On the Existence of Affine Maximal Maps

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In affine surfaces theory, Blaschke ([B]) found that the Euler-Lagrange equation of the equiaffine area functional is of fourth order and nonlinear. He also showed that this equation is equivalent to the vanishing of the affine mean curvature, which led to the notion of affine minimal surface without a previous study of the second variation formula. But Calabi proved in [C] that, for locally strongly convex surfaces, the second variation is always negative and since then, locally strongly convex surfaces with vanishing affine mean curvature are called affine maximal surfaces.

After Calabi's work this class of surfaces has become a fashion research topic and it has received many interesting contributions.

In this poster we present the resolution of the problem of existence and uniqueness of affine maximal surfaces containing a regular analytic curve and with a given affine normal along it, see [AMM2]. As applications we give results about symmetries, characterize when a curve in $\mathbb{R}^3$ can be a geodesic of a such surface and study helicoidal affine maximal surfaces, that is, surfaces invariant under a one-parametric group of equiaffine transformations. We obtain new examples with an analytic curve in its singular set, which have been studied in [AMM3]. To do that, we introduce the notion of affine map which allows us to analyze global problems regarding to affine maximal surfaces admitting some natural singularities.

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On the existence of Affine Maximal Maps

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Abstract

We present the resolution of the problem of existence and uniqueness of affine maximal surfaces containing a regular analytic curve and with a given affine normal along it, see [1]. As applications we give results about symmetries, characterize when a curve in $\mathbb{R}^3$ can be a geodesic of a such surface and study helicoidal affine maximal surfaces, that is, surfaces invariant under a one-parametric group of equiaffine transformations. We obtain new examples with an analytic curve in its singular set, which have been studied in [2].

1. Affine Maximal Surfaces

The equiaffine area functional

$$\int dA = \int |K_e|^{\frac{1}{4}} dA_e,$$

with K_e the euclidean Gauss curvature and dA_e the element of euclidean area, has attracted to many geometers since the beginning of the last century.

Well-known Facts:

- Blaschke (1923): the associated fourth order Euler-Lagrange equation is equivalent to the vanishing of the affine mean curvature.
- Calabi (1982): locally strongly convex surfaces have always a negative second variation (affine maximal surfaces).

1.1. Recent developments

- Affine Weierstrass formulas that have provided an important tool in their study, (Calabi, Li, 1990).
- Entire solutions of the fourth order affine maximal surface equation

$$\phi_{yy}\omega_{xx} - 2\phi_{xy}\omega_{xy} + \phi_{xx}\omega_{yy} = 0, \quad \omega = \left(det\left(\nabla^2\phi\right)\right)^{-3/4}, \qquad (1..1)$$

  are always quadratic polynomials (Trudinger-Wang, 2000).
- Every Affine complete affine maximal surface is an elliptic paraboloid, (Li-Jia, 2001, Trudinger-Wang, 2002).
- The Affine Plateau Problem (Trudinger-Wang, 2005).
- Their extension to different nonlinear fourth order equations (Li-Jia, 2003, Trudinger-Wang, 2002).
- The validity of the results in affine maximal surfaces with some natural singularities that may arise (Aledo, Chaves, Gálvez, Martínez, Milán, Mira, 2005-2008).

1.2. Basic Notations

Let $\psi : \Sigma \to \mathbb{R}^3$ be a l.s.c immersion, with second fundamental form σ_e definite positive,

$$\begin{aligned} g &= K_e^{-\frac{1}{4}}\sigma_e, && \text{Berwald-Blaschke metric} \\ \xi &= \frac{1}{2}\Delta_g\psi, && \text{affine normal} \end{aligned}$$

with Δ_g:= Laplace-Beltrami operator associated to g.

The affine conormal field $N := K_e^{-1/4}N_e$, satisfies

$$\langle N, \xi\rangle = 1, \qquad \langle N, d\psi(v)\rangle = 0, \quad v \in T_p\Sigma, \qquad (1..2)$$

and the Euler-Lagrange equation:= $\Delta_g N = 0$.

1.3. Weierstrass-type Representation Formulas

In the simply-connected case ψ can be recovered from N and the conformal class of the Blaschke metric:

Lelieuvre formula

$$\psi = 2\,\mathrm{Re}\int \imath\, N \times N_z dz$$

Calabi's Representation ψ determine a holomorphic curve $\Phi : \Omega \subset \Sigma \to \mathbb{C}^3$ s.t.

$$N = \Phi + \overline{\Phi}, \qquad g = -\imath Det\left[\Phi + \overline{\Phi}, \Phi_z, \overline{\Phi_z}\,\right] dz d\overline{z}. \qquad (1..3)$$

ψ is determined, up to real translation, by a holomorphic curve Φ satisfying $-\imath Det\left[\Phi + \overline{\Phi}, \Phi_z, \overline{\Phi_z}\,\right] > 0$. To be precise,

$$\psi = 2\,\mathrm{Re}\int \imath\, \left(\Phi + \overline{\Phi}\right) \times \Phi_z dz = -\imath\,(\,\Phi \times \overline{\Phi} - \int \Phi \times d\Phi + \int \overline{\Phi} \times \overline{d\Phi}\,).$$

2. The Affine Björling Problem

Let $\beta : I \to \Sigma$ be a regular analytic curve. $\alpha = \psi \circ \beta$, $Y = \xi \circ \beta$ and $U = N \circ \beta$, then, along the curve α

$$\left.\begin{array}{l} 0 = \langle\alpha'(s), U(s)\rangle, \\ 1 = \langle Y(s), U(s)\rangle, \\ 0 = \langle Y'(s), U(s)\rangle, \\ 0 < \lambda(s) = -\langle\alpha'(s), U'(s)\rangle = \langle\alpha''(s), U(s)\rangle, \end{array}\right\} \qquad (2..1)$$

where by prime we indicate derivation respect to s, for all $s \in I$.

Definition Given $Y, U, \alpha : I \longrightarrow \mathbb{R}^3$ regular analytic curves.
$\{Y, U\}$ is an *analytic equiaffine normalization* of α if there is an analytic positive function $\lambda : I \to \mathbb{R}^+$ such that all the equations in (2..1) hold on I.

Theorem Let $\{Y, U\}$ be an analytic equiaffine normalization of α, then there exists a unique affine maximal surface ψ containing $\alpha(I)$, with conormal field and Blaschke normal along α, U and Y respectively.
(ψ:= a.m.s. along α generated by $\{Y, U\}$).

Outline of the Proof

- By the Inverse Function Theorem $\exists z : s + \imath t$, $s \in I$
- Identity Principle: $N_z = \frac{1}{2}\left(U_z + \imath Y \times \alpha_z\right), \qquad z \in \Omega$
- Via Calabi's representation.

$$\psi = \alpha(s_0) + 2\,\mathrm{Re}\int_{s_0}^{z} \imath(\Phi + \overline{\Phi}) \times \Phi_\zeta d\zeta, \qquad (2..2)$$

where, $\Phi(z) = \frac{1}{2}\left(U + \imath\int_{s_0}^{z} Y \times \alpha_\zeta d\zeta\right), \qquad z \in \Omega, \quad s_0 \in I$, on a complex domain Ω containing I.

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Corollary Let $\alpha, Y : I \to \mathbb{R}^3$ be two regular analytic curves

$$Det[\,Y', \alpha', Y\,]Det[\,Y', \alpha', \alpha''\,] > 0, \qquad \text{on} \quad I. \qquad (2..3)$$

$\Rightarrow \exists_1\, \psi$ containing $\alpha(I)$ with Y as Blaschke normal along α.

Proof $\exists_1\, U$ and λ,

$$U = \frac{Y' \times \alpha'}{Det[\,Y', \alpha', Y\,]}, \qquad 0 < \lambda = \frac{Det[\,Y', \alpha', \alpha''\,]}{Det[\,Y', \alpha', Y\,]}$$

s.t. $\{Y, U\}$ is an a.e.n. of α. The result follows from above Theorem, taking in Calabi's representation,

$$\Phi(z) = \frac{Y_z \times \alpha_z}{2Det[\,Y_z, \alpha_z, Y\,]} + \frac{\mathrm{i}}{2}\int_{s_0}^{z} Y \times \alpha_\zeta d\zeta, \qquad z \in \Omega, \quad s_0 \in I.$$

Corollary $\alpha, Y : I \to \mathbb{R}^3$ regular analytic curves

$$[\,Y, \alpha', \alpha''\,] \neq 0, \qquad Y' \times \alpha' = 0, \qquad \text{on} \quad I. \qquad (2..4)$$

Given $\lambda : I \to \mathbb{R}^+$, $\exists_1\, \psi$ containing $\alpha(I)$, such that its Blaschke normal along α is Y and $g(\alpha', \alpha') = \lambda$.

3. Applications

3.1. The Cauchy Problem

If $\psi : \Omega \longrightarrow \mathbb{R}^3$ is the graph of a l.s.c. function $\phi(x, y)$, $(x, y) \in \Omega$. The Euler-Lagrange equation for the affine area functional is

$$\phi_{yy}\omega_{xx} - 2\phi_{xy}\omega_{xy} + \phi_{xx}\omega_{yy} = 0, \qquad \omega = \left(det\left(\nabla^2\phi\right)\right)^{-3/4}.$$

In this situation

$$\begin{aligned} g_\phi &= \sqrt[3]{\omega}\left(\phi_{xx}\, dx^2 + 2\phi_{xy}\, dx\, dy + \phi_{yy}\, dy^2\right), \\ N &= \sqrt[3]{\omega}\,(-\phi_x, -\phi_y, 1), \\ \xi &= \left(\varphi_y, -\varphi_x, \frac{1}{\sqrt[3]{\omega}} - \phi_y\varphi_x + \phi_x\varphi_y\right), \end{aligned} \qquad (3..1)$$

where $\varphi_x = \frac{1}{3}\left(\phi_{xy}\omega_x - \phi_{xx}\omega_y\right)$ *and* $\varphi_y = \frac{1}{3}\left(\phi_{yy}\omega_x - \phi_{xy}\omega_y\right)$.

The Cauchy Problem

$$\begin{cases} \phi_{yy}\omega_{xx} - 2\phi_{xy}\omega_{xy} + \phi_{xx}\omega_{yy} = 0, \qquad \omega = \left(det\left(\nabla^2\phi\right)\right)^{-3/4} \\ \phi(x, 0) = a(x), \\ \phi_y(x, 0) = b(x), \\ \phi_{yy}(x, 0) = c(x), \\ \phi_{yyy}(x, 0) = d(x), \\ c(x)a''(x) - b'(x)^2 > 0 \end{cases}$$

where a, b, c, d are analytic functions on I, ϕ is defined on Ω containing $I \times \{0\}$, **has solution**

$$(x, y, \phi(x, y)) = (s_0, 0, a(s_0)) + 2\,\mathrm{Re}\int_{s_0}^{z = s + \mathrm{i}\,t} \left(\Phi + \overline{\Phi}\right) \times \Phi_\zeta\, d\zeta,$$

with

$$\begin{aligned} \Phi(z) &= \frac{1}{2}\left(U(z) + \imath\int_{s_0}^{z} Y(\zeta) \times A(\zeta)\, d\zeta\right), \\ U(s) &= \left(c(s)a''(s) - b'(s)^2\right)^{-1/4}(-a'(s), -b(s), 1), \\ A(s) &= (1, 0, a'(s)), \\ Y(s) &= \frac{1}{4}\left(c(s)a''(s) - b'(s)^2\right)^{-7/4}\left(b'(da'' + 3cb'') - 2b'^2c' - c(c'a'' + ca'''),\right. \\ &\quad b'(3c'a'' + ca''') - 2b'^2b'' - a''(da'' + cb''), \\ &\quad +4b'^4 - 2b'^2(a'c' + 4ca'' + bb'') - a''((-4c^2 + bd)a'' + bcb'') \\ &\quad \left. - ca'(c'a'' + ca''') + b'\left(a'(da'' + 3cb'') + b(3c'a'' + ca''')\right)\right). \end{aligned}$$

3.2. Symmetry and Geodesics

Consider $T : \mathbb{R}^3 \to \mathbb{R}^3$, the equiaffine transformation given by $T(v) = Av + b$ and $\{Y, U\}$ an analytic equiaffine normalization of $\alpha : I \to \mathbb{R}^3$. We say T is a symmetry of $\{Y, U\}$ if $\exists\ \Gamma : I \to I$ analytic diffeomorphism such that $\alpha \circ \Gamma = T \circ \alpha, \quad Y \circ \Gamma = AY, \quad U \circ \Gamma = (A^t)^{-1}U.$

Theorem. (Generalized symmetry principle). Any symmetry of an analytic equiaffine normalization induces a global symmetry of the affine maximal surface generated by the equiaffine normalization.

If $\beta : I \to \Sigma$ is a regular curve s.t., $\alpha = \psi \circ \beta$, $Y = \xi \circ \beta$ and $U = N \circ \beta$ are analytic $\Rightarrow$ α is a pre-geodesic for the Blaschke metric if and only if

$$[\alpha', \alpha'', Y] + [U, U', U''] = 0 \qquad \text{on} \quad I. \qquad (3..2)$$

Then a regular analytic curve $\alpha : I \to \mathbb{R}^3$ is the geodesic of some affine maximal surface if and only if there exists an affine equiaffine normalization $\{Y, U\}$ of α satisfying $(3..2)$ and $\langle\alpha'', U\rangle = c$ for a positive constant c.

Thus, we can obtain that every planar analytic curve whose curvature does not vanish at any point is pre-geodesic of an affine maximal surface which has the plane containing the curve as a symmetry plane. Also, every analytic helix, (k/τ constant), is pre-geodesic of an affine maximal surface.

4. Helicoidal affine maximal surfaces

Consider $T_s(v) = A_sv + b_s$ a one-parametric subgroup of equiaffine transformations. From our existence Theorem and generalized symmetry Principle, an affine maximal surface invariant under T_s, $s \in \mathbb{R}$, is locally given as the surface generated by the following $\{T_s\}$-symmetric a.e.n $\{Y, U\}$, along the orbit $\alpha_p(s) = T_s(p)$ of a fixed point $p = (p_1, p_2, p_3)$,

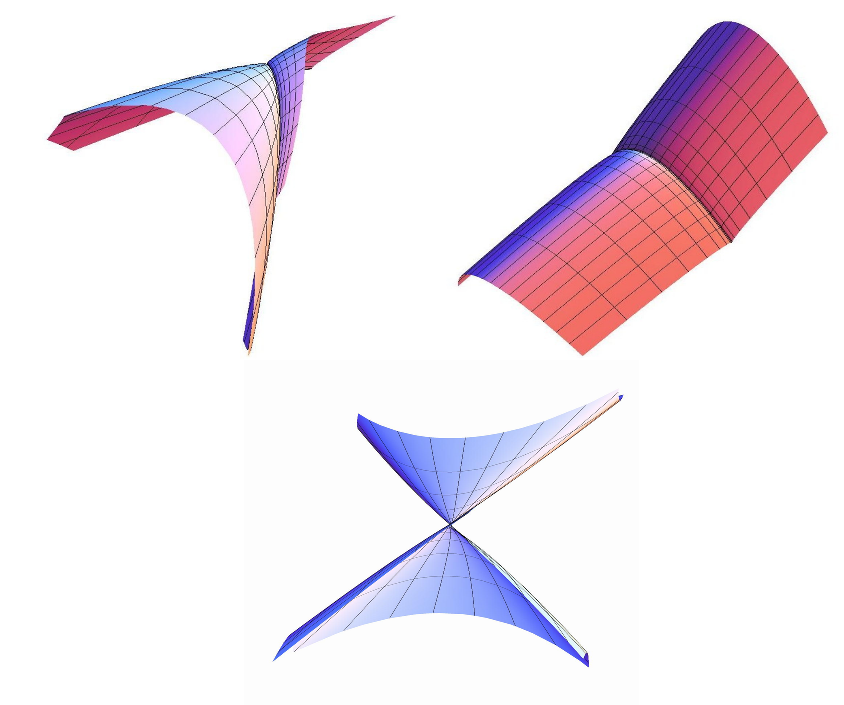
$$Y(s) = A_sY_p, \qquad U(s) = (A_s^t)^{-1}U_p$$

and $Y_p, U_p \in \mathbb{R}^3$ satisfy the necessary conditions for (2..1) holds. In particular, the Berwald-Blaschke metric must be constant along α_p.

We apply our representation to classify the affine maximal surfaces invariant under these groups.

4.1. Some $G_{1,a}$-invariant affine maximal surfaces

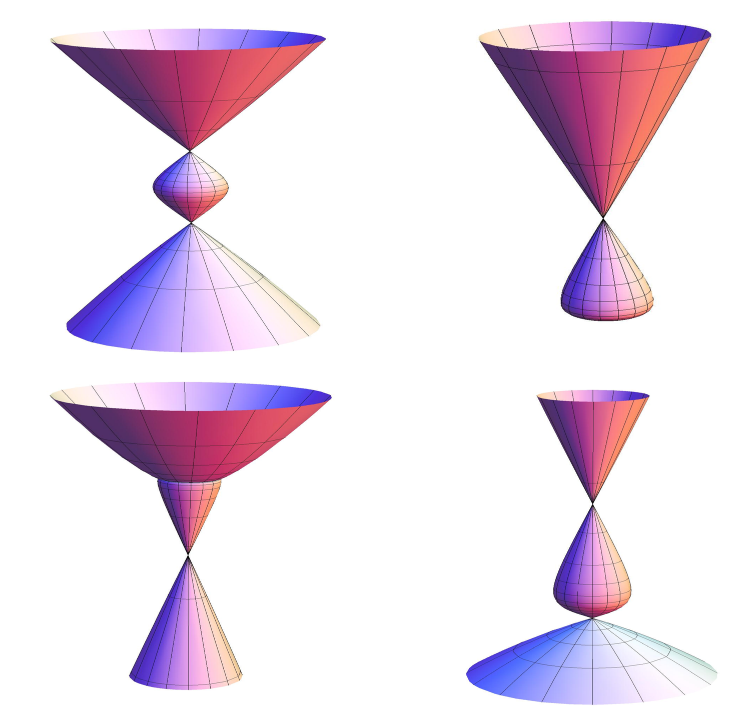
For this one-parametric group the orbit of a point p is given by $\alpha_p(s) = \left(p_1 + p_2as + p_3a\frac{s^2}{2} + a\frac{s^3}{6}, p_2 + p_3s + \frac{s^2}{2}, p_3 + s\right)$.



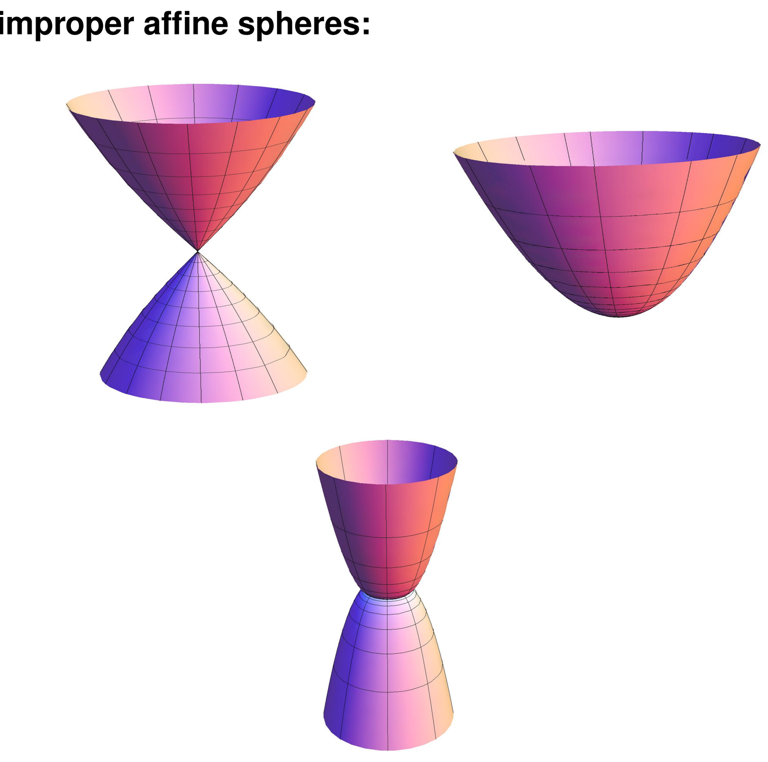
4.2. Some $G_{2,a}$-invariant affine maximal surfaces

In this case $\alpha_p(s) = (p_1\cos(s) + p_2\sin(s), -p_1\sin(s) + p_2\cos(s), p_3 + as)$.

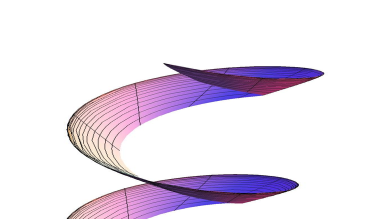
Rotational affine maximal surfaces:



Rotational improper affine spheres:



Non rotational $G_{2,a}$-invariant affine maximal surface:



5. Affine maximal maps

Some Helicoidal affine maximal surfaces:

- Are glued by analytic curves where the affine metric is degenerated but the affine normal and the affine conormal are well defined.
- Can be represented as in (2..2), where Φ is a well-defined holomorphic regular curve on the Riemann surface Σ.

Definition If a map $\psi : \Sigma \longrightarrow \mathbb{R}^3$ admits a representation as in (2..2) for a certain holomorphic curve Φ which satisfies that $[\Phi + \overline{\Phi}, \Phi_z, \overline{\Phi_z}]|dz|^2$ does not vanish identically, we say that ψ is an *affine maximal map*

Theorem $\alpha : I \longrightarrow \mathbb{R}^3$ a regular analytic curve with non-vanishing curvature. Then, for any non-vanishing regular analytic function $h : I \longrightarrow \mathbb{R}$ there exists a unique affine maximal map ψ_h containing $\alpha(I)$ in its set of singularities.

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