Some Geometric Aspects of the Hessian One Equation

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Abstract The Hessian one equation and its complex resolution provides an important tool in the study of improper affine spheres in $\mathbb{R}^3$ with some kind of singularities. The singular set can be characterized and, in most of the cases, it determines the surface. Here, we show how to obtain improper affine spheres with a prescribed singular set and construct some global examples with the desired singularities. We also classify improper affine spheres admitting a planar singular set.

1 Introduction

Differential geometry of surfaces and partial differential equations (PDEs) are related by a productive tie by means of which both theories out mutually benefited.

Many classic partial differential equations (PDEs) are link to interesting geometric problems, [18, 20, 27]. Sometimes, the geometry allows to establish non trivial properties of the solutions and to determine new solutions in terms of already known solutions.

One of the biggest contributions from geometry to the theory of partial differential equations is the Monge Ampère equation. Among the most outstanding Monge Ampère equation we can quote the Hessian one equation

$$\phi_{xx}\phi_{yy} - \phi_{xy}^2 = \varepsilon, \qquad \varepsilon \in \{-1, 1\}. \tag{1}$$

This is the easiest Monge Ampère equation and it appears, among others, in problems of affine differential geometry, flat surfaces or special Kähler manifolds.

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The equation (1) has been studied from a global perspective and the situation changes completely if we take $\varepsilon = 1$ (definite case) or $\varepsilon = -1$ (indefinite case). When $\varepsilon = 1$, Jörgens , [13, 14], proved that revolution surfaces provide the only entire solutions with at most an isolated singularity and solutions in $\mathbb{R}^2$ with a finite set of points removed are classified in [9]. The indefinite case is more complicated and we can not expect a classification result as in the definite case. Actually, $\phi(x,y) = xy + g(x)$ is an entire solution for any function g.

Another important issue in the theory of geometric PDEs is the study of singularities. Concerning with (1), a geometric theory of smooth maps with singularities (improper affine maps) has been developed in [21, 25]. In most of the cases the singular set determines the surface and, generically, the singularities are cuspidal edges and swallowtails, see [1, 5, 12, 23, 24].

In this paper we show how to obtain easily improper affine maps with a prescribed singular set and construct some global examples with the desired singularities. We also classify definite improper affine maps admitting a planar singular set.

The paper is organized as follows. In section 2 we introduce some notations and give a complex resolution for the equation (1).

In Section 3 we discuss a priori conditions on a curve in $\mathbb{R}^3$ to be a singular curve of an improper affine map with prescribed cuspidal edges and swallowtails. We also study isolated singularities both from a local as a global view.

In Section 4 we describe the global behavior of embedded complete definite improper affine maps with a planar singular set and those with only a finite number of isolated singularities.

2 The conformal structure

Let $\phi : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a solution to (1) on a planar domain Ω. Then its graph

$$\psi = \{(x,y,\phi(x,y)) : \ (x,y) \in \Omega\}$$

describes an improper affine sphere in the affine 3-space $\mathbb{R}^3$ with constant affine normal $\xi = (0,0,1)$, affine metric h,

$$h := \phi_{xx}\, dx^2 + \phi_{yy}\, dy^2 + 2\phi_{xy}\, dxdy, \tag{2}$$

and affine conormal N,

$$N := (-\phi_x, -\phi_y, 1). \tag{3}$$

From (2) and (3) it is easy to check that the following relations hold,

$$h = - < dN, d\psi >, \quad < N, \xi >= 1, \quad < N, d\psi >= 0, \tag{4}$$

$$\sqrt{\det(h)} = \det[\psi_x, \psi_y, \xi] = -\det[N_x, N_y, N], \tag{5}$$

see [19, 26] for more details. Conversely, up to unimodular transformations, any improper affine sphere in $\mathbb{R}^3$ is, locally, the graph over a domain in the x,y-plane of a solution to (1).

When $\varepsilon = 1$ (resp. $\varepsilon = -1$) the affine metric h induces a Riemann (Lorentz) surface structure on Ω known as the *underlying conformal structure of* $\phi(x,y)$.

It follows from (1) that,

$$(d\phi_x)^2 + \varepsilon\, dy^2 = \phi_{xx}h, \qquad (d\phi_y)^2 + \varepsilon\, dx^2 = \phi_{yy}h, \tag{6}$$

and the expression (6) indicates that the two first coordinates of ψ and N provides conformal parameters for h. Actually, let consider $\mathbb{C}_\varepsilon$ the complex (split-complex) numbers according to $\varepsilon = 1$ (or $\varepsilon = -1$), that is

$$\mathbb{C}_\varepsilon = \{z = s + j\,t \ : \ s,t \in \mathbb{R},\ j^2 = -\varepsilon,\ j1 = 1j\}, \tag{7}$$

see [4, 11] for more information, then it is not difficult to prove, see [3, 8, 23], that $\Phi : \Omega \longrightarrow \mathbb{C}^3_\varepsilon$,

$$\Phi := N + j\ \xi \times \psi, \tag{8}$$

is a planar holomorphic (split-holomorphic) curve. In fact, $\Phi = (-B, A, 1)$ where

$$A := -\phi_y + j\,x, \qquad B := \phi_x + j\,y, \tag{9}$$

are holomorphic (split-holomorphic) functions on Ω. Moreover, from (1) and (2),

$$|d\Phi|^2 = |dA|^2 + |dB|^2 = (\phi_{xx} + \phi_{yy})h, \tag{10}$$

and $|d\Phi|^2$ and h are in the same conformal class always that $\phi_{xx} + \phi_{yy}$ has a sign.

From (2) and (9), the metric h is given by

$$h := \mathrm{Im}(dA\overline{dB}) = |dG|^2 - |dF|^2 \tag{11}$$

where $2F = -B - \varepsilon j\,A$ and $2G = B - \varepsilon j\,A$, and the immersion ψ may be recovered as

$$\psi := \mathrm{Im}(A, B, \int A d\overline{B}) = -\frac{1}{2}\,\mathrm{Im}\int(\Phi + \overline{\Phi}) \times d\Phi. \tag{12}$$

or

$$\psi := (G + \overline{F}, \frac{|G|^2}{2} - \frac{|F|^2}{2} + 2\,\mathrm{Re}\int G d\overline{F}), \tag{13}$$

where in (13) the two first coordinates of ψ are identified as numbers of $\mathbb{C}_\varepsilon$ in the standard way.

Remark 1. The complex representation (12) is similar to the introduced in [6] and (13) was studied in [7, 8, 21, 25].

3 Allowing singularities

In this section we discuss improper affine spheres admitting some kind of singularities. First, we study when a prescribed curve of singularities determines the surface and then we deal with the case of isolated singularities both from a local as a global view.

Definition 1. Let Σ be a Riemann (Lorentz) surface and $\psi : \Sigma \longrightarrow \mathbb{R}^3$ be a differentiable map, ψ is called an improper affine map with constant affine normal $\xi = (0,0,1)$, if ψ is given as in (12) for some holomorphic (split-holomorphic) curve $\Phi = (-B, A, 1) : \Sigma \longrightarrow \mathbb{C}^3_\varepsilon$ satisfying that $\mathrm{Im}(dA\overline{dB})$ does not vanish identically on Σ.

Remark 2. Equivalent definitions of improper affine maps (also called improper affine fronts for other authors) has been introduced in [21, 25, 15].

From (3), (9) and (12) one may write $\Phi = N + j\ \xi \times \psi$, where N is the affine conormal of ψ and we have

$$h = \mathrm{Im}(dA\overline{dB}) = -\frac{\varepsilon j}{4}\det[\Phi + \overline{\Phi}, d\Phi, d\overline{\Phi}].$$

The singular set of ψ is the set of points where h degenerates. A singular point z_0 is called non degenerate if, writting $h = \rho|dz|^2$ around z_0, then

$$\rho(z_0) = 0, \qquad d\rho|_{z_0} \neq 0.$$

When z_0 is a non degenerate singular point, $\psi(z_0)$ is either an isolated singularity or the singular set of ψ around z_0 becomes a regular curve $\gamma : I \subset \mathbb{R} \longrightarrow \Sigma$. Generically, the image of these curves are singular curves with cuspidal edges and swallowtails, see [5, 25, 12]. In [17], we have the following criterion for the singular curve $\alpha = \psi \circ \gamma$,

Theorem 1 ([17]**).** *If η is a vector field along γ, with $\eta(s) \neq 0$ in the kernel of $d\psi_{\gamma(s)}$ for any s in the interval I, then*

1. *$\gamma(0) = z_0$ is a cuspidal edge if and only if $\det[\gamma'(0), \eta(0)] \neq 0$, where $\det$ denotes the usual determinant and prime indicates differentiation with respect to s.*
2. *$\gamma(0) = z_0$ is a swallowtail if and only if $\det[\gamma'(0), \eta(0)] = 0$ and*

$$\frac{d}{ds}\Big|_{s=0} \det[\gamma'(s), \eta(s)] \neq 0.$$

3.1 *Prescribing singular curves*

The *affine Björling problem* of finding an improper affine map containing a curve α with a prescribed affine conormal U along it has been discuss in [1, 23] and its solution is applied to see that a non constant singular curve determines the surface.

Actually, if we assume that $\alpha : I \longrightarrow \mathbb{R}^3$ is an analytic curve, which is in the singular set of a definite (indefinite) improper affine map ψ^ε, then from (4) the affine conormal U along α satisfies

$$< \alpha', U >= 0, \qquad < U, \xi >= 1, \qquad < \alpha'', U >= 0.$$

Hence, if α is non constant but $\det[\alpha', \alpha'', \xi] \equiv 0$ on I, then $\alpha' \times \alpha'' \equiv 0$ and α is an straight line with a constant tangent vector v. In this case, $< N, v >= 0$ and the conormal N of ψ^ε satisfies $\det[N, N_z, N_{\bar{z}}] \equiv 0$ on a neighborhood of α which is a contradiction.
But, if $\det[\alpha', \alpha'', \xi] \neq 0$ on I, then U is uniquely determined by α and it may be written as

$$U = \frac{\alpha' \times \alpha''}{\det[\alpha', \alpha'', \xi]}. \tag{14}$$

Then, ψ^ε is uniquely determined as in (12) by the holomorphic (split-holomorphic) curve

$$\Phi_\varepsilon = \frac{\alpha_z \times \alpha_{zz}}{\det[\alpha_z, \alpha_{zz}, \xi]} + j\, \xi \times \alpha, \tag{15}$$

which is defined in a neighborhood of I in $\mathbb{C}_\varepsilon$ where the holomorphic (split-holomorphic) extension of α is well defined.

Theorem 2. *Let* $\alpha : I \longrightarrow \mathbb{R}^3$ *be an analytic curve satisfying* $\det[\alpha', \alpha'', \xi] \neq 0$ *on* I*. Then the following items hold*

- *there exists a unique definite improper affine map containing* $\alpha(I)$ *in its singular set.*
- *if* $\det[\alpha', \alpha'', \alpha''']^2 \neq \det[\alpha', \alpha'', \xi]^4$ *on* I*, then there exists a unique indefinite improper affine map containing* $\alpha(I)$ *in its singular set.*

Moreover, in both cases $\alpha(s)$ *is a cuspidal edge for all* $s \in I$*.*

Proof. From (12) and (14), we have that along I the improper affine map ψ^ε given by Φ_ε satisfies

$$\begin{aligned}\psi^\varepsilon_z &= \frac{\varepsilon\, j}{4}((\Phi + \overline{\Phi}) \times \Phi_z) = \frac{\varepsilon\, j}{2}\, U \times U' - \frac{1}{2} U \times \left(\xi \times \alpha'\right)\\ &= \frac{1}{2}\alpha' + \frac{\varepsilon\, j}{2}\, U \times U' = \frac{1}{2}\alpha' + \frac{\varepsilon\, j}{2}\, \frac{\det[\alpha', \alpha'', \alpha''']}{\det[\alpha', \alpha'', \xi]^2}\alpha'\end{aligned}$$

and ψ^ε contains the curve α with

$$\psi^\varepsilon_s = \alpha', \qquad \psi^\varepsilon_t = -\frac{\det[\alpha', \alpha'', \alpha''']}{\det[\alpha', \alpha'', \xi]^2}\alpha'. \tag{16}$$

Thus, from (4), (5) and (16), we get $\det[\psi^\varepsilon_s, \psi^\varepsilon_t, \xi](s,0) = 0,\ \forall s \in I$, and

$$\frac{d}{dt}\Big|_{(s,0)} \det[\psi^\varepsilon_s, \psi^\varepsilon_t, \xi] = \det[\psi^\varepsilon_{ts}, \psi^\varepsilon_t, \xi](s,0) - \varepsilon \det[\psi^\varepsilon_s, \psi^\varepsilon_{ss}, \xi](s,0)$$
$$= \det[\alpha', \alpha'', \xi]\Big(-\varepsilon - \frac{\det[\alpha', \alpha'', \alpha''']^2}{\det[\alpha', \alpha'', \xi]^4}\Big) \neq 0.$$

That is, α is a non degenerate singular curve and the kernel of $d\psi^\varepsilon$ at $\gamma(s) = (s,0)$ is spanned by $\eta = (\det[\alpha', \alpha'', \alpha'''], \det[\alpha', \alpha'', \xi]^2)$. We conclude that $\det(\gamma', \eta) = \det[\alpha', \alpha'', \xi]^2 \neq 0$ and $\alpha(s)$ is a cuspidal edge for all $s \in I$ from Theorem 1.

Fig. 1 Indefinite improper affine maps whose singular set contains $\alpha(s) = (\cos(s), \sin(s), as)$ with $a = 0.2$ and $a = 0$

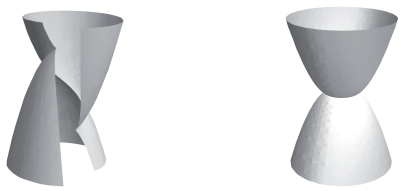

Fig. 2 Definite improper affine maps whose singular set contains $\alpha(s) = (\cos(s), \sin(s), as)$ with $a = 0.5$ and $a = 0$

Theorem 3. *Let* $\alpha : I \longrightarrow \mathbb{R}^3$ *be an analytic curve satisfying* $\det[\alpha', \alpha'', \xi] \neq 0$ *on* $I \setminus \{0\}$ *and such that* $0 \in I$ *is a zero of* α', $\alpha' \times \alpha''$, $\det[\alpha', \alpha'', \xi]$ *and* $\det[\alpha', \alpha'', \alpha''']$ *of order* 1, 2, 2 *and* 3 *respectively. Then the following items hold*

- *there exists a unique definite improper affine map containing* $\alpha(I)$ *in its singular set.*
- *if* $\det[\alpha', \alpha'', \alpha''']^2 \neq \det[\alpha', \alpha'', \xi]^4$ *on* $I \setminus \{0\}$*, then there exists a unique indefinite improper affine map containing* $\alpha(I)$ *in its singular set.*

Moreover, in both cases $\alpha(0)$ *is a swallowtail.*

Proof. Following the same arguments as in the proof of Theorem 2, we have that α is a non degenerate singular curve of ψ^ε and the kernel of $d\psi^\varepsilon$ at $\gamma(s) = (s,0)$

is spanned by $\eta = (1, \det[\alpha', \alpha'', \xi]^2 / \det[\alpha', \alpha'', \alpha'''])$. But from the hypothesis, 0 is a zero of order 1 of $\det(\gamma', \eta) = \det[\alpha', \alpha'', \xi]^2 / \det[\alpha', \alpha'', \alpha''']$ and $\alpha(0)$ is a swallowtail from Theorem 1.

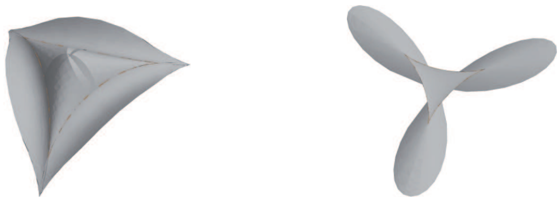

Fig. 3 Improper affine maps with three swallowtails

3.2 Isolated singularities

It is well known, see [1], that the conformal structure of the affine metric around any isolated singularity of a definite improper affine map is that of an annulus. Moreover, any definite improper affine map must be symmetric with respect to point reflection in $\mathbb{R}^3$ through any isolated embedded singularity.
In the case of indefinite improper affine maps and when the conformal structure of the affine metric around an isolated singularity is that of an annulus we have as application of the affine Björling problem that (see [1, 24]),

Theorem 4 ([1, 24]). *Let $U : \mathbb{R} \longrightarrow \mathbb{R}^2 \times \{1\}$ be a 2π-periodic regular analytic parameterization of a convex curve. Then, there exists a unique (definite) indefinite improper affine map ψ, with a non removable isolated singularity, where the affine conormal is tending to U. Moreover, it is embedded if and only if $U(\mathbb{R})$ is a Jordan curve.*

Fig. 4 Improper affine maps with isolated singularities

But in the indefinite case, one may construct improper affine maps with non removable isolated singularities around which the conformal structure of the affine metric is a punctured disk $\mathscr{D}^*$. Actually, from (11) and (12) it is easy to see the following result,

Theorem 5. *Let $A : \mathscr{D} \longrightarrow \mathbb{C}_{-1}$ be a split-holomorphic function satisfying $A_z = H^2$ for some split-holomorphic function $H : \mathscr{D} \longrightarrow \mathbb{C}_{-1}$. If $z_0 \in \mathscr{D}$ is an isolated zero of F, then the indefinite improper affine map $\psi : \mathscr{D} \longrightarrow \mathbb{R}^3$ given, as in* (12)*, by the split-holomorphic curve $\Phi(z) = (jz, A(z), 1)$, is well defined on $\mathscr{D}^* = \mathscr{D} - \{z_0\}$ and it has a non removable isolated singularity at z_0.*

Remark 3. By using the Theorem 5 we can construct indefinite improper affine map $\psi : \mathbb{C}_{-1} \longrightarrow \mathbb{R}^3$ with a finite number of prescribed isolated singularities at the points $\{z_1, \cdots, z_n\}$. For this is enough to consider a split-holomorphic function $H : \mathbb{C}_{-1} \longrightarrow \mathbb{C}_{-1}$ with zeros at the points $\{z_1, \cdots, z_n\}$.

Fig. 5 Entire solutions on the puncture plane obtained by taking $H(z) = z$ in Theorem 5

4 Global results

The aim of this section is to determine the global behavior of embedded complete definite improper affine maps such that any connected component of its singular set is mapped on a plane in $\mathbb{R}^3$ and those with only a finite number of isolated singularities.

4.1 The case of finitely many isolated singularities.

In [9] is proved the existence of entire solutions of (1) with any finite number of isolated singularities. The situation is totally different for an embedded complete definite improper affine map, where complete means that the affine metric is complete outside a compact subset.

Actually, from the generalized symmetry principle one has, [1, Theorem 4.2], any definite improper affine map must be symmetric with respect to point reflection in $\mathbb{R}^3$ through any isolated embedded singularity. As immediate consequence we has

Theorem 6. *Any embedded complete definite improper affine map whose singular set is a finite number of isolated singularities must be rotational, see Figure 4.*

Proof. An easy application of the Maximum Principle let us to see that any embedded complete improper affine map with only one isolated singularity must be rotational. Consequently, it is enough to prove that if a complete improper affine map $\psi : \Sigma \longrightarrow \mathbb{R}^3$ has two different isolated singularities p_1 and p_2, then it has infinitely many isolated singularities.

In fact, having in mind that $\psi(\Sigma)$ is symmetric with respect the reflections, s_1 and s_2, in $\mathbb{R}^3$ through the points p_1 and p_2, respectively, we get that

$$s_1(p_2), s_2(p_1), s_2 \circ s_1(p_2), s_1 \circ s_2(p_1), s_1 \circ s_2 \circ s_1(p_2), \cdots$$

also are isolated singularities of the map.

4.2 Embedded complete definite improper affine map with a planar singular set

We shall prove the following result:

Theorem 7. *Let $\psi : \Sigma \longrightarrow \mathbb{R}^3$ be an embedded complete definite improper affine map with a non-degenerate analytic singular set $\mathscr{S} \subset \Sigma$ such that $\psi(\mathscr{S})$ lies on a plane Π in $\mathbb{R}^3$. Then ψ is a snowman rotational improper affine map (see Figure 3.1)*

Proof. Let $\mathscr{K} \subset \Sigma$ be a compact containing $\mathscr{S}$ in its interior. Thanks to a classical result of Huber, [10], $\Sigma \setminus \mathrm{int}(\mathscr{K})$ is conformally a compact Riemann surface with compact boundary and finitely many points removed which are the ends of ψ.

But ψ is an embedding and then each end is asymptotic to one of rotational type (see [8]). Consider Σ^+ a connected component of $\psi(\Sigma) \setminus \psi(\mathscr{S})$, if we add to $\Sigma^+ \cup \partial\Sigma^+$ the planar bounded regions determined by the convex Jordan curves of its boundary, we get a globally convex surface $\widetilde{\Sigma}^+$ in $\mathbb{R}^3$.
It is clear than Σ^+ has at least one end, otherwise adding its reflexion respect the plane Π we get a compact flat improper affine map without boundary, which is impossible see [21].

Consider $\Phi = (B, A, 1)$ the holomorphic curve associated to Σ^+ and denote by Σ^+_* the corresponding improper affine map associate to the holomorphic curve $\Phi_*(-jA, -jB, 1)$, then Σ^+_* has the following properties:

1. the boundary of Σ^+_* is a singular point $a \in \mathbb{H}^3$.
2. Having in mind that any embedded complete end is of rotational type, see [8], any end of Σ^+_* is also embedded and complete, moreover Σ^+_* has the same number of ends as Σ^+.

In other words, $\Sigma_*^+ \cup \{a\}$ is a non compact complete definite improper affine map with only one isolated singularity. An easy application of the Maximum Principle says it must be rotational and, consequently, Σ^+ is also rotational and then the Theorem follows easily

Definite improper affine map with a planar singular set also are symmetric. Actually, we have

Proposition 1. *Any improper affine map containing an analytic singular curve lying on a plane Π in $\mathbb{R}^3$ must be symmetric with respect to the plane Π.*

Proof. Let $\psi : \Sigma \longrightarrow \mathbb{R}^3$ be the improper affine map having

$$\alpha = (\alpha_1, \alpha_2, 0) : I \longrightarrow \mathbb{R}^3$$

as a singular curve with affine conormal $V = (0,0,1)$ along α, then from (15), ψ is determined by the holomorphic (split-holomorphic) curve $\Phi_\varepsilon(z) = (\alpha_1(z), \alpha_2(z), 1)$, z in a neighborhood Ω_ε of I in $\mathbb{C}_\varepsilon$ where the holomorphic (split-holomorphic) extension of α is well determined. Then, from the Riemann-Schwarz symmetry principle we have that $\overline{\alpha_i(z)} = \alpha_i \bar{z}$ and , we conclude $\psi(\Omega_\varepsilon)$ is symmetric respect to the plane Π in $\mathbb{R}^3$.

Using this fact, we can generalize the Theorem 7 as follows:

Theorem 8. *Let $\psi : \Sigma \longrightarrow \mathbb{R}^3$ be an embedded complete improper affine map with a non-degenerate analytic singular set $\mathscr{S} \subset \Sigma$ such that any connected component of $\psi(\mathscr{S})$ lies on a plane in $\mathbb{R}^3$. Then ψ is a snowman rotational improper affine map.*

Remark 4. There is a flat metric associated with (1) that connects the equation to another interesting family of surfaces. Actually, if we consider on Ω de Riemannian metric

$$ds^2 = dx^2 + dy^2, \tag{17}$$

one may checks, from (2) and (17), that h satisfies the Codazzi-Mainardi equations of classical surface theory with respect to the metric ds^2. In other words, the pair (ds^2, h) of real quadratic forms is a Codazzi pair on Ω (see for instance [2, 16] for more information about Codazzi pairs). Moreover, from (1), (2) and (17), (ds^2, h) has constant extrinsic curvature $K(ds^2, h) = \varepsilon$ and from the existence and uniqueness theorem of surfaces in a space form we have that, locally, (Ω, ds^2) is isometrically immersed in the 3-dimensional space form $\mathbb{M}^3(-\varepsilon)$ of constant sectional curvature $-\varepsilon$. Conversely, any flat surface in $\mathbb{M}^3(-\varepsilon)$ has around any point a local coordinates (x,y) such that its second fundamental form may be written as in (2) where ϕ is solution to (1).

In [22] you can find similar results to the above mentioned theorems for flat surfaces in the hyperbolic space.

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