# Singular curves of affine maximal maps [1]

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**Abstract**

We analyze the solution of the affine Cauchy problem with a given analytic curve in its singular set. Then, we show how to obtain affine maximal maps with prescribed cuspidal edges and swallowtails. As consequence, we extend our recent work about improper affine spheres with singular curves.

## 1 Introduction

The family of affine maximal surfaces in $\mathbb{R}^3$ is an important subject in geometric analysis, since they are extremals of a geometric functional and the associated Euler-Lagrange equation is a non linear fourth order partial differential equation, which generalizes the Hessian one equation, see [B, LJSX].

In particular, the improper affine spheres are the umbilical affine maximal surfaces. Thus, the famous result by Jörgens in [J] motivated that Calabi proposed the affine Bernstein problem, which asked if the elliptic paraboloid is the only global example.

He also proved that, for locally strongly convex affine maximal surfaces, the second variation of the equiaffine area functional is always negative and obtained an affine Weierstrass formula, in terms of harmonic vector fields and holomorphic curves, see [C1, C2, C3].

After Calabi's work, the use of geometric methods in studying PDEs of affine differential geometry was continued by different authors and the affine Bernstein problem was solved affirmatively, see [LJ, MM, TW1, TW2, TW3].

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This lack of global regular examples has led to a recent study of affine maximal maps, that is, affine maximal surfaces with some singularities. This has revealed an interesting global theory, where the solution of the affine Cauchy problem shows the existence of an important amount of affine maximal surfaces with singular curves and isolated singularities, see [ACG, AMM1, AMM2, AMM3, AMM4, C, IM, M].

Here, we characterize when an analytic curve of $\mathbb{R}^3$ is the singular curve of some affine maximal map with prescribed cuspidal edges and swallowtails.

First, in Section 2, we introduce the notion of affine maximal map with a conformal representation, which generalizes the Weierstrass formula for improper affine spheres.

Thus, in Section 3, we can take the solution of the affine Cauchy problem and give the conditions to the existence and uniqueness of affine maximal maps with the desired singularities. In particular, we also characterize when an analytic curve of $\mathbb{R}^3$ is the singular curve of some improper affine sphere with prescribed cuspidal edges and swallowtails.

# 2 Improper affine spheres and affine maximal maps

Consider $\psi : \Sigma \longrightarrow \mathbb{R}^3$ a definite improper affine sphere, that is, an immersion with constant affine normal $\xi$ and Riemannian affine metric $h$. Then, see [LSZ, NS], up to an equiaffine transformation, one has $\xi = (0,0,1)$ and $\psi$ can be locally seen as the graph of a solution $f(x,y)$ of the Hessian one equation

$$f_{xx}f_{yy} - f_{xy}^2 = 1. \tag{2.1}$$

In such case, the definite affine metric $h$ of $\psi$ is given by

$$h = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2, \tag{2.2}$$

the affine conormal $N$ is

$$N = (-f_x, -f_y, 1) \tag{2.3}$$

and (2.1) is equivalent to

$$\sqrt{det(h)} = [\psi_x, \psi_y, \xi] = [N_x, N_y, N], \tag{2.4}$$

that is, the volume element of $h$ coincides with the determinant $[.,.,\xi]$.

We also observe that $h = -\langle dN, d\psi\rangle$ and $N$ is determined by

$$\langle N, \xi\rangle = 1, \qquad \langle N, d\psi\rangle = 0, \tag{2.5}$$

with the standard inner product $\langle\ ,\ \rangle$ in $\mathbb{R}^3$. Moreover, from (2.1), (2.2) and (2.3), one can obtain

$$\Delta_h N = 0,$$

where $\Delta_h$ is the Laplace-Beltrami operator associated to $h$.

Actually, see [C3, FMM1, FMM2, L], if we take a conformal parameter $z = s+it \in \mathbb{C}$ for the Riemannian affine metric $h$, then from (2.4) and (2.5) we have

$$h = 2\rho\, dz\, d\overline{z}, \qquad \rho = \langle N, \psi_{z\overline{z}}\rangle = -i[\psi_z, \psi_{\overline{z}}, \xi] = -i[N, N_z, N_{\overline{z}}] > 0 \tag{2.6}$$

and

$$\xi = \frac{-i}{\rho} N_z \times N_{\overline{z}}, \qquad N = \frac{-i}{\rho}\psi_z \times \psi_{\overline{z}}, \tag{2.7}$$

where $\overline{z} = s - it$ and by $\times$ we denote the cross product in $\mathbb{C}^3$. Hence, we get

$$\psi_z = iN \times N_z, \qquad N_z = i\xi \times \psi_z \tag{2.8}$$

and the Laplacian

$$\psi_{z\overline{z}} = \rho\xi, \qquad N_{z\overline{z}} = 0. \tag{2.9}$$

**Remark 1.** *In general, the holomorphic curve $N_z : \Sigma \longrightarrow \mathbb{C}^2 \times \{0\}$ can be integrated locally and the harmonic affine conormal $N$ is the real part of a holomorphic curve. However, since the affine normal $\xi$ is constant, from (2.8), we have a global holomorphic curve $\Phi : \Sigma \longrightarrow \mathbb{C}^2 \times \{1/2\}$, such that*

$$N = \Phi + \overline{\Phi}, \qquad i\xi \times \psi = \Phi - \overline{\Phi}. \tag{2.10}$$

*Conversely, we can recover the improper affine sphere $\psi$ with its harmonic affine conormal $N : \Sigma \longrightarrow \mathbb{R}^2 \times \{1\}$ and the conformal class of its affine metric as*

$$\psi = 2\,\mathrm{Re}\int iN \times N_z dz = 2\,\mathrm{Re}\int i\left(\Phi + \overline{\Phi}\right) \times d\Phi. \tag{2.11}$$

Next, we change the plane $\mathbb{R}^2 \times \{1\}$ by the space $\mathbb{R}^3$ and obtain the following generalization of improper affine spheres.

**Definition 2.1.** *Let $\Sigma$ be a Riemann surface. We say that a map $\psi : \Sigma \longrightarrow \mathbb{R}^3$ is an* affine maximal map *if there exists a harmonic vector field $N : \Sigma \longrightarrow \mathbb{R}^3$ such that $[N, N_z, N_{\overline{z}}]$ does not vanish identically and $\psi$ is given as in (2.11).*

**Remark 2.** *The singular set $\mathcal{S}_\psi$ of an affine maximal map $\psi$ is the set of points where $\rho = -i[N, N_z, N_{\overline{z}}]$ vanishes. It is clear that $\Sigma \setminus \mathcal{S}_\psi$ is dense in $\Sigma$.*

*Observe that in this case the affine normal $\xi$ given by (2.7) may not be well-defined on $\mathcal{S}_\psi$.*

*Of course, when $\mathcal{S}_\psi = \emptyset$ one has an affine maximal surface, which is an improper affine sphere if $\xi$ is constant.*

From now on, we consider an affine maximal map $\psi : \Sigma \longrightarrow \mathbb{R}^3$ with affine conormal $N : \Sigma \longrightarrow \mathbb{R}^3$ and a regular analytic curve $\gamma : I \longrightarrow \Sigma$, for an interval $I$.

Also, we denote $\alpha = \psi \circ \gamma$ and $U = N \circ \gamma$, with parameter $s \in I$. Thus, by the Inverse Function Theorem, there exists a conformal parameter $z = s + it$ and we can parameterize a piece of the affine maximal map by $\psi : \Omega \subset \mathbb{C} \longrightarrow \mathbb{R}^3$, with $I \subset \Omega$,

$$\psi(s,0) = \alpha(s), \qquad N(s,0) = U(s).$$

Then, from the Identity Principle and (2.11), we obtain that the map $\psi$ can be recovered as

$$\psi = \alpha(s_0) + 2\,\mathrm{Re} \int_{s_0}^{z} iN \times N_z dz, \qquad z \in \Omega \subset \mathbb{C}, \quad s_0 \in I, \tag{2.12}$$

where the affine conormal $N$ is given by

$$N(z) = \mathrm{Re}\left( U(z) - \mathrm{i} \int_{s_0}^{z} \eta(\zeta) d\zeta \right), \qquad z \in \Omega \subset \mathbb{C}, \quad s_0 \in I, \tag{2.13}$$

with $U(z)$ and $\eta(z)$ the holomorphic extensions of $U(s) = N(s,0)$ and $\eta(s) = N_t(s,0)$ to a neighborhood $\Omega$ of $I$.

Moreover, from (2.11) we have that, along $\alpha$,

$$\eta \times U = -\alpha' \tag{2.14}$$

and $s_0 \in I$ is a singular point of $\psi$ if

$$\rho(s_0) = \langle U, \alpha'' \rangle(s_0) = -\langle U', \alpha' \rangle(s_0) = 0, \tag{2.15}$$

where by prime we indicate derivative with respect to $s$.

Conversely, if $\alpha, \eta, U : I \longrightarrow \mathbb{R}^3$ are analytic maps satisfying (2.14) on $I$, then there exists a unique solution $\psi$ to the geometric Cauchy problem for affine maximal maps with these data, see [AMM1, AMM2].

# 3 Singular curves of affine maximal maps

We analyze the solution of the above problem when $\alpha : I \longrightarrow \mathbb{R}^3$ is a singular curve of $\psi$. This case is interesting because the data $U$ and $\eta$ are determined by $\alpha$ and two analytic functions. In fact, from (2.14) and (2.15) we have

$$U = \frac{\alpha' \times \alpha''}{\lambda} \tag{3.1}$$

and

$$\eta = \phi \alpha' \times \alpha'' - \frac{\lambda}{|\alpha' \times \alpha''|^2} (\alpha' \times \alpha'') \times \alpha'. \tag{3.2}$$

Thus, we can obtain the following results.

**Theorem 3.1.** *Let $\alpha : I \longrightarrow \mathbb{R}^3$ be an analytic curve with non-vanishing curvature on $I$. Then, for any analytic functions $\lambda, \phi : I \longrightarrow \mathbb{R}$, $\lambda > 0$, there is a unique affine maximal map $\psi$ with $U$ and $\eta$ given by (3.1) and (3.2), respectively.*

*Moreover, $\alpha$ is a singular curve of $\psi$ and $\alpha(s)$ is a cuspidal edge for all $s \in I$.*

*Proof.* From the hypothesis, we can define the affine maximal map $\psi$ as in (2.12), with the affine conormal $N$ given by (2.13). Now, since $N$ is harmonic, from (3.1) and (3.2), we get that, along $\alpha$,

$$[N, N_s, N_t] = [U, U', \eta] = -\langle \alpha', U' \rangle = 0$$

and

$$\begin{aligned} [N, N_s, N_t]_t &= [U, \eta', \eta] - [U, U', U''] \\ &= -\lambda - \frac{[\alpha', \alpha'', \alpha''']^2}{\lambda^3} < 0. \end{aligned}$$

Consequently, $[N, N_z, N_{\overline{z}}]$ does not vanish identically and the points of $\alpha$ are the unique singular points in a neighborhood of it.

Also, we have that, along $\alpha$,

$$\psi_s = \alpha', \qquad \psi_t = -\frac{[\alpha', \alpha'', \alpha''']}{\lambda^2}\alpha' \tag{3.3}$$

and the kernel of $d\psi$ at $\gamma(s) = (s, 0)$ is spanned by

$$\nu = ([\alpha', \alpha'', \alpha'''], \lambda^2).$$

Hence, $det(\gamma', \nu) = \lambda^2 \neq 0$ and we conclude that $\alpha(s)$ is a cuspidal edge for all $s \in I$, see [KRSUY]. □

**Example 3.2.** *If we take the curve $\alpha : \mathbb{R} \longrightarrow \mathbb{R}^3$ and the functions $\lambda, \phi : \mathbb{R} \longrightarrow \mathbb{R}$ given by*

$$\alpha(s) = (\cos(s), \sin(s), 0), \qquad \lambda(s) = \phi(s) = 1, \qquad \forall s \in \mathbb{R},$$

*then, from Theorem 3.1, the data*

$$U(s) = (0, 0, 1), \qquad \eta(s) = (\cos(s), \sin(s), 1),$$

*provide the harmonic affine conormal $N : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$,*

$$N(s, t) = (\cos(s)\sinh(t), \sin(s)\sinh(t), 1 + t)$$

*and the affine maximal map $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, with coordinates*

$$\begin{aligned} \psi_1(s, t) &= (1 + t)\cos(s)\cosh(t) - \cos(s)\sinh(t), \\ \psi_2(s, t) &= (1 + t)\sin(s)\cosh(t) - \sin(s)\sinh(t), \\ \psi_3(s, t) &= \frac{t}{2} - \frac{1}{4}\sinh(2t). \end{aligned}$$

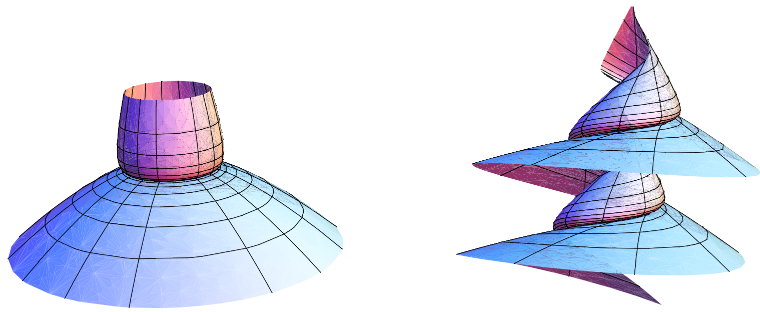

Figure 1: Affine maximal maps with cuspidal edges.

*So, around $t = 0$, the singular set of $\psi$ is the circle $\alpha(\mathbb{R}) = \psi(\mathbb{R} \times \{0\})$ and the singularities are cuspidal edges, (see Figure 1).*

*Similarly, we can obtain an affine maximal map $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ with*

$$\alpha(s) = (\cos(s), \sin(s), s), \qquad \lambda(s) = \phi(s) = 1, \qquad \forall s \in \mathbb{R}.$$

*That is, with the helix $\alpha(\mathbb{R}) = \psi(\mathbb{R} \times \{0\})$ in its singular set.*

**Theorem 3.3.** *Let $\alpha : I \longrightarrow \mathbb{R}^3$ be an analytic curve with non-vanishing curvature on $I - \{0\}$ and such that $0 \in I$ is a zero of $\alpha'$, $\alpha' \times \alpha''$ and $[\alpha', \alpha'', \alpha''']$ of order $1$, $2$ and $3$ respectively.*

*Then, for any analytic functions $\lambda, \phi : I \longrightarrow \mathbb{R}$, $\lambda > 0$ on $I - \{0\}$ and with a zero of order $2$ in $0$, there is a unique affine maximal map $\psi$ with $U$ and $\eta$ given by (3.1) and (3.2), respectively.*

*Moreover, $\alpha$ is a singular curve of $\psi$ and $\alpha(0)$ is a swallowtail.*

*Proof.* We follow the arguments of the above proof, from (3.1) to (3.3). Note that $U$, $\eta$ and $\psi_t$ are well defined by the hypothesis.

However, in this case, the kernel of $d\psi$ at $\gamma(s) = (s, 0)$ is spanned by

$$\nu = \Big(1, \frac{\lambda^2}{[\alpha', \alpha'', \alpha''']}\Big)$$

and $\alpha(0)$ is a swallowtail, because $0$ is a zero of order $1$ of

$$det(\gamma', \nu) = \frac{\lambda^2}{[\alpha', \alpha'', \alpha''']}.$$

$\square$

**Example 3.4.** *The curve $\alpha : \mathbb{R} \longrightarrow \mathbb{R}^3$ defined by*

$$\alpha(s) = \Big(\cos(s) + \frac{1}{2}\cos(2s),\ -\sin(s) + \frac{1}{2}\sin(2s),\ \frac{1}{6}\cos(3s)\Big)$$

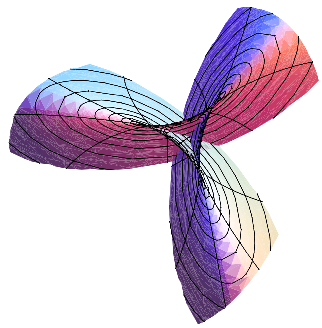

Figure 2: Affine maximal map with 3 swallowtails.

*has*

$$[\alpha', \alpha'', \alpha'''] = \sin(3s) - \frac{1}{2}\sin(6s),$$

*with the same* $2\pi$*-periodic zeros,* $\frac{2}{3}\pi$*,* $\frac{4}{3}\pi$ *and* $2\pi$*, that the function* $\lambda : \mathbb{R} \longrightarrow \mathbb{R}$ *given by*

$$\lambda(s) = 1 - \cos(3s).$$

*Thus, from Theorem 3.3, we can obtain an affine maximal map with* $\alpha$ *as a singular curve with three swallowtails connected by three arcs with cuspidal edges, (see Figure 2).*

Finally, from (2.5), (2.8) and (3.1), if we take $\lambda = [\alpha', \alpha'', \xi_0]$ and $\eta = -\xi_0 \times \alpha'$, with $\xi_0 = (0, 0, 1)$, then we can deduce the following results for definite improper affine spheres with singular curves, see [M1, M2] for the indefinite case.

**Corollary 3.5.** *Let* $\alpha : I \longrightarrow \mathbb{R}^3$ *be an analytic curve with* $[\alpha', \alpha'', \xi_0] \neq 0$ *on* $I$*. Then, there is a unique definite improper affine map containing* $\alpha(I)$ *in its singular set.*

*Moreover,* $\alpha(s)$ *is a cuspidal edge for all* $s \in I$*.*

**Corollary 3.6.** *Let* $\alpha : I \longrightarrow \mathbb{R}^3$ *be an analytic curve with* $[\alpha', \alpha'', \xi_0] \neq 0$ *on* $I - \{0\}$ *and such that* $0 \in I$ *is a zero of* $\alpha'$*,* $\alpha' \times \alpha''$*,* $[\alpha', \alpha'', \xi_0]$ *and* $[\alpha', \alpha'', \alpha''']$ *of order* 1, 2, 2 *and* 3 *respectively.*

*Then, there is a unique definite improper affine map containing* $\alpha(I)$ *in its singular set and* $\alpha(0)$ *is a swallowtail.*

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