-^{r_0}$ and the immersion $\widetilde{\psi}$ can be recovered from N, up to a translation, as in (2.3). Consequently, $\widetilde{\psi}$ and N satisfy,

$$\left.\begin{array}{ll} x_u = -N_3^2\,(N_2/N_3)_v\,, & x_v = N_3^2\,(N_2/N_3)_u\,, \\ y_u = N_3^2\,(N_1/N_3)_v\,, & y_v = -N_3^2\,(N_1/N_3)_u\,, \\ \phi_u = N_1 N_{2v} - N_2 N_{1v}, & \phi_v = -N_1 N_{2u} + N_2 N_{1u}, \end{array}\right\}$$

or equivalently,

$$\left.\begin{array}{l} X_{uu} + X_{vv} - f_u X_u - f_v X_v = (0,0), \\ Y_{uu} + Y_{vv} + f_u Y_u + f_v Y_v = (0,0,0), \\ \phi_{uu} + \phi_{vv} - f_u \phi_u - f_v \phi_v = 2N_3^2[Y_u, Y_v, Y] = 2[N_u, N_v, N]/N_3 > 0, \end{array}\right\} \tag{5.3}$$

where $f = 2\log(N_3)$, $X = (x, y)$ and $Y = (N_1/N_3, N_2/N_3, 1)$.

5.1 Bounded Behavior

If $0 < E_0 < \infty$, we can assume $E_0 = 1$; otherwise we argue in a similar way. Then, from (2.1) and (5.1), $N_3 \equiv 1$ in $\operatorname{Im}(z) = 0$, and so N_3 extends in a harmonic way to $\Delta^{r_0} = \{z = u + \mathrm{i}\, v \mid -r_0 < v < r_0\}$. From (5.3) (see [Le], [SS]), X, Y and ϕ also extend to Δ^{r_0} as solutions of (5.3). In particular, $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^3$ given by $\gamma(u) = Y(u,0) = N(u,0)$ is a 2π-periodic analytic parametrization of an embedded planar curve with non negative curvature.

On the other hand, (2.3) and (5.2) tell us that there exists a 2π-periodic regular analytic function $\lambda : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying $N_v(u,0) = \lambda(u) N(u,0)$, $u \in \mathbb{R}$.

When $\gamma(\mathbb{R})$ is a regular analytic strictly convex Jordan curve, we are able to recover the solution in terms of the pair $\{\gamma, \lambda\}$.

More precisely, let $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^3$, $\gamma(u) = (\gamma_1(u), \gamma_2(u), 1)$ be a 2π-periodic regular analytic parametrization of a strictly convex Jordan curve and $\lambda : \mathbb{R} \longrightarrow \mathbb{R}$ be a 2π-periodic regular analytic function. Observe that $\gamma(\mathbb{R})$ encloses a convex planar domain and it has non-zero curvature at every point, that is, changing of orientation if it was necessary we can assume that

$$[\gamma(u), \gamma'(u), \gamma''(u)] > 0, \qquad u \in \mathbb{R}, \tag{5.4}$$

where we shall use prime to denote derivation respect to u. Then, it is clear the existence of a positive real number r such that the holomorphic extensions $\lambda(z)$ and $\gamma(z)$ of λ and γ, respectively, are well defined in

$$\Delta^r = \{z = u + \mathrm{i}\, v \mid -r < v < r\}.$$

The pair $\{\gamma, \lambda\}$ let us define a holomorphic curve $F^{\gamma\lambda} : \Delta^r \longrightarrow \mathbb{C}^3$,

$$F^{\gamma\lambda}(z) = \gamma(z) - \mathrm{i} \int_0^z \lambda(w)\gamma(w)dw, \qquad z \in \Delta^r. \tag{5.5}$$

We shall denote by $\mathcal{F}$ the family of this kind of holomorphic curves. It is clear from the definition that $F^{\gamma\lambda}$ is 2π-periodic and it induces a well defined holomorphic curve in the annulus $\Delta^r/2\pi\mathbb{Z}$.

Let us note that from (5.5), the real part N of a holomorphic curve $F^{\gamma\lambda} \in \mathcal{F}$ satisfies

$$N(u,0) = \gamma(u), \quad N_u(u,0) = \gamma'(u), \quad N_v(u,0) = \lambda(u)\gamma(u), \quad u \in \mathbb{R}. \tag{5.6}$$

Moreover, if we set $\Phi = [N, N_u, N_v]$ then from (5.4), (5.5) and (5.6),

$$\Phi(u,0) = 0, \qquad -\Phi_v(u,0) = [\gamma(u), \gamma'(u), \gamma''(u)] > 0, \qquad u \in \mathbb{R}. \tag{5.7}$$

Applying the Implicit Function Theorem we have, from (5.7), that there exists $r_0 > 0$ such that N satisfies the conditions (2.2) in

$$\Omega = \Delta_-^{r_0} = \Delta^{r_0} \cap \{\mathrm{Im}(z) < 0\}.$$

Via (2.3), and having in mind that $N_3 \equiv 1$ in $\mathbb{R}$, we get a locally strongly convex affine maximal immersion $\psi : \Delta_-^{r_0} \longrightarrow \mathbb{R}^3$ which is a vertical graph around each point (see Figure 4).

Since $\psi_u = N \times N_v$, we get from (5.6) that $\lim_{v\to 0} \psi_u(u,v) = (0,0,0)$ and so ψ extends continuously to $\{\mathrm{Im}(z) = 0\}$. Up to a suitable translation we may assume that

$$\psi(u,0) = (0,0,0), \qquad u \in \mathbb{R}. \tag{5.8}$$

From (2.6), (2.7), (5.6) and (5.8) we obtain

$$\lim_{v\to 0} L^\psi(u,v) = \Gamma(u), \qquad \lim_{v\to 0} n^L(u,v) = (0,0,1),$$

where $\Gamma(u) = (\gamma_1(u), \gamma_2(u), 0)$ is the x_1x_2-projection of γ. Consequently, for $r, \varepsilon > 0$ small enough, $L^\psi(\Delta_-^r)$ lies in the half space $\mathbb{R}^3_+ = \{x_3 > 0\}$ and $\gamma^\varepsilon = L^\psi(\Delta_-^r) \cap \{x_3 = \varepsilon\}$ are regular convex Jordan curves tending to $\Gamma(\mathbb{R})$ when ε tends to 0. Hence, $L^\psi(\Delta_-^r) \cap \{0 < x_3 < \varepsilon\}$ is globally the graph of a convex function $\phi^L(y_1, y_2)$, $(y_1, y_2) \in \Omega^L$, for some domain Ω^L in the exterior of $\Gamma(\mathbb{R})$ which contains $\Gamma(\mathbb{R})$ in its boundary. But then, via the Legendre transform of (y_1, y_2, ϕ^L) we conclude that ψ is also a graph of a solution $\phi(x_1, x_2)$ of (1.1), where $(x_1, x_2) \in \mathcal{U}^\star$ for some neighbourhood $\mathcal{U}$ of the origin in $\mathbb{R}^2$. Therefore, we have proved

Theorem 8 *Let N be the real part of a holomorphic curve $F^{\gamma\lambda} \in \mathcal{F}$. Then there exists a solution ϕ of (1.4) such that*

1. *N is the affine conormal vector field of the graph of ϕ.*
2. *ϕ extends continuously at the origin and it has the underlying conformal structure of an annulus.*
3. *ϕ is not $\mathcal{C}^1$ at the origin and $(-\nabla\phi, 1)$ tends to the convex Jordan curve γ at the origin.*

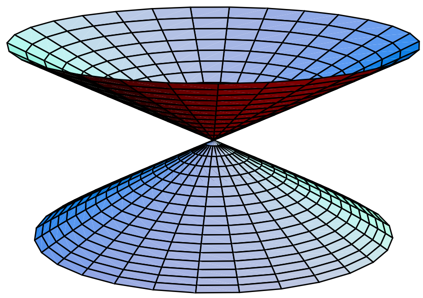

Figure 4: Bounded behavior with $N = (e^v \cos(u), -e^v \sin(u), 2 + 2v)$

5.2 Unbounded Behavior

If

$$\lim_{(x,y)\to(0,0)} \det(\nabla^2\phi) = \infty,$$

then, from (2.1) and (2.6), N vanishes identically in $\mathrm{Im}(z) = 0$, and so it reflects in a harmonic way to Δ^{r_0} and ψ extends also around the singularity. From (2.6), $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^3$ given by $\gamma(u) = (N_1/N_3, N_2/N_3, 1)(u, 0)$ is a 2π-periodic analytic parametrization of an embedded planar curve with non negative curvature and, from (2.3) and (5.2), there exists a 2π-periodic positive regular analytic function $\alpha : \mathbb{R}\longrightarrow\mathbb{R}$ satisfying $N_v(u, 0) = \alpha(u)\gamma(u)$, $u \in \mathbb{R}$.

Similarly to the above case, when $\gamma(\mathbb{R})$ is a regular analytic strictly convex Jordan curve, we are able to recover the solution in terms of the pair $\{\gamma, \alpha\}$. Indeed, if $\gamma : \mathbb{R}\longrightarrow\mathbb{R}^3$, $\gamma(u) = (\gamma_1(u), \gamma_2(u), 1)$, is a 2π-periodic regular analytic parametrization of a strictly convex Jordan curve satisfying (5.4), and $\alpha : \mathbb{R}\longrightarrow\mathbb{R}$ is a 2π-periodic positive analytic function, then the pair $\{\gamma, \alpha\}$ defines a holomorphic curve $G^{\gamma\alpha} : \Delta^r\longrightarrow\mathbb{C}^3$, given by

$$G^{\gamma\alpha}(z) = -\,\mathrm{i}\int_0^z \alpha(w)\gamma(w)dw, \qquad z \in \Delta^r. \tag{5.9}$$

The family of this kind of holomorphic curves will be denoted by $\mathcal{G}$.

It is also clear that $G^{\gamma\alpha}$ is 2π-periodic and it induces a well defined holomorphic curve in the annulus $\Delta^r/2\pi\mathbb{Z}$.

Consider now N the real part of a holomorphic curve $G^{\gamma\alpha} \in \mathcal{G}$. Then from (5.9) we have

$$N(u, 0) = N_u(u, 0) = (0, 0, 0), \qquad N_v(u, 0) = \alpha(u)\gamma(u), \quad u \in \mathbb{R}.$$

If we write $N = N_3T$, we obtain from (5.9) and the harmonicity of N that

$$\left.\begin{array}{l} T(u,0) = \gamma(u), \qquad T_v(u,0) = (0,0,0) \quad u \in \mathbb{R}, \\ N_3(T_{uu} + T_{vv}) + 2(N_3)_uT_u + 2(N_3)_vT_v = (0,0,0). \end{array}\right\} \tag{5.10}$$

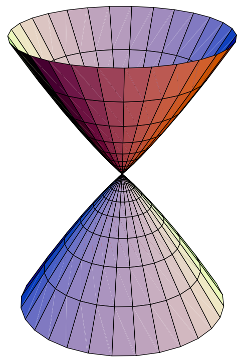

Figure 5: Unbounded behavior with $N(u,v) = (\sin(u)\sinh(v), \cos(u)\sinh(v), v)$

Since $0 < (N_3)_v(u,0) = \alpha(u)$, we can assume that $N_3 > 0$ in Δ^r_- (otherwise we work in $\Delta^r_+ = \Delta^r \cap \{\mathrm{Im}(z) > 0\}$).

Thus, $\Phi = [N, N_u, N_v] = (N_3)^3[T, T_u, T_v]$ vanishes in a point of Δ^r_- if and only if $[T, T_u, T_v]$ so does. By taking derivatives with respect to the normal direction in the second equation of (5.10), we have

$$(3(N_3)_v T_{vv} + (N_3)_v T_{uu} + 2(N_3)_{uv} T_u)\,(u,0) = 0,$$

which together to (5.4) and (5.10) gives

$$[T, T_u, T_v]_v(u,0) = -\frac{1}{3}[T, T_u, T_{uu}](u,0) = -\frac{1}{3}[\gamma(u), \gamma'(u), \gamma''(u)] < 0.$$

The Implicit Function Theorem let us conclude that Φ never vanishes in Δ^r_- for some r small enough and we can assume that N satisfies the conditions in (2.2).

Via (2.3) we have a locally strongly convex affine maximal immersion $\psi : \Delta^r_- \longrightarrow \mathbb{R}^3$ which is a vertical graph around each point (see Figure 5).

By taking the Legendre transform of ψ and arguing as in the case of the family $\mathcal{F}$, we have the following:

Theorem 9 *Let N be the real part of a holomorphic curve $G^{\gamma\alpha} \in \mathcal{G}$. Then there exists a solution ϕ of (1.4) such that ϕ satisfies the properties in Theorem 8.*

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